

THE WU METRIC IS NOT UPPER SEMICONTINUOUS

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ABSTRACT. We give an example of a bounded pseudoconvex domain in \mathbb{C}^n , the Wu metric of which (associated to the Kobayashi–Royden or the Azukawa metric) is not upper semicontinuous.

H. Wu defined in [Wu 2] an invariant metric which was to combine invariant properties of the Kobayashi–Royden metric and regularity properties of Kähler metrics. The metric, called later the Wu metric, was studied in several papers (e.g. [Che–Kim 1], [Che–Kim 2], [Juc], [Jar–Pfl 3], [Jar–Pfl 2]). Jarnicki and Pflug pointed out ([Jar–Pfl 3], [Jar–Pfl 2]) that such an elementary property as its upper semicontinuity had not been completely understood. It was claimed ([Wu 2], [Wu 1]) or cited ([Che–Kim 1], [Juc]) without proof that the Wu metric is an upper semicontinuous pseudo-Hermitian metric defined on every complex manifold. We are going to present an example of a bounded pseudoconvex domain in \mathbb{C}^n which contradicts that statement.

1. DEFINITION

We present the sketch of the definition of the Wu metric in an abstract setting ([Jar–Pfl 3]). For detailed discussion we refer the reader to [Jar–Pfl 3] (or [Jar–Pfl 2]) and [Wu 2].

Let $p : \mathbb{C}^n \rightarrow \mathbb{R}_+$ be a \mathbb{C} -seminorm. Put:

$$I(p) := \{X \in \mathbb{C}^n : p(X) < 1\},$$

$$V(p) := \{X \in \mathbb{C}^n : p(X) = 0\},$$

$U(p) :=$ the orthogonal complement of V with respect to
the standard scalar product in \mathbb{C}^n ;

$$I_0(p) := I(p) \cap U(p).$$

Notice that $I_0(p)$ is bounded.

For any pseudo-Hermitian scalar product $s : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$, define

$$q_s(X) := \sqrt{s(X, X)}, \quad X \in \mathbb{C}^n.$$

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Let $\mathcal{F}(p)$ be the set of all pseudo-Hermitian scalar products $s : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ such that $q_s \leq p$ (or, equivalently, $I(p) \subset I(q_s)$). There exists a unique (!) element $s(p) \in \mathcal{F}(p)$ which is maximal with respect to the partial ordering \prec defined for $a, b \in \mathcal{F}(p)$:

$$a \prec b \text{ if } \det[a(e_j, e_k)]_{j,k=1,\dots,m} \leq \det[b(e_j, e_k)]_{j,k=1,\dots,m},$$

for any basis (e_1, \dots, e_m) of $U(p)$. In other words, $I_0(q_{s(p)})$ is an ellipsoid that has minimal volume of all ellipsoids containing the p -ball $I_0(p)$. This ellipsoid is also called (cf. [Che–Kim 1], [Che–Kim 2]) “the best fitting ellipsoid”.

We define

$$\begin{aligned} \widetilde{\mathbb{W}}p(X) &:= q_{s(p)}(X), \quad X \in \mathbb{C}^n; \\ \mathbb{W}p(X) &:= \sqrt{m} \widetilde{\mathbb{W}}p(X), \quad X \in \mathbb{C}^n, \end{aligned}$$

where $m := \dim U(p)$.

For a domain $D \subset \mathbb{C}^n$, denote by $\mathcal{M}(D)$ the space of all pseudometrics

$$\eta : D \times \mathbb{C}^n \rightarrow \mathbb{R}_+, \quad \eta(a; \lambda X) = |\lambda| \eta(a; X), \quad \lambda \in \mathbb{C}, (a, X) \in D \times \mathbb{C}^n,$$

such that

$$\forall a \in D \exists M, r > 0 : \eta(z; X) \leq M \|X\|, \quad z \in \mathbb{B}(a, r) \subset D, X \in \mathbb{C}^n,$$

where $\mathbb{B}(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}$ and $\|\cdot\|$ is the standard Euclidean norm.

For $\eta \in \mathcal{M}(D)$ we define the Wu pseudometric:

$$(\mathbb{W}\eta)(a; X) := (\mathbb{W}\widehat{\eta}(a; \cdot))(X), \quad a \in D, X \in \mathbb{C}^n.$$

Here $\widehat{\eta}$ denotes the Busemann pseudometric associated to η (cf. e.g. [Jar–Pfl 1]), i.e.

$$\widehat{\eta}(a; X) := \sup\{p(X)\}, \quad a \in D, X \in \mathbb{C}^n,$$

where the supremum is taken over all \mathbb{C} -seminorms p such that $p \leq \eta(a; \cdot)$.

For convenience we define also

$$(\widetilde{\mathbb{W}}\eta)(a; X) := (\widetilde{\mathbb{W}}\widehat{\eta}(a; \cdot))(X), \quad a \in D, X \in \mathbb{C}^n.$$

Some basic properties of the Wu pseudometric are listed in Proposition 1.

We denote by Δ the open unit disk in \mathbb{C} . Let $(\eta_D)_D$ be a family of pseudometrics defined for all domains $D \subset \mathbb{C}^n, n \geq 1$. We call $(\eta_D)_D$ a *holomorphically contractible family of pseudometrics* (cf. [Jar–Pfl 1], [Jar–Pfl 2]) if we have

$$\eta_\Delta(z; X) = \frac{|X|}{1 - |z|^2}, \quad z \in \Delta, X \in \mathbb{C},$$

$$\eta_{D_2}(F(z); F'(z)X) \leq \eta_{D_1}(z; X), \quad z \in D_1, X \in \mathbb{C}^{n_1},$$

for domains $D_1 \subset \mathbb{C}^{n_1}, D_2 \subset \mathbb{C}^{n_2}$ and every holomorphic mapping $F : D_1 \rightarrow D_2$.

We say that the family $(\eta_D)_D$ has the *product property* if

$$\begin{aligned} (1) \quad \eta_{D_1 \times D_2}((z, w); (X, Y)) &= \max(\eta_{D_1}(z; X), \eta_{D_2}(w; Y)), \\ &(z, w) \in D_1 \times D_2, (X, Y) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \end{aligned}$$

for any domains $D_1 \subset \mathbb{C}^{n_1}, D_2 \subset \mathbb{C}^{n_2}, n_1, n_2 \geq 1$.

Proposition 1 (cf. [Wu 2], [Jar–Pfl 3], [Jar–Pfl 2]).

(a) *If $\eta \in \mathcal{M}(D)$ is a continuous metric, then so is $\mathbb{W}\eta$.*

- (b) If $(\eta_D)_D$ is a holomorphically contractible family of pseudometrics, then for any biholomorphic mapping $F : D_1 \rightarrow D_2$ ($D_j \subset \mathbb{C}^n$, $j = 1, 2$) we have

$$\mathbb{W}_{\eta_{D_2}}(F(z); F'(z)X) = \mathbb{W}_{\eta_{D_1}}(z; X), \quad z \in D_1, X \in \mathbb{C}^n.$$

- (c) If $(\eta_D)_D$ is a holomorphically contractible family of pseudometrics, then for any holomorphic mapping $F : D_1 \rightarrow D_2$ ($D_1 \subset \mathbb{C}^{n_1}$, $D_2 \subset \mathbb{C}^{n_2}$) we have

$$\mathbb{W}_{\eta_{D_2}}(F(z); F'(z)X) \leq \sqrt{n_2} \mathbb{W}_{\eta_{D_1}}(z; X), \quad z \in D_1, X \in \mathbb{C}^n.$$

- (d) If $(\eta_D)_D$ is a family of pseudometrics satisfying the product property, then

$$\begin{aligned} \mathbb{W}_{\eta_{D_1 \times D_2}}((z, w); (X, Y)) &= \left((\mathbb{W}_{\eta_{D_1}}(z; X))^2 + (\mathbb{W}_{\eta_{D_2}}(w; Y))^2 \right)^{\frac{1}{2}}, \\ (z, w) \in D_1 \times D_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}, (X, Y) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}. \end{aligned}$$

Recall definitions of the k -th Carathéodory–Reiffen $(\gamma^{(k)})$, Azukawa (A) and Kobayashi–Royden (κ) pseudometrics. For details and properties see for example [Jar–Pfl 1], [Jar–Pfl 2].

For a domain $D \subset \mathbb{C}^n$ and $a \in D$, $X \in \mathbb{C}^n$, $k \in \mathbb{N} \setminus \{0\}$ define

$$\begin{aligned} \gamma_D^{(k)}(a; X) &:= \sup \left\{ \left| \frac{1}{k!} f^{(k)}(a) X \right|^{\frac{1}{k}} : f \in \mathcal{O}(D, \Delta), \text{ord}_a f \geq k \right\}; \\ A_D(a; X) &:= \sup \left\{ \limsup_{0 \neq \lambda \rightarrow 0} \frac{v(a + \lambda X)}{|\lambda|} : v : D \rightarrow [0, 1] \text{ is log-psh}, \right. \\ &\quad \left. \exists M, r > 0 : v(z) \leq M \|z - a\| \text{ if } \|z - a\| < r \right\}; \\ \kappa_D(a; X) &:= \inf \{ t > 0 : \exists \varphi \in \mathcal{O}(\Delta, D) : \varphi(0) = a, t\varphi'(0) = X \}. \end{aligned}$$

2. PROBLEM

Jarnicki and Pflug pointed out that the continuity of $\mathbb{W}\eta$ could be proved only under the assumption that η is continuous. The question of the upper semicontinuity of the Wu metric appears in a natural way, for instance in the definition of the integrated form $\int(\mathbb{W}\eta)$. In general, the upper semicontinuity of η does not imply the upper semicontinuity of $\mathbb{W}\eta$ (cf. [Jar–Pfl 3], [Jar–Pfl 2]). In the case where η is the Kobayashi–Royden (pseudo)metric the problem has remained open, even though Wu (cf. [Wu 1], [Wu 2]) and Cheung and Kim (cf. [Che–Kim 1]) claimed (without proof) the upper semicontinuity of $\mathbb{W}\kappa_D$.

Jarnicki and Pflug ([Jar–Pfl 3], [Jar–Pfl 2]) stated the following question:

Let $\alpha \in \{\gamma_D^{(k)}, A_D, \kappa_D\}$. Is $\mathbb{W}\alpha$ upper semicontinuous?

We construct a bounded pseudoconvex domain $G \subset \mathbb{C}^n$ such that $\mathbb{W}\kappa_G$ and $\mathbb{W}A_G$ are not upper semicontinuous. We get even more: if $(\alpha_D)_D$ is a holomorphically contractible family of metrics such that $\alpha_D \in \mathcal{M}(D)$ and $A_D \leq \alpha_D \leq \kappa_D$ for any domain $D \subset \mathbb{C}^n$, then $\mathbb{W}\alpha_G$ is not upper semicontinuous.

In view of Proposition 1, $\mathbb{W}\gamma_D^{(k)}$ is a continuous metric for a bounded domain D , since $\gamma_D^{(k)}$ is a continuous metric (cf. [Jar–Pfl 1]).

The problem remains open for $\gamma^{(k)}$ in the class of domains that are not biholomorphic to any bounded domain.

3. COUNTEREXAMPLE

To simplify the notation let $\mathbb{B}_{\eta_D}(a) := I(\eta_D(a; \cdot))$ denote the unit η_D -ball centered at a point $a \in D$ (for $\eta_D \in \mathcal{M}(D)$) and let $\mathcal{E}(B) := I(\widetilde{\mathbb{W}h}_B)$ stand for the smallest ellipsoid containing the balanced domain B . Here, h_B is the Minkowski functional of the domain B .

Remark 2. The construction of the Wu metric determines that

- (a) there exists a unique $\mathcal{E}(B)$ for any balanced domain $B \subset \mathbb{C}^n$;
- (b) if p is a \mathbb{C} -seminorm, then $\widetilde{\mathbb{W}p} = h_{\mathcal{E}(I(p))}$. Consequently, finding $\mathcal{E}(\mathbb{B}_{\eta_D}(a))$ is equivalent to finding $\widetilde{\mathbb{W}\eta_D}(a; \cdot)$ and $\mathbb{W}\eta_D(a; \cdot)$ for $\eta_D \in \mathcal{M}(D)$, $D \subset \mathbb{C}^n$, $a \in D$.

Remark 3. If B is a balanced Reinhardt domain, then

- (a) $\mathcal{E}(B)$ is a complete Reinhardt domain;
- (b) there exist $a_1, \dots, a_n \geq 0$ such that

$$\widetilde{\mathbb{W}h}_B(X) = (s(\widehat{h}_B)(X, X))^{\frac{1}{2}} = h_{\mathcal{E}(B)}(X) = \left(\sum_{j=1}^n a_j |X_j|^2 \right)^{\frac{1}{2}}, \quad X \in \mathbb{C}^n.$$

Proof of Remark 3. The set B is invariant under the action of

$$\Phi_\Lambda(X) := (\lambda_1 X_1, \dots, \lambda_n X_n), \quad X \in \mathbb{C}^n,$$

where $\Lambda = (\lambda_1, \dots, \lambda_n) \in (\partial\Delta)^n$. Thus $\Phi_\Lambda(\mathcal{E}(B)) = \mathcal{E}(\Phi_\Lambda(B)) = \mathcal{E}(B)$. Since the ellipsoid $\mathcal{E}(B)$ is convex, it has to be complete Reinhardt.

Let $[a_{jk}]_{j,k=1,\dots,n}$ be the matrix representation of the Hermitian scalar product $s(\widehat{h}_B)$ (recall $s(\widehat{h}_B)(X, X) = (\widetilde{\mathbb{W}h}_B)^2(X)$) in the canonical basis of \mathbb{C}^n , i.e. $s(\widehat{h}_B)(X, Y) = \sum_{j,k=1}^n a_{jk} X_j \bar{Y}_k$, $X, Y \in \mathbb{C}^n$. Then we have

$$\mathcal{E}(B) = \{X \in \mathbb{C}^n : \sum_{j=1}^n a_{jj} |X_j|^2 + 2 \operatorname{Re} \sum_{j < k} a_{jk} X_j \bar{X}_k < 1\},$$

which is not invariant under Φ_Λ unless $a_{jk} = 0$ for $j \neq k$. □

From now on, we restrict ourselves to \mathbb{C}^2 . However, some of the remarks can be easily transferred to the n -dimensional case.

Consider the mapping (cf. [Che–Kim 1], [Che–Kim 2])

$$\Psi : \mathbb{C}^2 \rightarrow \mathbb{R}_+^2, \quad \Psi(z, w) := (|z|^2, |w|^2), \quad (z, w) \in \mathbb{C}^2,$$

which helps to develop the intuition how the ellipsoids $\mathcal{E}(B)$ look and will be useful in all computations performed in the sequel.

Notice that Ψ transforms any bounded complete Reinhardt ellipsoid in \mathbb{C}^2 into a triangle

$$T_{a,b} := \{(u, v) \in \mathbb{R}_+^2 : \frac{u}{a} + \frac{v}{b} < 1\}$$

for some $a, b > 0$. In fact, the mapping Ψ determines the one-to-one correspondence between bounded complete Reinhardt ellipsoids in \mathbb{C}^2 and triangles $T_{a,b} \subset \mathbb{R}_+^2$.

Let $\partial^\circ T_{a,b}$ denote the hypotenuse of the triangle $T_{a,b}$, i.e.

$$\partial^\circ T_{a,b} := \{(u, v) \in \bar{T}_{a,b} : \frac{u}{a} + \frac{v}{b} = 1\}.$$

Lemma 4.

- (a) $T_{a,b} \subset T_{c,d}$ if and only if $a \leq c$ and $b \leq d$.
- (b) If $B \subset\subset \mathbb{C}^2$ is a balanced Reinhardt domain, then $\Psi(\mathcal{E}(B))$ is a (unique) triangle $T_{a,b}$ of smallest area that contains $\Psi(B)$. Moreover, $h_{\mathcal{E}(B)}^2(X) = \frac{|X_1|^2}{a} + \frac{|X_2|^2}{b}$, $X \in \mathbb{C}^2$.
- (c) $\Psi(\mathcal{E}(r_1\Delta \times r_2\Delta)) = T_{2r_1^2, 2r_2^2}$ for $r_1, r_2 > 0$.
- (d) $\mathcal{E}(\frac{\sqrt{3}}{2}\Delta \times \frac{1}{2}\Delta) \not\subset \mathcal{E}(B_{\|\cdot\|}(0))$.

Proof. (a) is trivial.

(b) Use Remark 3. Let $a, b > 0$ be such that $(h_{\mathcal{E}(B)})^2(X) = \frac{|X_1|^2}{a} + \frac{|X_2|^2}{b}$, $X \in \mathbb{C}^2$. We have $\text{vol}(\mathcal{E}(B)) = \beta ab$, where β is the volume of the Euclidean unit ball in \mathbb{C}^2 . Therefore $\text{vol}(\Psi(\mathcal{E}(B))) = \text{vol}(T_{a,b}) = \frac{1}{2}ab = \frac{1}{2\beta} \text{vol}(\mathcal{E}(B))$.

(c) Let $c, d > 0$ be such that $T_{c,d} = \Psi(\mathcal{E}(r_1\Delta \times r_2\Delta))$. Note that $\Psi(r_1\Delta \times r_2\Delta) = [0, r_1^2] \times [0, r_2^2]$ and $(r_1^2, r_2^2) \in \partial^\circ T_{c,d}$, i.e. $\frac{r_1^2}{c} + \frac{r_2^2}{d} = 1$. Finding c, d boils down now to finding the minimum of the function

$$(2) \quad V(c) := \text{vol}(T_{c,d(c)}) = \frac{c^2 r_2^2}{2(c - r_1^2)}.$$

(d) It is a straightforward consequence of (a)–(c), namely we get

$$\Psi(\mathcal{E}(\frac{\sqrt{3}}{2}\Delta \times \frac{1}{2}\Delta)) = T_{\frac{3}{2}, \frac{1}{2}}, \quad \Psi(\mathcal{E}(B_{\|\cdot\|}(0))) = T_{1,1}. \quad \square$$

Lemma 4 (d) shows that the inclusion $B_1 \subset\subset B_2$ does not imply $\mathcal{E}(B_1) \subset \mathcal{E}(B_2)$, even for very regular domains B_1, B_2 . It is where the idea of a counterexample comes from. For a large R we are going to construct a domain $G \subset \mathbb{C}^2$ such that $(1, C) \in \overline{\mathbb{B}_{\alpha_G}(0)} \subset \overline{\Delta} \times 2\overline{\Delta}$ for some $C = C(R) > 0$ and at the same time $(0, R) \in \overline{\mathbb{B}_{\alpha_G}(z_\nu)} \subset \overline{\Delta} \times R\overline{\Delta}$ for $z_\nu \in G, z_\nu \rightarrow 0$. The mentioned \mathbb{B}_{α_G} will be Reinhardt. So $\Psi(\mathcal{E}(\mathbb{B}_{\alpha_G}(0)))$ and $\Psi(\mathcal{E}(\mathbb{B}_{\alpha_G}(z_\nu)))$ will be equal to, resp., $T_{a,b}$ and T_{a_ν, b_ν} for some $a, b, a_\nu, b_\nu > 0$ with $b_\nu \geq R^2 \gg 1$ and desirably $a_\nu < a$. It means that even if $\mathbb{B}_{\alpha_G}(0) \subset \mathbb{B}_{\alpha_G}(z_\nu)$ for large ν (because of the possible upper semicontinuity of α_G) we are able to arrange everything to end with $T_{a,b} \not\subset T_{a_\nu, b_\nu}$ (for sufficiently large ν). Having in mind that the Wu metric at a point z is simply a Minkowski functional of the ellipsoid $\mathcal{E}(\mathbb{B}_{\alpha_G}(z))$ we conclude that $W\alpha_G$ is not upper semicontinuous at the point $0 \in G$.

Define a subharmonic function $u : \Delta \rightarrow (-\infty, 1]$

$$u(z) := 1 + \sum_{j=4}^{\infty} \frac{1}{2^j} \max\{\log \frac{|2^{-j} - z|}{2}, -2^{2j}\}, \quad z \in \Delta,$$

and put for $R > 1$

$$\begin{aligned} G_R &:= \{(z, w) \in \Delta \times R\Delta : 2R|w|e^{u(z)} < 1\}, \\ \Omega_R &:= \left(\Delta \times \frac{1}{2R}\Delta\right) \cup \left(\frac{1}{2}\Delta \times R\Delta\right), \\ \widehat{\Omega}_R &:= \text{the envelope of holomorphy of } \Omega_R, \\ C = C_R &:= \frac{1}{2eR}. \end{aligned}$$

Let us also fix a holomorphically contractible family of pseudometrics $(\alpha_D)_D$ for $D \subset \mathbb{C}^n$, $n \geq 1$, satisfying the following condition for any domain $D \subset \mathbb{C}^n$:

$$(3) \quad A_D(a; X) \leq \alpha_D(a; X) \leq \kappa_D(a; X), \quad a \in D, X \in \mathbb{C}^n.$$

Clearly $\alpha_D \in \mathcal{M}(D)$.

We can precisely describe neither balls \mathbb{B}_{α_G} nor ellipsoids $\mathcal{E}(\mathbb{B}_{\alpha_G})$. Nevertheless, the next proposition gives enough information to prove the main result (Proposition 6). We emphasize that $\mathbb{B}_{\alpha_{G_R}}(0) \subset (\Delta \times 2\Delta)$ independently of the choice of the number R . The suitable R will be fixed later in the proof of Proposition 6.

Proposition 5. *Let $R > 1$ be a fixed number.*

- (a) $u(0) = 1 - \frac{3}{4} \log 2 > 0$ and $u(z) > 0$ for $\frac{1}{4} \leq |z| < 1$.
- (b) $\Delta \times C\Delta \subset G_R \subset (\Delta \times \frac{1}{2R}\Delta) \cup (\frac{1}{4}\Delta \times R\Delta) \subset \Omega_R$.
- (c) $\mathcal{E}(\mathbb{B}_{\alpha_{G_R}}((z, 0)))$ is a complete Reinhardt domain for any $z \in \Delta$.
- (d) $\mathbb{B}_{\alpha_{G_R}}(0) \subset (\Delta \times 2\Delta) \cap \widehat{\Omega}_R$. Moreover, $(1, C) \in \partial\mathbb{B}_{\alpha_{G_R}}(0)$.
- (e) $\mathbb{B}_{\alpha_{G_R}}((2^{-\nu}, 0)) \subset \widehat{\Omega}_R$ and $(0, R) \in \partial\mathbb{B}_{\alpha_{G_R}}((2^{-\nu}, 0))$ for sufficiently large $\nu \in \mathbb{N}$.

Proof. (a) $u(0) = 1 + \sum_{j=4}^{\infty} \frac{\log 2^{-j-1}}{2^j} = 1 - \log 2 \sum_{j=4}^{\infty} \frac{j+1}{2^j} = 1 - \frac{3}{4} \log 2$. On the other hand, for $|z| \geq \frac{1}{4}$, $j \geq 3$, we have $|z - \frac{1}{2^j}| \geq \frac{1}{8}$ and $u(z) \geq 1 + \sum_{j=4}^{\infty} \frac{\log 2^{-4}}{2^j} = 1 - \frac{1}{2} \log 2 > 0$.

(b) If $|z| < 1, |w| < C$, then $2R|w|e^{u(z)} < 1$ because $u(z) < 1$. To prove the second inclusion, it is enough to show $|z| \geq \frac{1}{4} \implies |w| < \frac{1}{2R}$ for $(z, w) \in G_R$. But by (a) we have $1 > 2R|w|e^{u(z)} > 2R|w|$.

(c) $\mathbb{B}_{\alpha_{G_R}}((z, 0))$ is a balanced domain; then it is invariant under the action of $\Phi_{(\lambda, \lambda)}$, $\lambda \in \overline{\Delta}$. On the other hand, the domain G_R is invariant under $\Phi_{(1, \mu)}$, $\mu \in \overline{\Delta}$, and so is $\mathbb{B}_{\alpha_{G_R}}((z, 0))$. By Remark 3 the ellipsoid $\mathcal{E}(\mathbb{B}_{\alpha_{G_R}}((z, 0)))$ is complete Reinhardt.

(d) Let $v(z, w) := \frac{1}{2}(|w|e^{u(z)} + |z|)$, $(z, w) \in G_R$. Notice that v is a log-psh function, $v(z, w) \leq e\|(z, w)\|$, and moreover, $v(z, w) \leq \frac{1}{4R} + \frac{1}{2} < 1$ for all $(z, w) \in G_R$. Hence, we obtain for $X = (X_1, X_2) \in \mathbb{C}^2$

$$\alpha_{G_R}(0; X) \geq A_{G_R}(0; X) \geq \limsup_{0 \neq \lambda \rightarrow 0} \frac{v(\lambda X)}{|\lambda|} = \frac{1}{2}(|X_2|e^{u(0)} + |X_1|).$$

The domain $\widehat{\Omega}_R$ is a pseudoconvex Reinhardt domain, thus $\mathbb{B}_{A_{\widehat{\Omega}_R}}(0) = \widehat{\Omega}_R$ (cf. [Jar-Pfl 1]). Then, by (b) and condition (3), we get that $\mathbb{B}_{\alpha_{G_R}}(0) \subset \widehat{\Omega}_R$. Since $(1, C) \in \partial\widehat{\Omega}_R$, it is necessarily $\alpha_{G_R}(0; (1, C)) \geq 1$. In fact, the equality holds here. Indeed, take $\varphi(\lambda) := (\lambda, C\lambda)$, $\lambda \in \Delta$. By (b) we have $\varphi(\Delta) \subset G_R$ and, consequently, $\alpha_{G_R}(0; (1, C)) \leq \kappa_{G_R}(0; (1, C)) \leq 1$.

(e) For natural numbers $\nu \geq 4$ define the biholomorphic mappings

$$\Phi_\nu(z, w) := \left(\frac{z - 2^{-\nu}}{1 - 2^{-\nu}z}, w \right), \quad (z, w) \in \Delta \times \mathbb{C},$$

and notice that $\Phi_\nu(\frac{1}{2^\nu}, 0) = (0, 0)$, $\Phi_\nu(G_R) \subset \widehat{\Omega}_R$, $\nu \geq 4$. Indeed, in view of Proposition 5 (b), it is enough to verify by a direct computation that $|z| < \frac{1}{4} \implies \frac{|z - 2^{-\nu}|}{|1 - 2^{-\nu}z|} < \frac{1}{2}$. Take $X = (X_1, X_2) \in \mathbb{B}_{\alpha_{G_R}}((2^{-\nu}, 0))$. Due to the contractibility of

α we get

$$\alpha_{\widehat{\Omega}_R} \left((0, 0); \left(\frac{1}{1 - 2^{-2\nu}} X_1, X_2 \right) \right) = \alpha_{\widehat{\Omega}_R} (\Phi_\nu(2^{-\nu}, 0); \Phi'_\nu(2^{-\nu}, 0)X) \leq \alpha_{G_R}((2^{-\nu}, 0); X) < 1.$$

The domain $\widehat{\Omega}_R$ is complete Reinhardt and so is the ball $\mathbb{B}_{\alpha_{\widehat{\Omega}_R}}(0)$. Therefore $X \in \mathbb{B}_{\alpha_{\widehat{\Omega}_R}}(0)$. In virtue of condition (3), we obtain

$$\mathbb{B}_{\alpha_{G_R}}((2^{-\nu}, 0)) \subset \mathbb{B}_{\alpha_{\widehat{\Omega}_R}}(0) \subset \mathbb{B}_{A_{\widehat{\Omega}_R}}(0) = \widehat{\Omega}_R.$$

The latter part of the statement can be obtained by repeating the argument from the proof of (d) with the mappings $\varphi_\nu(\lambda) := (2^{-\nu}, R\lambda)$, $\lambda \in \Delta$, for sufficiently large $\nu \in \mathbb{N}$. □

For a number $R > 1$ put

$$(4) \quad A = A(R) := \frac{4 - \frac{1}{2}C^2}{4 - C^2} = \frac{32e^2R^2 - 1}{32e^2R^2 - 2}.$$

The number A is chosen in such a way that the points $(\frac{1}{2}, 4)$, $(1, C^2)$, $(A, 0)$ are collinear. In other words $(\frac{1}{2}, 4), (1, C^2) \in \partial^\circ T_{A, b_0}$ for some b_0 (precisely, $b_0 = 8 - C^2$).

Proposition 6.

- (a) For any $R > 1$ we have $(\sqrt{A}, 0) \in \mathcal{E}(\mathbb{B}_{\alpha_{G_R}}(0))$.
- (b) For any $R \geq 5$ there exists $\nu_0(R) \in \mathbb{N}$ such that $(\sqrt{A}, 0) \notin \mathcal{E}(\mathbb{B}_{\alpha_{G_R}}((2^{-\nu}, 0)))$ for $\nu \geq \nu_0(R)$.

Corollary 7. Let $R \geq 5$ be a fixed number.

- (a) $\mathbb{W}\alpha_{G_R}$ is not upper semicontinuous at the point $(0; (1, 0))$.
- (b) If the family of pseudometrics $(\alpha_D)_D$ has the product property (1), then $\mathbb{W}\alpha_{G_R \times \Delta^{n-2}}$ ($n \geq 2$) is not upper semicontinuous at $(0; (1, 0, \dots, 0))$.

Proof of Corollary 7. (a) Take a number $\nu_0(R)$ such as in Proposition 6 (b). Since $\widetilde{\mathbb{W}}\alpha_{G_R}$ is the Minkowski functional of $\mathcal{E}(\mathbb{B}_{\alpha_{G_R}})$ (cf. Remark 2 (b)), then for $\nu \geq \nu_0(R)$ we get

$$\begin{aligned} \mathbb{W}\alpha_{G_R}(0; (1, 0)) &= \sqrt{2} \widetilde{\mathbb{W}}\alpha_{G_R}(0; (1, 0)) < \frac{\sqrt{2}}{\sqrt{A}} \\ &\leq \sqrt{2} \widetilde{\mathbb{W}}\alpha_{G_R}((2^{-\nu}, 0); (1, 0)) = \mathbb{W}\alpha_{G_R}((2^{-\nu}, 0); (1, 0)). \end{aligned}$$

Conclude with $\nu \rightarrow \infty$.

(b) From Proposition 1 (d) we get

$$\mathbb{W}\alpha_{G_R \times \Delta^{n-2}}((z, w); (1, 0, \dots, 0)) = \mathbb{W}\alpha_{G_R}(z; (1, 0))$$

for $(z, w) \in G_R \times \Delta^{n-2}$ and we apply (a). □

Proof of Proposition 6. Crucial for our consideration will be the geometry of the domain $\widehat{\Omega}_R$. We give here the precise description:

$$\begin{aligned}
 \log(\widehat{\Omega}_R) &= \{(x, y) \in \mathbb{R}^2 : x < 0, y < \log R, \\
 &\quad y < -\frac{\log(2R^2)}{\log 2}x - \log(2R)\}, \\
 (5) \quad \widehat{\Omega}_R &= \{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < R, 2R|z|^{\frac{\log(2R^2)}{\log 2}}|w| < 1\}, \\
 \Psi(\widehat{\Omega}_R) &= \{(u, v) \in \mathbb{R}_+^2 : u < 1, v < R^2, 4R^2u^{\frac{\log(2R^2)}{\log 2}}v < 1\}.
 \end{aligned}$$

Notice a very similar “shape” of $\Psi(\widehat{\Omega}_R)$ and $|\widehat{\Omega}_R| := \{(|z|, |w|) \in \mathbb{R}_+^2 : (z, w) \in \widehat{\Omega}_R\}$. We will especially make use of the convexity of the function $v(u) = \left(4R^2u^{\frac{\log(2R^2)}{\log 2}}\right)^{-1}$ for $\frac{1}{4} \leq u \leq 1$.

Recall that by Proposition 5 (c) the ellipsoids of our interest are bounded complete Reinhardt domains. Thus, we are allowed to use triangles $T_{c,d}$ for proving or disproving the minimality of those ellipsoids. To prove both (a) and (b) we are going to proceed in the following way: we fix T being a triangle corresponding to the ellipsoid $\mathcal{E}(\mathbb{B}_{\alpha_G})$ and assume for a contradiction that the claim does not hold. Then we manipulate the triangle rotating its hypotenuse slightly around a certain fixed point from $\partial^\circ T$. Performing the rotation, we are able to decrease the area of the triangle and provide at the same time (with the use of Proposition 5 (d),(e)) that it still contains $\Psi(\mathbb{B}_{\alpha_G})$. It obviously contradicts the minimality of the initial triangle T .

Recall that the number $A = A(R) > 1$ (cf. (4)) was chosen such that $(\frac{1}{2}, 4), (1, C^2) \in \partial^\circ T_{A,b_0}$. □

Proof of Proposition 6 (a). Let $a, b > 0$ be such that

$$(6) \quad T_{a,b} = \Psi(\mathcal{E}(\mathbb{B}_{\alpha_{G_R}}(0))).$$

Suppose that the statement (a) does not hold, i.e. $a \leq A$.

Consider the following condition:

$$\begin{aligned}
 (7) \quad &\text{there exist } v_1 > 0, u_2 > \frac{2}{3}, v_2 > 0 \text{ such that} \\
 &\overline{\Psi(\widehat{\Omega}_R)} \cap ([0, 1] \times [0, 4]) \cap \partial^\circ T_{a,b} = [(u_2, v_2), (1, v_1)].
 \end{aligned}$$

First we show that condition (7) is sufficient to establish a contradiction, then we prove (7).

Note that condition (7) remains true if we substitute (a, b) by (\tilde{a}, \tilde{b}) from a small neighborhood of (a, b) . We will rotate the hypotenuse of the triangle $T_{a,b}$ around the point (u_2, v_2) . Define the family

$$(8) \quad \mathfrak{T}(u_2, v_2) := \{T_{c,d(c)} : [0, u_2] \times [0, v_2] \subset \overline{T}_{c,d(c)}, (u_2, v_2) \in \partial^\circ T_{c,d(c)}\},$$

where $d(c)$ is a function of c . Certainly, $T_{a,b} \in \mathfrak{T}(u_2, v_2)$.

By Lemma 4 (c), $T_{2u_2, 2v_2}$ is a triangle of minimal area in $\mathfrak{T}(u_2, v_2)$. Moreover, the volume function $c \mapsto \text{vol}(T_{c,d(c)})$ is decreasing for $c \in (a, 2u_2)$. (Note that $a \leq A < \frac{4}{3} < 2u_2$.) On the other hand, by Proposition 5 (d) and properties (5),

(7), we see that for sufficiently small $\varepsilon > 0$

$$\begin{aligned} \Psi(\mathbb{B}_{\alpha_{GR}}(0)) &\subset ([0, 1] \times [0, 4]) \cap \Psi(\widehat{\Omega}_R) \cap T_{a,b} \\ &\subset ([0, 1] \times [0, 4]) \cap \Psi(\widehat{\Omega}_R) \cap T_{a+\varepsilon, d(a+\varepsilon)}. \end{aligned}$$

This contradicts the minimality of the triangle $T_{a,b}$ (condition (6)).

To prove (7), we show that there exist $u_0 \in (\frac{2}{3}, 1]$, $v_0 > 0$ such that

$$(9) \quad \overline{\Psi(\widehat{\Omega}_R)} \cap ([0, 1] \times [0, 4]) \cap \partial^\circ T_{A,b_0} = [(u_0, v_0), (1, C^2)].$$

This can be seen by taking a look at the logarithmic image of $\widehat{\Omega}_R$ (cf. (5)) and verifying that e.g. $(\frac{1}{2} \log \frac{2}{3}, 0) \notin \overline{\log(\widehat{\Omega}_R)}$. Thus, $(\frac{2}{3}, 1) \in T_{A,b_0} \setminus \Psi(\widehat{\Omega}_R)$, which, together with $(\frac{1}{2}, 4) \in \overline{T_{A,b_0}} \setminus \overline{\Psi(\widehat{\Omega}_R)}$, means that

$$\left[\left(\frac{1}{2}, 4 \right), \left(\frac{2}{3}, 1 \right) \right] \subset \overline{T_{A,b_0}} \setminus \overline{\Psi(\widehat{\Omega}_R)}.$$

This implies (9) because $[(\frac{1}{2}, 4), (1, C^2)] \in \partial^\circ T_{A,b_0}$.

Since we assumed that $a \leq A$ and we have $(1, C) \in \partial \mathbb{B}_{\alpha_{GR}}(0)$ (cf. Proposition 5 (d)), the slope of $\partial^\circ T_{a,b}$ is steeper than $\partial^\circ T_{A,b_0}$ and $(1, C^2) \in \overline{T_{a,b}}$. This fact and conditions (5), (9) imply (7). \square

Proof of Proposition 6 (b). Let $a_\nu, b_\nu > 0$ be such that

$$(10) \quad T_{a_\nu, b_\nu} = \Psi\left(\mathcal{E}(\mathbb{B}_{\alpha_{GR}}((2^{-\nu}, 0)))\right), \quad \nu \geq 1.$$

We show that for $R \geq 5$

$$(11) \quad \left[\left(\frac{1}{3}, \frac{R^2}{2} \right), \left(1, \frac{1}{4R^2} \right) \right] \subset T_{A,R^2} \setminus \overline{\Psi(\widehat{\Omega}_R)}.$$

First, let us check that $(\frac{1}{3}, \frac{R^2}{2}) \in T_{A,R^2} \setminus \overline{\Psi(\widehat{\Omega}_R)}$. It is equivalent to

$$\begin{cases} \frac{1}{3A} + \frac{1}{2} < 1, \\ 2R^4 > 3^{\frac{\log(2R^2)}{\log 2}}, \end{cases}$$

which holds for $R \geq 2$. Obviously, $(1, \frac{1}{4R^2}) \in \partial \Psi(\widehat{\Omega}_R)$. On the other hand, the condition $(1, \frac{1}{4R^2}) \in T_{A,R^2}$ reduces to the inequality

$$\frac{1}{A} + \frac{1}{4R^4} < 1 \iff -4R^4 + 32eR^2 - 1 < 0,$$

which holds for $R \geq 5$. This proves (11).

Fix an R such that condition (11) is satisfied and take a $\nu_0(R)$ so large that Proposition 5 (e) holds for $\nu \geq \nu_0(R)$.

Suppose that for a certain fixed $\nu \geq \nu_0(R)$ the claim of (b) does not hold. That is equivalent to $a_\nu > A$. We have necessarily $b_\nu \geq R^2$ because $(0, R) \in \partial \mathbb{B}_{\alpha_{GR}}((2^{-\nu}, 0))$ (cf. Proposition 5 (e)). Thus, by Lemma 4 (a) we have

$$(12) \quad T_{A,R^2} \subset T_{a_\nu, b_\nu}.$$

We claim that there exist some $0 < u_3 < \frac{1}{3}$, $v_3 > 0$ such that

$$(13) \quad (u_3, v_3) \in \overline{\Psi(\widehat{\Omega}_R)} \cap \partial^\circ T_{a_\nu, b_\nu} \subset [(0, b_\nu), (u_3, v_3)].$$

Indeed, it suffices to see that e.g.

$$\left[\left(\frac{1}{3}, \frac{R^2}{2} \right), \left(1, \frac{1}{4R^2} \right) \right] \subset T_{a_\nu, b_\nu} \setminus \overline{\Psi(\widehat{\Omega}_R)},$$

which follows by (11) and (12).

Now, take the family $\mathfrak{T}(u_3, v_3)$ defined as in (8). Obviously, $T_{a_\nu, b_\nu} \in \mathfrak{T}(u_3, v_3)$. We repeat (almost literally) the argument from the proof of (a).

By Lemma 4 (c), $T_{2u_3, 2v_3}$ is a triangle of minimal area in $\mathfrak{T}(u_3, v_3)$. Moreover, the volume function $c \mapsto \text{vol}(T_{c, d(c)})$ is increasing for $c \in (2u_3, a_\nu)$. (Recall that $2u_3 < A < a_\nu$.) On the other hand, by Proposition 5 (e) and properties (5), (13), we obtain for sufficiently small $\varepsilon > 0$

$$\Psi(\mathbb{B}_{\alpha_{GR}}(2^{-\nu}, 0)) \subset \Psi(\widehat{\Omega}_R) \cap T_{a_\nu, b_\nu} \subset \Psi(\widehat{\Omega}_R) \cap T_{a_\nu - \varepsilon, d(a_\nu - \varepsilon)}.$$

This contradicts the minimality of T_{a_ν, b_ν} (condition (10)). \square

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