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SOLUTION OF A QUESTION OF KNARR

KOEN THAS

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ABSTRACT. The Moufang condition is one of the central group theoretical conditions in Incidence Geometry, and was introduced by Jacques Tits in his famous lecture notes (1974).

About ten years ago, Norbert Knarr studied generalized quadrangles (buildings of Type $\mathbf{B}_2$) which satisfy one of the Moufang conditions locally at one point. He then posed the fundamental question whether the group generated by the root-elations with its root containing that point is always a sharply transitive group on the points opposite this point, that is, whether this group is an elation group.

In this paper, we solve the question and a more general version affirmatively for finite generalized quadrangles.

Moreover, we show that this group is necessarily nilpotent (which was only known up till now when both Moufang conditions are satisfied for all points and lines).

In fact, as a corollary, we will prove that these groups always have to be p-groups for some prime p.

1. KNARR'S QUESTION

The Moufang condition is one of the central group theoretical conditions in Incidence Geometry, and was introduced by Jacques Tits in his lecture notes "Buildings of Spherical Type and Finite BN-Pairs" [19]. It was noted by him that spherical buildings of rank at least 3 satisfy the so-called "Moufang property", implying these structures to have a lot of symmetry. When the rank of these buildings is 2, i.e., when one is dealing with *generalized n-gons* [21], this is not necessarily the case; many examples exist which are not Moufang.

Already in the 60's, Tits started a program to obtain all Moufang generalized n-gons, and much later, J. Tits and R. M. Weiss [20] eventually finished the classification of finite and infinite Moufang generalized n-gons. For the finite case, this result was already obtained by P. Fong and G. M. Seitz in [1], the most difficult case by far being the case $n = 4$, and for this latter case, there is also a geometrical proof which is a culmination of work by S. E. Payne and J. A. Thas [8, Chapter 9], W. M. Kantor [6] and the author [10, Appendix].

In the aforementioned work of S. E. Payne and J. A. Thas, the importance of local Moufang conditions became obvious — not only numerous characterizations

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of known classes of generalized 4-gons, or also "generalized quadrangles", came out; also the theory of *translation generalized quadrangles* arose from it, and the abstraction to *elation generalized quadrangles* (see below) eventually led to many new classes of generalized quadrangles. We refer to K. Thas and H. Van Maldeghem [17] for a survey on old and new results on Moufang generalized quadrangles.

About ten years ago, Norbert Knarr studied generalized quadrangles which satisfy one of the Moufang conditions locally at one point. He then posed the fundamental question whether the group generated by the root-elations with root containing that point is always a sharply transitive group on the points opposite this point, that is, whether this group is an *elation group* and the point an *elation point* [7].

One of his motivations to do so was to find a good definition for elation generalized quadrangle, as an alternative to the existing one, and as natural generalization of the concept of translation plane [5] (and so also as an alternative to the theory of translation generalized quadrangles developed in Chapter 8 of [8] in order to classify finite Moufang quadrangles).

More formally, let (x, A, y) be a *root* of a generalized quadrangle, that is, A is a line and x, y are distinct points for which $xIAIy$. Then this root is "Moufang" if there is a group of automorphisms of the quadrangle that fixes A pointwise and x and y linewise, which acts regularly on the points not incident with A of any line through x. Such automorphisms are called "root-elations". Interchanging the role of points and lines, we obtain *dual roots* and *dual root-elations*.

A generalized quadrangle $\mathcal{S}$ is a *Moufang quadrangle* if all its roots and all its dual roots are Moufang. In that case, the group H generated by all root-elations and dual root-elations corresponding to roots and dual roots containing a fixed point x is a group that fixes x linewise and acts sharply transitively on the points not collinear with x. Also, if the number of points of the quadrangle is finite, H can be proved to be nilpotent, and it is even a p-group.

By definition, the fact that a GQ $\mathcal{S}$ contains a point x and admits an automorphism group H that fixes x linewise and acts sharply transitively on the points not collinear with x, means that $\mathcal{S}$ (or $\mathcal{S}^x$ or $(\mathcal{S}^x, H)$) is an *elation generalized quadrangle* ("EGQ" for short) with *elation point* x and *elation group* H (cf. Chapter 8 of [8]).

We can now formulate the question of Knarr as follows:

Knarr's Question. *Let $\mathcal{S}$ be a thick generalized quadrangle, and let (∞) be a point of $\mathcal{S}$. Suppose that any root and dual root containing (∞) is a Moufang (dual) root. Is $(\mathcal{S}^{(\infty)}, W)$ an elation generalized quadrangle with elation point (∞), where W is the group generated by the root-elations and dual root-elations associated to these (dual) Moufang roots?*

In this paper, we solve the question affirmatively for finite generalized quadrangles.

We will first show

Theorem 1.1. *Let $\mathcal{S}$ be a thick finite generalized quadrangle, and let (∞) be a point of $\mathcal{S}$. Suppose that any root containing (∞) is a Moufang root. Then $(\mathcal{S}^{(\infty)}, W)$ is an elation generalized quadrangle with elation point (∞), where W is the group generated by the root-elations associated to these Moufang roots.*

From Theorem 1.1 and its proof, we will deduce the solution to Knarr's question.

Moreover, we solve another long-standing open question by showing that the group W is necessarily nilpotent, both in the situation of Theorem 1.1 and in that of Knarr's question. As already mentioned, this was only known up till now when both Moufang conditions are satisfied for *all* points and lines, that is, when the quadrangle is a Moufang quadrangle.

In fact, we will even prove that these groups always have to be p-groups for some prime p.

However, the nilpotency will, in the course of the proof, not follow "directly" from the fact that W is a p-group.

The proof of the main theorem will consist of a blend of group theoretical arguments, combinatorial observations and synthetic incidence geometric reasoning.

In the next section, we quickly recall a few basic combinatorial notions which will be needed in the course of the proof. We will refer to the standard work [8], especially Chapters 1, 2 and 8, when we use results not mentioned here. We also refer the reader to the book [3] for the group theoretical notions which will be used and not defined.

2. Introductory combinatorics

We tersely review some basic notions taken from the theory of generalized quadrangles, for the sake of convenience.

Finite generalized quadrangles. A (finite) *generalized quadrangle* (GQ) *of order* (s,t) is a point-line incidence structure $\mathcal{S} = (P, B, I)$ in which P and B are disjoint (non-empty) sets of objects called *points* and *lines* respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line;
(ii) each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point;
(iii) if p is a point and L is a line not incident with p, then there is a unique point-line pair (q, M) such that $pIMIqIL$.

If $s = t$, then $\mathcal{S}$ is also said to be *of order* s. If $s, t > 1$, $\mathcal{S}$ is *thick*. There is a *point-line duality* for GQ's of order (s,t) for which in any definition or theorem the words "point" and "line" are interchanged, and also the parameters.

Collinearity/concurrency/regularity. Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$ and call these points *collinear*, provided that there is some line L such that $pILIq$. Dually, for $L, M \in B$, we write $L \sim M$ when L and M are *concurrent*.

For $p \in P$, put $p^{\perp} = \{q \in P \parallel q \sim p\}$ (and note that $p \in p^{\perp}$). For a pair of distinct points $\{p, q\}$, we also denote $p^{\perp} \cap q^{\perp}$ by $\{p,q\}^{\perp}$. Then $|\{p,q\}^{\perp}| = s+1$ or $t+1$, according to whether $p \sim q$ or $p \not\sim q$, respectively.

For $p \neq q$, we define $\{p,q\}^{\perp\perp} = \{r \in P \parallel r \in s^{\perp}$ for all $s \in \{p,q\}^{\perp}\}$. When $p \not\sim q$, we have that $|\{p,q\}^{\perp\perp}| = s+1$ or $|\{p,q\}^{\perp\perp}| \leq t+1$ according to whether $p \sim q$ or $p \not\sim q$, respectively. If $p \sim q$, $p \neq q$, or if $p \not\sim q$ and $|\{p,q\}^{\perp\perp}| = t+1$, we say that the pair $\{p,q\}$ is *regular*. The point p is *regular* provided $\{p,q\}$ is regular for every $q \in P \setminus \{p\}$. Regularity for lines is dually defined. One easily proves that either $s = 1$ or $t \leq s$ if $\mathcal{S}$ has a regular pair of non-collinear points; see 1.3.6 of [8].

The GQ $W(q)$**.** Consider the 3-dimensional projective space $\mathbf{PG}(3,q)$ over the finite field with q elements $\mathbf{GF}(q)$. The points of $\mathbf{PG}(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity form a GQ of order q, denoted $W(q)$. All the points of $W(q)$ are regular [8, dual of 3.3.1(i)], and this characterizes $W(q)$ as the only GQ of order $s = q \neq 1$ with this property.

SubGQ's. A *subquadrangle*, or also *subGQ*, $\mathcal{S}' = (P', B', I')$ of a GQ $\mathcal{S} = (P, B, I)$ is a GQ for which $P' \subseteq P$, $B' \subseteq B$, and where I' is the restriction of I to $(P' \times B') \cup (B' \times P')$.

3. Proof of Theorem 1.1

Setting. *$\mathcal{S} = (P, B, I)$ is a thick GQ of order (s,t) satisfying the hypotheses of Theorem* 1.1*. We suppose $s > 2$, since $s = 2$ implies that the GQ is isomorphic to a GQ arising as the points and lines of a non-singular quadric of Witt index* 2 *in $\mathbf{PG}(d,2)$, $d \in \{4,5\}$ (which is denoted by $\mathcal{Q}(d,2)$), and then the theorem is well-known to hold since these GQ's are Moufang. Suppose $r \sim (\infty) \neq r$; then the group of s root-elations with root $(r, r(\infty), (\infty))$ is denoted by R_r throughout. The group generated by all R_r's is W.*

We proceed in a number of steps to obtain Theorem 1.1. Recall that a *center of transitivity* of a GQ is a point which is fixed linewise by an automorphism group that acts transitively on the points not collinear with it. Also, it is easy to prove that W is as such, an exercise which we leave for the reader (a stronger statement will be obtained later on).

If (∞) **is a regular point, the theorem is proved.** Suppose (∞) is regular. By the proof of Proposition 7.3 of K. Thas and H. Van Maldeghem [18], either there is a subgroup of W of size t, the elements of which fix each point of $(\infty)^{\perp}$ (called "symmetries") — and then (∞) is called a "center of symmetry", or there is a subGQ $\mathcal{S}'$ of order t which contains (∞), and then $s = t^2$.

Suppose we are in the latter case. Suppose L is a line of $\mathcal{S}'$ not containing (∞), and let $LIz \sim (\infty)$. Then the t elements of R_z which map a point of $L \cap \mathcal{S}'$ onto a point of $L \cap \mathcal{S}'$ stabilize $\mathcal{S}'$. Whence $\mathcal{S}'$ satisfies the hypotheses of the main theorem, and so (∞) is also a center of transitivity for $\mathcal{S}'$. By S. E. Payne and K. Thas [9, Theorem 7.1], $\mathcal{S}'$ is an EGQ, and by the dual of Theorem 1 of [2], it now follows that s is the power of a prime, say p. By S. E. Payne and K. Thas [9, Lemma 5.1] we then have two possibilities:

(a) W has a subgroup which acts regularly on $P \setminus (\infty)^{\perp}$, and this subgroup contains all elation subgroups of the full group of automorphisms that fix (∞) linewise, so also all root and dual root groups with root containing (∞). In particular, in this case Theorem 1.1 and an affirmative answer to Knarr's question are verified. Note that since $s = t^2$, W is a p-group.

(b) $p = 2$, and we may suppose $\mathcal{S}'$ to be isomorphic to $W(t)$. Moreover, there is an involutory automorphism fixing $\mathcal{S}'$ pointwise, and (∞) is a center of symmetry.

Now suppose (∞) is a center of symmetry, and denote the group of symmetries with center (∞) by $\mathfrak{C}$. Take two arbitrary points $x, y \in (\infty)^{\perp}$ for which $x \not\sim y$.

Then clearly $[R_x, \mathfrak{C}] = [R_y, \mathfrak{C}] = \{\mathbf{1}\}$. As W is generated by the root groups for which the root contains (∞), and as $\mathfrak{C}$ commutes with these root groups, $\mathfrak{C}$ is contained in the center $Z(W)$ of W. Also, $[R_x, R_y]$ fixes $(\infty)x$ and $(\infty)y$ pointwise, so that by K. Thas [11, Theorem 2.6], $[R_x, R_y] \leq \mathfrak{C}$. It now easily follows that $R_x\mathfrak{C}R_y = \langle R_x, \mathfrak{C}, R_y\rangle$ is a group of order s^2t.

Suppose $\theta \in \langle R_x, \mathfrak{C}, R_y\rangle$ fixes some point l of $P \setminus (\infty)^\perp$. Write $\theta = g_xg_cg_y$, with $g_x \in R_x, g_c \in \mathfrak{C}, g_y \in R_y$. As θ fixes each point of $\{(\infty), l\}^\perp$, it follows that $g_x = \mathbf{1} = g_y$. But a symmetry g_c can only fix l if it is the identity. So, $\langle R_x, \mathfrak{C}, R_y\rangle$ is an elation group which acts regularly on $P \setminus (\infty)^\perp$. Take $a \in R_x$ and $b \in R_y$. Then $[a, b] \in \mathfrak{C}$. So $\langle R_x, \mathfrak{C}, R_y\rangle$ has the property that as soon as an element fixes a point different from (∞) on a line through (∞), the line is pointwise fixed by the element ($\langle R_x, \mathfrak{C}, R_y\rangle$ induces a regular group on such a line minus (∞)). It now follows easily that all root-elations with root containing (∞) are contained in $\langle R_x, \mathfrak{C}, R_y\rangle$, so that again Theorem 1.1 and Knarr's question are affirmatively verified for this case.

Moreover, the main result of D. Hachenberger [4] now tells us that

$$W = \langle R_x, \mathfrak{C}, R_y\rangle$$

is a p-group for some prime p.

Fixed points structure of elements of W. Let x not be collinear with (∞), and put $H = W_x$. Let LIx be a fixed line, and suppose $r, r' \neq r$ are on L so that there is some element α in H mapping r onto r'. Let z be the point of $(\infty)^\perp$ incident with L. Let θ, θ' be root-elations of R_z. Then $r^{\theta^{-1}\alpha\theta} = r^{\theta'^{-1}\alpha\theta'}$ if and only if $\alpha^\theta(\alpha^{\theta'})^{-1} = \phi$ fixes r. It is clear that ϕ fixes r and z linewise, so that ϕ fixes a subGQ of order (s', t) by [8, 8.1.1]. But ϕ also fixes $z(\infty)$ pointwise, leading to $\phi = \mathbf{1}$. So $\alpha^\theta = \alpha^{\theta'}$ (or $\theta'\theta^{-1}\alpha\theta\theta'^{-1} = \alpha$). As α fixes x, this is only possible if α fixes some subGQ of order (s'', t) pointwise, where $s'' > 1$; cf. [8, 8.1.1].

In the rest of the proof, we will distinguish two cases for x:

(a) *There is an $\alpha \in W_x$ not fixing a thick subGQ of order (s', t) pointwise.*

(b) *Each element θ of $W_x \setminus \{\mathbf{1}\}$ fixes some thick subGQ of order (s'_θ, t) pointwise.* If two of these subGQs would be distinct, they would have to intersect in a subGQ of order $(1, t)$ (by 2.2.2 of [8]), leading to the fact that (∞) is regular. But by the previous section we can exclude this case. So $s'_\theta = s'_{\theta'} = s$ for any $\theta, \theta' \in W_x \setminus \{\mathbf{1}\}$, and all elements of $W_x \setminus \{\mathbf{1}\}$ fix the same subGQ pointwise.

Case 1: Suppose that there is no such subGQ for (x, L) (w.r.t. α). Then $\alpha^\theta \neq \alpha^{\theta'}$ for $\theta \neq \theta'$. So $[a, \alpha] = [b, \alpha]$ (where the latter expressions are commutators), with a, b root-elations in R_z, if and only if $a = b$. Note that such a commutator $[a, \alpha]$ is itself a root-elation. It now follows that

$$\langle [a, \alpha] \parallel a \in R_z\rangle = R_z.$$

As the root-elations generate W, it holds that W equals its derived group.

Lemma 3.1. *W equals its derived group.* □

Recall an elementary property for commutators, which we will call $(*)$: if x, y, z are elements of a group G, then

$$(*) \qquad [xy, z] = [x, z]^y[y, z].$$

Let E be the set of all root-elations in W of which the root contains (∞). Then this is a normal set in W (that is, for any $w \in W$, $E^w = E$). Call an *E-commutator* of W a commutator of the form $[a, b]$, where $a, b \in E$.

So, if g, h, i are elements of E, then $[gh, i] = [g, i]^h[h, i] = [g^h, i^h][h, i]$, so $[gh, i]$ is a product of E-commutators.

Since E generates W, the following lemma now follows inductively:

Lemma 3.2. *For any $g, h \in W$, the commutator $[g, h]$ is a product of E-commutators. Whence any element of the derived group W' is also a product of E-commutators.* □

Let a be an element of E. We write $A\mathfrak{R}a$ or $a\mathfrak{R}A$, with $AI(\infty)$, if A is the unique line through (∞) of "the" root of a.

Before proceeding, we first observe a property.

Lemma 3.3. *Let L be a line incident with (∞). Then W is generated by the root-elations of which the root does not contain L.*

Proof. Let $L' \sim L$ be a line not incident with (∞). Let $W(L)$ be the group generated by the root-elations of which the root does not contain L. We must show that $W(L)$ contains all root groups of type R_z, where $(\infty) \neq zIL$. Let x, y be arbitrary distinct points on L' not collinear with (∞). Suppose XIx is different from L', and YIy is also different from L'. Let $o \neq x$ be a point on X not collinear with (∞). Since $s > 2$, we can choose o such that the unique line O through o that meets Y has $O \cap Y \not\sim (\infty)^\perp$. Also, it is easy to see that we can choose X and Y in such a way that $O \not\sim L$. Put $X \cap (\infty)^\perp = x', Y \cap (\infty)^\perp = y'$ and $O \cap (\infty)^\perp = o'$. Take $\alpha \in R_{x'}$ such that $x^\alpha = o$, $\alpha' \in R_{o'}$ such that $o^{\alpha'} = Y \cap O$ and $\alpha'' \in R_{y'}$ such that $(Y \cap O)^{\alpha''} = y$. Then $\alpha\alpha'\alpha'' \in W(L)$ and it maps x onto y. So $W(L)$ acts transitively on the points of L' not in $(\infty)^\perp$. Call the latter point set L'^*.

As no element fixes a thick subGQ pointwise, $W(L)$ in its action on L^* is a Frobenius group, so that the Frobenius kernel consists of all elements of $W(L)_L$ that do not fix a point of L^*. On the other hand, W_L is also a Frobenius group on L^*, so that the Frobenius kernel of this action coincides with that of the action of $W(L)_L$ on L^*. It follows easily that the elements of R_z induce this Frobenius kernel, so that $W(L)$ contains R_z. The lemma follows. □

Lemma 3.4. *Suppose some non-trivial automorphism α' fixes the lines $F, F' \neq F$ incident with (∞) pointwise, and let $\beta \neq \mathbf{1}$ be a root-elation for which $\beta\mathfrak{R}F$. Then $[\alpha', \beta] = \mathbf{1}$.*

Proof. Clearly, $[\alpha', \beta]$ fixes F and F' pointwise, and (∞) linewise. Since β is a root-elation, there is a point $(\infty)'IF$, different from (∞), fixed linewise by β. So $[\alpha', \beta]$ also fixes $(\infty)'$ linewise. By [8, 8.1.1] we conclude that $[\alpha', \beta]$ is the identity element. □

Fix $AI(\infty)$. Now suppose a, b, c are (non-identity) root-elations for which $a\mathfrak{R}A\mathfrak{R}b\mathfrak{R}A\mathfrak{R}c$. By Lemma 3.3, we can write $a = \prod_i g_i$, where i ranges over a finite set $\{1, 2, \ldots, r\}$ and the $g_i \in E$ have roots not containing A.

So

$$[a, [b, c]] = [\prod_i g_i, [b, c]] = \prod_i [g_i, [b, c]]^{\prod_{j>i} g_j}.$$

Let $A_i \mathfrak{R} g_i$, and note that for $k \neq k'$ in $\{1, 2, \ldots, r\}$, A_k can equal $A_{k'}$. Then $[g_i, [b, c]]^{\prod_{j>i} g_j} = \ell_i$ fixes A_i and A pointwise. Now let d be a root-elation with again $d \mathfrak{R} A$. Then

$$[d, [a, [b, c]]] = [[a, [b, c]], d]^{-1} = [\prod_i \ell_i, d]^{-1} = (\prod_i [\ell_i, d]^{\prod_{j>i} \ell_j})^{-1}.$$

Now consider $[\ell_i, d]$ for $i \in \{1, 2, \ldots, r\}$. As d is a root-elation with root containing A, $[\ell_i, d]$ also is as such. On the other hand, ℓ_i fixes A and $A_i \neq A$ pointwise. It follows that $[\ell_i, d]$ is the identity (by Lemma 3.4), and so also $[\ell_i, d]^{\prod_{j>i} \ell_j}$, and hence

$$[d, [a, [b, c]]] = \mathbf{1}.$$

Lemma 3.5. *For $a, b, c, d \in E$ non-trivial root-elations with root containing A, we have*

$$[d, [a, [b, c]]] = \mathbf{1}. \qquad \square$$

Now we calculate the *lower central series* $\{L_1(W), L_2(W), \ldots\}$ of W, defined by the rules

$$L_1(W) = W, \quad L_2(W) = W' = [W, W], \quad L_i(W) = [W, L_{i-1}(W)] \text{ for } i > 2.$$

One notes that $L_2(W)$ is generated by E-commutators, while $L_3(W)$ is generated by commutators of the form $[g, [h, i]]$ (call such commutators E^2-commutators), with $g, h, i \in E$, $L_4(W)$ is generated by commutators of the form $[g, [h, [i, j]]]$ (E^3-commutators) with $g, h, i, j \in E$, etc.

Let $a, b \in E$ be arbitrary and not trivial, and suppose $A \mathfrak{R} a$ and $B \mathfrak{R} b$. Then we have two possibilities for $[a, b]$: (1) either $A = B$ and $[a, b]$ fixes A pointwise, or (2) $A \neq B$ and A and B are pointwise fixed by $[a, b]$. Take $c \in E$ non-trivial, $C \mathfrak{R} c$, and consider $[c, [a, b]]$.

First suppose that $A = B$. Then either (1.a) $C \neq A$ and $[c, [a, b]]$ fixes A and C pointwise, or (1.b) $C = A$ and A is fixed pointwise by $[c, [a, b]]$.

Now suppose $A \neq B$; then either (2.a) $C \in \{A, B\}$ and $[c, [a, b]]$ is the identity by Lemma 3.4, or (2.b) $C \notin \{A, B\}$ and $[c, [a, b]]$ fixes A, B, C pointwise.

Finally, let $d \in E$ be non-trivial, $D \mathfrak{R} d$. Then we obtain the following cases:

Case (1.a) if $D \in \{A, C\}$, $[d, [c, [a, b]]] = \mathbf{1}$ by Lemma 3.4;
if $D \notin \{A, C\}$, A, C, D are fixed pointwise by this commutator.
Case (1.b) if $D = C = B = A$, $[d, [c, [a, b]]] = \mathbf{1}$ by Lemma 3.5;
if $D \neq A$, D, A are fixed by $[d, [c, [a, b]]]$.
Case (2.b) if $D \in \{A, B, C\}$, $[d, [c, [a, b]]] = \mathbf{1}$ by Lemma 3.4;
if $D \notin \{A, B, C\}$, A, B, C, D are fixed pointwise by $[d, [c, [a, b]]]$.

So either $[d, [c, [a, b]]] = \mathbf{1}$ or it fixes (at least) $A, D \neq A$ pointwise.

It now easily follows inductively, using Lemma 3.4, that E^{i-1}-commutators, $i \geq 4$, which generate $L_i(W)$, fix at least $i - 2$ distinct lines incident with (∞) pointwise. Put $i = t+4$. Then $L_{t+4}(W)$ is generated by the set of E^{t+3}-generators, which are of the form

$$[a, \ell],$$

with $a \in E$ and ℓ an E^{t+2}-commutator. So ℓ fixes each point collinear with (∞). By Lemma 3.4, $[a, \ell] = \mathbf{1}$, and $L_{t+4}(W) = \{\mathbf{1}\}$.

Theorem 3.6. *W is nilpotent of class at most $t + 3$.* $\square$

But this latter theorem contradicts Lemma 3.1, so that if there is no subGQ of the aforementioned type, the stabilizer in W of a point is trivial, thus proving Theorem 1.1 for this case.

Case 2: Suppose that α fixes a subGQ $\mathcal{S}'$ of order (s', t), $s' > 1$, pointwise. Consider W_x; as there is only one thick subGQ of order (s', t) containing x (in fact, there is only one thick subGQ containing x with $t+1$ lines through a point), W_x stabilizes $\mathcal{S}'$. Also, as we do not want to be in the previous case, each element of W_x fixes $\mathcal{S}'$ pointwise.

Now define an incidence structure Π as follows:

- Points are the $(s/s')^2$ subGQ's of order (s', t) of ${\mathcal{S}'}^W$ (call subGQ's in this orbit "W-subGQ's").
- Lines are the point sets $\mathcal{S}'' \cap M$, where $\mathcal{S}'' \in {\mathcal{S}'}^W$ and M a line incident with (∞).
- Incidence is inverse containment.

Two distinct W-subGQ's intersect in the $s'+1$ points on some line through (∞), together with all lines on these points. So in Π, two distinct points are incident with exactly one line. Let $\mathcal{S}''$ be a point, and N be a line not containing $\mathcal{S}''$ in Π.

Suppose $RI(\infty)$ is the line which contains the point set N in $\mathcal{S}$. Then $\mathcal{S}'' \cap R$ defines the unique line of Π parallel to N and containing $\mathcal{S}''$. It easily follows that Π is an affine plane of order t. As its number of points is $(s/s')^2$, we have $s = s't$.

Let $\mathbf{R}$ be an arbitrary root containing (∞), and let $R(\mathbf{R})$ be the corresponding group of root-elations clearly $R(\mathbf{R})$ induces a collineation group in the plane Π that fixes the line at infinity pointwise, and that fixes all lines of the parallel class defined by the line of $\mathcal{S}$ in $\mathbf{R}$. The collineation group G of Π generated by all induced root-elations acts sharply transitively on the points of Π, see e.g. Hughes and Piper [5], so that Π is a translation plane with translation group G. Whence $t = s/s'$ is a prime power p^h, where h is integral.

Let $\mathcal{S}''$ be a W-subGQ. If z, z' are points of $\mathcal{S}''$ not collinear with (∞), and $\theta \in W$ maps z onto z', then clearly ${\mathcal{S}''}^\theta = \mathcal{S}''$. So $W_{\mathcal{S}''}/W_z$ induces an elation group of $\mathcal{S}''$, making $\mathcal{S}''$ into an EGQ with elation point (∞).

By D. Frohardt [2, dual of Theorem 1], it follows that s' and t are powers of p, so s is also a power of p, since $s' \leq t$ by [8, 2.2.2].

As the root groups interpreted as collineation groups of Π generate the translation group G, it follows that if an element ϕ of W stabilizes one W-subGQ, it stabilizes all W-subGQ's. In particular, for any $\mathcal{S}'' \in {\mathcal{S}'}^W$ we have that $W_{\mathcal{S}'}$ stabilizes all elements of ${\mathcal{S}'}^W$. So if $|W_y| = k$, $y \not\sim (\infty)$, k divides s'. Now take $\alpha \in W_x \setminus \{\mathbf{1}\}$, $x \not\sim (\infty)$, and consider again the set $\{[\phi, \alpha] \parallel \phi \in R_z\}$, where $LIz \sim (\infty)$, $x \sim L$. Then we observed that $[\phi, \alpha] = \mathbf{1}$ if and only if $\phi \in (R_z)_{\mathcal{S}'}$, that is, if and only if ϕ is one of the s' elements in the stabilizer of $\mathcal{S}'$ in R_z. So $|\{[\phi, \alpha] \parallel \phi \in R_z\}| = s/s'$.

Note that each of these elements are root-elations, and that $\langle [\phi, \alpha] \parallel \phi \in R_z \rangle = T$ stabilizes $\mathcal{S}'$, so T can contain at most s' root-elations. Whence

$$\frac{s}{s'} \leq s'.$$

As $s/s' = t$ and $s' \leq t$, we have $s = s't = {s'}^2$.

By Theorem 3.3 of [16], we now have that $|W_x| = k = 2$ (we suppose k not to be 1), and $\mathcal{S}' \cong W(t)$, $t = 2^h$, so that all of its points are regular. But then (∞) is also regular, and we already handled this case.

This ends the proof of Theorem 1.1. $\square$

Note that the nilpotency of W immediately follows from the proof.

By the dual of Lemma 3 of [2], it now follows that W is a p-group for some prime p.

To give a complete answer to Knarr's question, we still have to show that if (∞) is not a regular point, Theorem 1.1 and its proof imply that W, as defined in Theorem 1.1, contains all dual root groups for which the dual root contains (∞), if $\mathcal{S}$ satisfies the conditions of Knarr's question.

Suppose $A, B \neq A$ are any two lines incident with (∞), and let R' be the group of dual root-elations with dual root $(A, (\infty), B)$. First note that W is a normal subgroup of the stabilizer of (∞) in the automorphism group of $\mathcal{S}$ (as E is a normal set in this group). Since W is a p-group, $W^{\#} = \langle W, R' \rangle = WR'$ also is a p-group. If $W \neq W^{\#}$, the stabilizer of any point $z \not\sim (\infty)$ in $W^{\#}$ is a p-group, and $W^{\#}_z$ fixes z and (∞) linewise, while fixing at least three points on any line incident with (∞) (since s is a power of p).

So [8, 8.1.1] implies that $W^{\#}_z$ fixes a subGQ $\mathcal{S}'$ of order (s', t) of $\mathcal{S}$ pointwise, $1 < s' < s$.

Since we may suppose (∞) not to be regular, the last part of the proof of Theorem 1.1 (Case 2) implies that $t^2 = s$, t is even, and $\mathcal{S}' \cong W(t)$. But then again $\mathcal{S}'$, and so also $\mathcal{S}$, contains a regular point, contradicting our assumption.

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Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, S22, B-9000 Ghent, Belgium

E-mail address: `kthas@cage.UGent.be`