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A POLARIZED PARTITION RELATION FOR CARDINALS OF COUNTABLE COFINALITY

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ABSTRACT. We prove that if $\operatorname{cf}\kappa = \omega$ and $\lambda = 2^{<\kappa}$, then

$$\begin{pmatrix}\lambda^+\\ \lambda\end{pmatrix} \to \begin{pmatrix}\lambda^+ & \alpha\\ \lambda & \kappa\end{pmatrix}^{1,1}$$

for all $\alpha < \omega_1$. This polarized partition relation holds if for every partition $\lambda \times \lambda^+ = K_0 \cup K_1$ either there are $B_0 \in [\lambda]^\lambda$ and $A_0 \in [\lambda^+]^{\lambda^+}$ with $B_0 \times A_0 \subseteq K_0$ or there are $B_1 \in [\kappa]^\lambda$ and $A_1 \in [\alpha]^{\lambda^+}$ with $B_1 \times A_1 \subseteq K_1$.

1. AN INTRODUCTION

For ordinals α, β, γ_0, γ_1, δ_0, and δ_1, the balanced polarized partition relation

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \to \begin{pmatrix}\gamma_0 & \gamma_1\\ \delta_0 & \delta_1\end{pmatrix}^{1,1}$$

holds if for any partition of $\beta \times \alpha = K_0 \cup K_1$ into two classes either there are $D_0 \in [\beta]^{\delta_0}$ and $C_0 \in [\alpha]^{\gamma_0}$ with $D_0 \times C_0 \subseteq K_0$ or there are $D_1 \in [\beta]^{\delta_1}$ and $C_1 \in [\alpha]^{\gamma_1}$ with $D_1 \times C_1 \subseteq K_1$. Similarly, for ordinals α, β, γ_0, γ_1, δ_0, δ_1, and ε the unbalanced polarized partition relation

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \to \begin{pmatrix}\gamma_0 & \begin{pmatrix}\gamma_1\\ \delta_1\end{pmatrix}_\varepsilon\\ \delta_0 & \end{pmatrix}^{1,1}$$

holds if for any partition of $\beta \times \alpha = \bigcup_{\xi<1+\varepsilon} K_\xi$ into $1+\varepsilon$ classes either there are $D_0 \in [\beta]^{\delta_0}$ and $C_0 \in [\alpha]^{\gamma_0}$ with $D_0 \times C_0 \subseteq K_0$ or there are $\xi < \varepsilon$, $D_1 \in [\beta]^{\delta_1}$, and $C_1 \in [\alpha]^{\gamma_1}$ with $D_1 \times C_1 \subseteq K_{1+\xi}$. These relations are monotonic in the sense that they remain true if anything on the left side increases or anything on the right side decreases. They were first introduced and studied by P. Erdős and R. Rado in [4].

With A. Hajnal they proved in [3] that if one assumes the Generalized Continuum Hypothesis (GCH), then the relation

$$\begin{pmatrix}\aleph_{\alpha+1}\\ \aleph_\alpha\end{pmatrix} \to \begin{pmatrix}\aleph_{\alpha+1} & \aleph_0\\ \aleph_\alpha & \aleph_\alpha\end{pmatrix}^{1,1}$$

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holds for each infinite cardinal $\aleph_\alpha$ with $\operatorname{cf}\aleph_\alpha = \aleph_0$. We remove the assumption of GCH and provide a direct proof that if $\operatorname{cf}\kappa = \omega$ and $\lambda = 2^{<\kappa}$, then

$$\begin{pmatrix}\lambda^+\\ \lambda\end{pmatrix} \to \begin{pmatrix}\lambda^+ & \alpha\\ \lambda & \kappa\end{pmatrix}^{1,1}$$

for all $\alpha < \omega_1$.

2. The result

Suppose κ and λ are cardinals. A family $\{A_\xi \mid \xi \in \Theta\}$ is a *κ-complete λ-uniform filter base* or *(κ, λ)-f.b.* if $|\bigcap_{\xi\in X} A_\xi| \geq \lambda$ for all nonempty $X \in [\Theta]^{<\kappa}$. Note that any (κ, λ)-f.b. is also a (κ_0, λ_0)-f.b. for all $\kappa_0 \leq \kappa$ and $\lambda_0 \leq \lambda$.

Lemma 1. *Suppose κ and λ are infinite cardinals with λ regular and $\kappa < \lambda$. For $A \subseteq \kappa$, let $\overline{A} = \kappa \setminus A$. Then for any $\{A_\xi \mid \xi < \lambda\} \subseteq \mathcal{P}(\kappa)$ either*

(a) *there is $X \in [\lambda]^\lambda$ such that $\{A_\xi \mid \xi \in X\}$ is an (ω, κ)-f.b., or*

(b) *there is $Y \in [\lambda]^\lambda$ such that $|\bigcap_{\xi\in Y} \overline{A_\xi}| = \kappa$.*

Proof. For $x \in [\lambda]^{<\omega}$, let $A(x) = \bigcap_{\xi\in x} A_\xi$. Fix $X \subseteq \lambda$ maximal such that $\{A_\xi \mid \xi \in X\}$ is an (ω, κ)-f.b. Suppose that $|X| < \lambda$. Let $\mu = \max\{\kappa, |X|\}$. Fix $M \prec H_{\lambda^+}$ with $|M| = \mu$, $\mu \cup \{\mu\} \cup X \cup \{X\} \subseteq M$, and $\{A_\xi \mid \xi < \lambda\} \in M$. Let $\theta = \sup(\lambda \cap M)$. Choose $\delta \in \lambda \setminus \theta$ arbitrarily. By the maximality of X, there must be $x \in [X]^{<\omega}$ with $|A_\delta \cap A(x)| < \kappa$. Note that $A(x) \in M$ so there must be in M a pairwise disjoint collection $\{B_\eta \mid \eta < \kappa\} \subseteq [A(x)]^\kappa$ with $A(x) = \bigcup_{\eta<\kappa} B_\eta$. Because $|A_\delta \cap A(x)| < \kappa$, there must be $\eta < \kappa$ with $A_\delta \cap B_\eta = \emptyset$. Let $Y = \{\xi < \lambda \mid A_\xi \cap B_\eta = \emptyset\}$. Then $B_\eta \in [\kappa]^\kappa$ by construction, $Y \in [\lambda]^\lambda$ by elementarity, and $B_\eta \subseteq \bigcap_{\xi\in Y} \overline{A_\xi}$ by definition. Thus $|\bigcap_{\xi\in Y} \overline{A_\xi}| = \kappa$. □

Corollary 2 (P. Erdős, A. Hajnal, and R. Rado for $\lambda = \kappa^+$)**.** *The relation*

$$\begin{pmatrix}\lambda\\ \kappa\end{pmatrix} \to \begin{pmatrix}\lambda & n\\ \kappa & \kappa\end{pmatrix}^{1,1}$$

holds whenever $n < \omega \leq \kappa < \lambda$ and λ is regular.

Note that in part (b) of the lemma we could require the set Y to be stationary (by taking $\delta = \theta$), to be contained in any particular element of $[\lambda]^\lambda \cap M$ (by taking δ from that set and adding that property to the definition of Y), or perhaps both. We could, for example, require that Y be stationary in λ and consist only of ordinals of countable cofinality. This could be accomplished by choosing M with $\operatorname{cf}\theta = \omega$, taking $\delta = \theta$, and adding "$\operatorname{cf}\xi = \omega$" to the definition of Y.

Lemma 3. *If $\lambda = 2^{<\kappa}$ and $\{A_\xi \mid \xi < \lambda^+\} \subseteq [\lambda]^\lambda$ is a $(\operatorname{cf}\kappa, \kappa)$-f.b., then for every $\alpha < (\operatorname{cf}\kappa)^+$ there is $X \in [\lambda^+]^\alpha$ with $|\bigcap_{\xi\in X} A_\xi| \geq \kappa$.*

Proof. Fix a sequence of cardinals $\{\kappa_\iota \mid \iota < \operatorname{cf}\kappa\} \subseteq \kappa$ with

$$\kappa = \sum_{\iota<\operatorname{cf}\kappa} \kappa_\iota \quad \text{and} \quad \operatorname{cf}\kappa \notin \{\operatorname{cf}\kappa_\iota \mid \iota < \operatorname{cf}\kappa\}.$$

Choose a regular cardinal θ greater than λ^+. Construct a sequence $\{M_\eta \mid \eta \leq \alpha\}$ of elementary substructures of H_θ such that $|M_\eta| = \lambda$, $\lambda + 1 \subseteq M_\eta$, $[M_\eta]^{<\operatorname{cf}\kappa} \subseteq M_\eta$, and $\{A_\xi \mid \xi < \lambda^+\}, M_\zeta \in M_\eta$ whenever $\zeta < \eta \leq \alpha$. Choose $\delta \in \lambda^+ \setminus M_\alpha$ arbitrarily. For $x \in [\lambda^+]^{<\operatorname{cf}\kappa}$, let $A(x) = \bigcap_{\xi\in x} A_\xi$. For $a \in [\lambda]^{<\kappa}$, let $X(a) = \{\xi < \lambda^+ \mid a \subseteq A_\xi\}$.

Claim A. *Suppose that $\iota < \operatorname{cf}\kappa$. Then for any $x \in [\lambda]^{\kappa_\iota}$ there is $y \in [\lambda]^{\kappa_\iota} \cap M_0$ with $y \subseteq x$. Consequently, $[A]^{\kappa_\iota} \cap M_0$ is nonempty for all $A \subseteq \lambda$ with $|A| \geq \kappa$.*

Proof. Because $|x| = \kappa_\iota$ and $\operatorname{cf}\kappa_\iota \neq \operatorname{cf}\kappa$ there must be $\upsilon < \kappa$ with $|x \cap 2^{\kappa_\upsilon}| = \kappa_\iota$. Note that $[2^{\kappa_\upsilon}]^{\kappa_\iota} \subseteq M_0$ because $[2^{\kappa_\upsilon}]^{\kappa_\iota} \in M_0$ and $|\,[2^{\kappa_\upsilon}]^{\kappa_\iota}\,| = (2^{\kappa_\upsilon})^{\kappa_\iota} \leq 2^{<\kappa} = \lambda$. Let $y = x \cap 2^{\kappa_\upsilon}$. Then $y \in [\lambda]^{\kappa_\iota} \cap M_0$ and $y \subseteq x$. □

Claim B. *Suppose that $\iota < \operatorname{cf}\kappa$ and $\eta < \alpha$. Then for any $a \in [A_\delta]^{<\kappa} \cap M_0$, $x \in [X(a)]^{<\operatorname{cf}\kappa} \cap M_\alpha$, there are $\bar{a} \in [A_\delta]^{<\kappa} \cap M_0$ and $\bar{x} \in [X(\bar{a})]^{<\operatorname{cf}\kappa} \cap M_\alpha$ with $\bar{a} \supseteq a$, $\bar{x} \supseteq x$, $|\bar{a} \setminus a| \geq \kappa_\iota$, and $\bar{x} \cap M_{\eta+1} \setminus M_\eta \neq \emptyset$.*

Proof. Let $a \in [A_\delta]^{<\kappa} \cap M_0$ and $x \in [X(a)]^{<\kappa} \cap M_\alpha$ be given. Let $A = A(x) \cap A_\delta$. Note that $|A| \geq \kappa$, since $\{A_\xi \,|\, \xi < \lambda^+\}$ is a $(\operatorname{cf}\kappa, \kappa)$-f.b. By Claim A, $[A \setminus a]^{\kappa_\iota} \cap M_0$ is nonempty. Choose $b \in [A \setminus a]^{\kappa_\iota} \cap M_0$ arbitrarily and let $\bar{a} = a \cup b$. Let $X = X(\bar{a}) \setminus M_\eta$. Note that X is in $M_{\eta+1}$ (since it is definable in $M_{\eta+1}$) and nonempty (since it includes δ). By elementarity, $X \cap M_{\eta+1}$ must be nonempty. Choose $\xi \in X \cap M_{\eta+1}$ arbitrarily, and let $\bar{x} = x \cup \{\xi\}$. □

Let $\{\xi(\iota) \,|\, \iota < \operatorname{cf}\kappa\}$ be an enumeration of (the ordinals less than) α in order type $\operatorname{cf}\kappa$ (with repetitions, perhaps). Repeated use of Claim B (taking unions at the limits) allows us to construct sequences of sets

$$\emptyset = a_0 \subseteq a_1 \subseteq a_2 \subseteq \cdots \subseteq a_\iota \subseteq \cdots \subseteq A_\delta$$

and

$$\emptyset = x_0 \subseteq x_1 \subseteq x_2 \subseteq \cdots \subseteq x_\iota \subseteq \cdots \subseteq \lambda^+ \cap M_\alpha$$

such that $a_\iota \subseteq A_\delta$, $x_\iota \subseteq X(a_\iota)$, $\operatorname{ot}(a_\iota) \geq \kappa_\iota$, and $x_\iota \cap M_{\xi(\iota)+1} \setminus M_{\xi(\iota)} \neq \emptyset$ for each $\iota < \kappa$. Let $A = \bigcup_{\iota<\operatorname{cf}\kappa} a_\iota$ and $X = \bigcup_{\iota<\operatorname{cf}\kappa} x_\iota$. Then $\operatorname{ot}(A) \geq \kappa$, $\operatorname{ot}(X) \geq \alpha$, and $A \subseteq \bigcap_{\xi\in X} A_\xi$. □

Among other things, this lemma immediately implies that if $\kappa^{<\kappa} = \kappa$ and there is a κ-dense ideal on κ, then

$$\begin{pmatrix}\kappa^+\\ \kappa\end{pmatrix} \to \begin{pmatrix}\alpha\\ \kappa\end{pmatrix}^{1,1}$$

for all $\alpha < \kappa^+$. This is true for $\kappa = \omega$ or for κ measurable and uncountable, for example. These last two relations were first proven by Baumgartner and Hajnal in [1] and Čudnovskiĭ in [2], respectively.

Proposition 4. *If $\operatorname{cf}\kappa = \omega$ and $\lambda = 2^{<\kappa}$, then*

$$\begin{pmatrix}\lambda^+\\ \lambda\end{pmatrix} \to \begin{pmatrix}\lambda^+ & \alpha\\ \lambda & \kappa\end{pmatrix}^{1,1}$$

for all $\alpha < \omega_1$. Consequently,

$$\begin{pmatrix}\lambda^+\\ \lambda\end{pmatrix} \to \begin{pmatrix}\lambda^+ & \begin{pmatrix}\alpha\\ \kappa\end{pmatrix}_m\\ \lambda & \end{pmatrix}^{1,1}$$

for all $\alpha < \omega_1$ and $m < \omega$.

Proof. Suppose $\lambda \times \lambda^+ = K_0 \cup K_1$. For $\xi < \lambda^+$ and $i \in \{0, 1\}$ put

$$K_i^\xi = \{\eta < \lambda \,|\, (\eta, \xi) \in K_i\}.$$

By Lemma 1 either (a) there is $X \in [\lambda^+]^{\lambda^+}$ such that $\{K_1^\xi \,|\, \xi \in X\}$ is an (ω, λ)-f.b., or (b) there is $Y \in [\lambda^+]^{\lambda^+}$ such that $|\bigcap_{\xi\in Y} K_0^\xi| = \lambda$. If the former, then by

Lemma 3 there are $A_1 \in [\lambda^+]^\alpha$ and $B_1 \in [\lambda]^\lambda$ with $B_1 \times A_1 \subseteq K_1$. If the latter, then clearly there are $A_0 \in [\lambda^+]^{\lambda^+}$ and $B_0 \in [\lambda]^\lambda$ with $B_0 \times A_0 \subseteq K_0$. □

The comments following Lemma 1 yield the following.

Corollary 5. *If* $\operatorname{cf}\kappa = \omega$, $\lambda = 2^{<\kappa}$, *and* $\mu \leq \lambda$, *then*

$$\begin{pmatrix} \text{stat}\,\lambda^+ \\ \lambda \end{pmatrix} \to \begin{pmatrix} \text{stat}\,\lambda^+ & \begin{pmatrix} \alpha \\ \kappa \end{pmatrix}_m \\ \lambda & \end{pmatrix}^{1,1}$$

for all $\alpha < \omega_1$ *and all* $m < \omega$. *In other words, for any stationary* $S \subseteq \lambda^+$ *and partition* $\lambda \times S = \bigcup_{i<1+m} K_i$, *either there are* $B_0 \in [\lambda]^\lambda$ *and stationary* $S_0 \subseteq S$ *with* $B_0 \times S_0 \subseteq K_0$ *or there are* $i < m$, $B_1 \in [\lambda]^\kappa$, *and* $A_1 \in [S]^\alpha$ *with* $B_1 \times A_1 \subseteq K_{1+i}$.

3. A conclusion

These results are almost optimal in the sense that in [3] it is proven from GCH that if $\operatorname{cf}\aleph_\alpha > \aleph_0$, then

$$\begin{pmatrix} \aleph_{\alpha+1} \\ \aleph_\alpha \end{pmatrix} \not\to \begin{pmatrix} \aleph_{\alpha+1} & \aleph_0 \\ \aleph_\alpha & \aleph_\alpha \end{pmatrix}^{1,1}$$

and if $\operatorname{cf}\aleph_\alpha = \aleph_0$, then

$$\begin{pmatrix} \aleph_{\alpha+1} \\ \aleph_\alpha \end{pmatrix} \not\to \begin{pmatrix} \aleph_{\alpha+1} & \aleph_2 \\ \aleph_\alpha & \aleph_\alpha \end{pmatrix}^{1,1}.$$

To our knowledge, however, the following question of [3] remains unanswered.

Question 1. Does GCH imply that

$$\begin{pmatrix} \aleph_{\omega+1} \\ \aleph_\omega \end{pmatrix} \to \begin{pmatrix} \aleph_{\omega+1} & \aleph_1 \\ \aleph_\omega & \aleph_\omega \end{pmatrix}^{1,1}?$$

The negative resolution of the next two questions would considerably sharpen the results presented above.

Question 2. Is it consistent that

$$\begin{pmatrix} (2^{<\kappa})^+ \\ 2^{<\kappa} \end{pmatrix} \to \begin{pmatrix} (2^{<\kappa})^+ & \omega_1 \\ 2^{<\kappa} & \kappa \end{pmatrix}^{1,1}$$

for all κ with $\operatorname{cf}\kappa = \omega$?

Question 3. Is it possible that there is κ with $\operatorname{cf}\kappa > \omega$ for which

$$\begin{pmatrix} (2^{<\kappa})^+ \\ 2^{<\kappa} \end{pmatrix} \to \begin{pmatrix} (2^{<\kappa})^+ & \omega \\ 2^{<\kappa} & \kappa \end{pmatrix}^{1,1}?$$

And finally, a related question of independent interest:

Question 4. Is it consistent that there is $\kappa > \omega$ for which

$$\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \kappa^+ & \kappa \\ \kappa & \kappa \end{pmatrix}^{1,1}?$$

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