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KRASNOSELSKII TYPE FIXED POINT THEOREMS AND APPLICATIONS

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ABSTRACT. In this paper, we establish two fixed point theorems of Krasnoselskii type for the sum of $A+B$, where A is a compact operator and $I-B$ may not be injective. Our results extend previous ones. As an application, we apply such results to obtain some existence results of periodic solutions for delay integral equations and then give three instructive examples.

1. Introduction and preliminaries

It is well known that Krasnoselskii's theorem may be combined with Banach and Schauder's fixed point theorems. In a certain sense, we can interpret this as follows: if a compact operator has the fixed point property, under a small perturbation, then this property can be inherited. The sum of operators is clearly seen in delay integral equations and neutral functional equations, which have been discussed extensively in [9, 16], for example. Krasnoselskii proved that the sum of $A+B$ has a fixed point in M, if (i) A is continuous and compact, (ii) $Ax+By \in M$ for every $x, y \in M$ and (iii) B is a strict contraction. However, in several applications, the verification of (ii) is quite hard to do and assumption (iii) is also quite restrictive. Recently, as a tentative approach to overcoming such difficulties, many interesting works have appeared with different ways and directions of weakening conditions (ii) and (iii). In [5], in order to improve condition (iii), Burton introduced the concept of a large contraction mapping and generalized this well-known result in a wide setting. On the other hand, Burton and Colleen Kirk [7] extended Krasnoselskii's idea by combining a result of Schaefer [13] on fixed points from a priori bounds with Banach's theorem. In [3], Barroso proposed the following improvement for (ii). If $\lambda \in (0,1)$, $u = \lambda Bu + Av$ for some $v \in M$, then $u \in M$. About (iii), many authors [1, 2, 4, 8] achieved their results by directly or indirectly making the assumption that $I-B$ is continuously invertible. Indeed, it would be interesting to investigate the case when $I-B$ is not injective. In this paper, we explore this kind of generalization by looking for the multi-valued operator $(I-B)^{-1}A$ achieving a fixed point in M. We should mention that other authors have already studied Krasnoselskii type results in locally convex spaces [4, 17].

Now we present some definitions and recall some basic facts.

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Let $(X, \|\cdot\|)$ be a Banach space and $M \subset X$; set

$$\mathsf{P}(M) := \{N, N \subset M, N \neq \phi\}.$$

A *multi-valued mapping* (or *multi-function*) $F : M \to \mathsf{P}(X)$ is said to be:

(1) upper semi-continuous if for each closed set $B \subset X$, $F^{-1}(B) = \{x \in X : F(x) \cap B \neq \phi\}$ is closed in M;

(2) closed if its graph $G(F) = \{(x, y) \in M \times X : y \in F(x)\}$ is closed;

(3) compact if $clF(M)$ is a compact subset of X.

This paper is organized by following three parts: in section 1, we show the background of this topic. Several abstract fixed point theorems are given in section 2. As an application, in section 3, we prove some existence results of periodic solutions for some nonlinear delay integral equations. In the last section, three instructive examples are given.

For the remainder of the introduction, we state the following two theorems as a prototype in this paper.

Theorem 1.1 ([17, p. 452]). *Let M be a closed and convex subset of a Banach space $(X, ||\cdot||)$ and $F : M \to \mathsf{P}(M)$ a multi-valued mapping. Suppose that*

(i) the set $F(M)$ is relatively compact;

(ii) F is upper semi-continuous on M;

(iii) the set $F(x)$ is nonempty, closed and convex for all $x \in M$.

Then there exists $x \in M$ with $x \in F(x)$.

Theorem 1.2 ([1]). *Let $(X, ||\cdot||)$ be a Banach space and $F : X \to \mathsf{P}(X)$ a multi-valued mapping. Suppose that*

(i) F is a compact multi-valued mapping;

(ii) F is upper semi-continuous on X;

(iii) the set $F(x)$ is closed and convex for all $x \in X$.

Then either the set $D = \{x \in X : \exists\lambda \in (0, 1), x \in \lambda F(x)\}$ is unbounded or there exists $x \in X$ with $x \in F(x)$.

2. Krasnoselskii type fixed point theorems

Theorem 2.1. *Let M be a bounded, closed and convex nonempty subset of a Banach space $(X, ||\cdot||)$. Suppose that A and B are continuous and map M into X such that*

(a) $A(M) \subset (I - B)(M)$;

(b) AM is contained in a compact subset of M;

(c) if $(I - B)x_n \to y$, then there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$;

(d) for every y in the range of $I - B$, $D_y = \{x \in M : (I - B)x = y\}$ is a convex set. Then there exists $y \in M$ with $y = Ay + By$.

Proof. First, we assume that $I - B$ is invertible. By item (c), it is easy to see that $(I - B)^{-1}$ is continuous. For any given $y \in M$, define $L : M \to M$ by $Ly := (I - B)^{-1}Ay$. L is well defined by assumption (a). Since AM is contained in a compact subset of M and $(I - B)^{-1}$ is a continuous mapping of AM into M, we see that $(I - B)^{-1}AM$ is contained in a compact subset of M. By Schauder's second theorem [[14]; p. 25] there exists a fixed point $y = (I - B)^{-1}Ay$. In this case, Theorem 2.1 is valid.

If $I - B$ is not invertible, $(I - B)^{-1}$ could be seen as a multi-valued mapping. For any given $y \in M$, define $H : M \to \mathsf{P}(M)$ by $Hy := (I - B)^{-1}Ay$. H is well defined by assumption (a). We should prove that H fulfills the hypotheses of Theorem 1.1.

Step 1. $H(x)$ is a convex set for each $x \in M$. This is an immediate consequence of assumption (d).

Step 2. H is a closed multi-valued mapping on M. Let $x \in M$ and $\{x_n\} \subset M$ such that $\lim_{n\to\infty} x_n = x$. Let $y_n \in H(x_n)$ such that $\lim_{n\to\infty} y_n = y$. By the definition of H, we have $(I - B)y_n = Ax_n$. Based on the continuousness of A and $I - B$, we obtain $(I - B)y = Ax$. Thus $y \in (I - B)^{-1}Ax$. This implies that the graph $G(H)$ is closed. Hence H is a closed multi-valued mapping.

Step 3. $H(x)$ is a nonempty closed set for each $x \in M$. This assertion follows from Step 2 and hypothesis (a) immediately.

Step 4. $H(M)$ is relatively compact. For any $\{y_n\} \subset H(M)$, we choose $\{x_n\} \subset M$ such that $y_n \in H(x_n)$. By the definition of H, we have $(I-B)y_n = Ax_n$. Based on the assumption (b), we obtain $\lim_{n\to\infty}(I - B)y_n = z$ for some $z \in M$. Thus there exists a subsequence $\{y_{n_k}\}$ converging to y_0 in M.

The results contained in Step 2 and Step 4 allow us to conclude that H is upper semi-continuous on M. By applying Theorem 1.1 to operator H, we obtain $y \in H(y)$ for some $y \in M$. Thus there exists $y \in M$ with $y = Ay + By$. The proof is complete. □

Theorem 2.2. *Let $(X, ||\cdot||)$ be a Banach space. $A, B : X \to X$ are continuous satisfying*

(a) $A(X) \subset (I - B)(X)$;

(b) A is a compact mapping;

(c) if $(I - B)x_n \to y$, then there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$;

(d) for every y in the range of $I - B$, $D_y = \{x \in X : (I - B)x = y\}$ is a convex set. Then either the set $D = \{x \in X : \exists\lambda \in (0, 1), x = \lambda B(\frac{x}{\lambda}) + \lambda Ax\}$ is unbounded or there exists $x \in X$ with $x = Ax + Bx$.

Proof. This result follows immediately from Theorem 1.2 and the technique used in Theorem 2.1. □

Remark 2.3. If $I-B$ is a 1-1 mapping, then the assumptions (c) and (d) in Theorem 2.1 or 2.2 hold if and only if $(I - B)^{-1}$ is continuous. The hypothesis that $(I - B)$ is continuous and invertible has been discussed extensively in [1, 4].

Similarly, we have the following results, which will be proved by using the methods used in Theorem 2.1.

Theorem 2.4. *Let M be a bounded, closed and convex nonempty subset of a Banach space $(X, ||\cdot||)$. Suppose that A and B are continuous and map M into X such that*

(a) $(I - A)(M) \subset B(M)$;

(b) $(I - A)M$ is contained in a compact subset of M;

(c) if $Bx_n \to y$, then there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$;

(d) for every y in the range of B, $D_y = \{x \in M : Bx = y\}$ is a convex set. Then there exists $y \in M$ with $y = Ay + By$.

It is well known that if B is a contraction mapping, then the conditions (a), (c) and (d) in Theorem 2.1 hold. Another way of checking such assumptions is to ask if B is a separate contraction mapping in the following sense.

Definition 2.5. Let (X, d) be a metric space. $f : X \to X$ is said to be a *separate contraction mapping* if there exist two functions $\varphi, \psi : R^+ \to R^+$ satisfying (1)

$\psi(0) = 0$, ψ is strictly increasing; (2) $d(f(x), f(y)) \leq \varphi(d(x,y))$; (3) $\psi(r) \leq r - \varphi(r)$ for $r = d(x,y) > 0$.

It is easy to see that if f is a contraction mapping, then f is a separate contraction mapping. Moreover, we state the following results concerning the separate contraction mapping.

Theorem 2.6 ([11, Lemma 1.1]). *If f is a large contraction mapping, then f is a separate contraction mapping.*

Theorem 2.7 ([11, Theorem 2.2]). *Let M be a bounded, convex nonempty subset of a Banach space $(X, ||\cdot||)$. Suppose that A and B map M into X such that*
(a) if $x = Ay + Bx$, for some $y \in M$, then $x \in M$;
(b) A is continuous and AM is contained in a compact subset of M;
(c) B is a separate contraction mapping.
Then there exists $y \in M$ with $y = Ay + By$.

Theorem 2.8 ([11, Theorem 2.3]). *Let $(X, ||\cdot||)$ be a normed space, $A, B : X \to X$, B a separate contraction mapping and A continuous and maps bounded sets into compact sets. Then either*
(i) the set $D = \{x \in X : \exists\lambda \in (0,1), x = \lambda B(\frac{x}{\lambda}) + \lambda Ax\}$ is unbounded or
(ii) there exists $x \in X$ with $x = Ax + Bx$.

Theorem 2.9 ([11, Theorem 2.1]). *Let (X, d) be a complete metric space. Suppose $f : X \to X$ is a separate contraction mapping. Then f has a unique fixed point in X.*

Remark 2.10. (a) Theorem 2.1 and Theorem 2.2 generalize and extend the corresponding results in [1, 2, 11] respectively.

(b) Theorem 2.9 slightly extends the corresponding result in [[5], Theorem 1]. If B is a large contraction mapping, then Theorem 2.7 deduces the corresponding result in [[5], Theorem 2].

3. Periodic solutions for delay integral equations

In this section, we are interested in obtaining the existence results of periodic solutions for some nonlinear delay integral equations given by

$$x(t) = f(t, x(t)) - \int_{t-\tau}^{t} D(t,s)g(s, x(s))ds \tag{3.1}$$

and

$$x(t) = f(t, x(t), x(t-\tau)) - \int_{t-\tau}^{t} D(t,s)g(s, x(s))ds. \tag{3.2}$$

Let $(X_T, ||\cdot||)$ be the space of continuous T-periodic functions $\varphi : R \to R$ with the sup-norm. Consider (3.1) and suppose that:

(i) f, g, D_{st} are continuous and there exists $k > 0$ such that $|f(t,x)| \leq k$ and

$$f(t+T, x) = f(t,x), \quad D(t+T, s+T) = D(t,s), \quad g(t+T, x) = g(t,x),$$

(ii) $t - \tau \leq s \leq t$ implies that $D_s(t, t-\tau) \geq 0$, $D_{st}(t,s) \leq 0$, $D(t, t-\tau) = 0$,

(iii) $(Bx)(t) = f(t, x(t))$ is a separate contraction mapping,

(iv) $xg(t,x) \geq 0$, there exists $\beta > 0$ and $L > 0$ with

$$-xg(t,x) + k|g(t,x)| \leq L - \beta|g(t,x)|.$$

Theorem 3.1. *Let all conditions (i)-(iv) hold. Then (3.1) has a T-periodic solution.*

Proof. Using the same technique in [[7], Lemma 3] or [[6], p.3987], we can show this theorem immediately. □

Consider (3.2) and suppose that:
(1) f, g, D_{st} are continuous and

$$f(t+T,x,y) = f(t,x,y), \quad D(t+T,s+T) = D(t,s), \quad g(t+T,x) = g(t,x),$$

(2) $|f(t,x_1,y_1) - f(t,x_2,y_2)| \le a|x_1 - x_2| + b|y_1 - y_2|$, $a + b < 1$,
(3) $xg(t,x) \ge 0$, there exists $\beta > 0$ and $L > 0$ with

$$-(1-a)x(t)g(t,x(t)) + b|x(t-\tau)g(t,x(t))| + k|g(t,x)| \le L - \beta|g(t,x)|,$$

where $k = \max_{t\in[0,T]} |f(t,0,0)|$.

Theorem 3.2. *Let conditions (1)-(3) and (ii) hold. Then (3.2) has a T-periodic solution.*

Proof. Define $(Bx)(t) = f(t,x(t),x(t-\tau))$. From the assumption (3), we know B is a contraction mapping. The rest of the proof of this result is similar to the corresponding results in [[7], Lemma 3] or [[6], p. 3987]. □

Remark 3.3. The equation (3.1) has been investigated by Burton in [7]. Here we substitute "separate contraction" for "contraction" in assumption (iii) to show that Theorem 2.7 is an improvement of the Burton result [[5], Theorem 2].

4. Some examples

In this section, we give some examples to illustrate that the separate contraction mappings generate the contraction and large contraction mappings. Firstly, we give an example that satisfies the condition of separate contraction, but is not a contraction.

Example 4.1. We consider the function $f : R \times [0,1] \to [0,1]$ defined by

$$f(t,x) = x - \frac{x^4}{4} + \frac{\sin^2(t)}{4}.$$

Let $X = C(R,[0,1])$ and $B : X \to X$ be defined by $(Bx)(t) = f(t,x(t))$. It is easy to verify that the operator B is well defined. Moreover, we have the following conclusion.

Conclusion 4.2. B is a separate contraction mapping, but not a contraction mapping.

In fact, for any $x, y \in X$ and each $t \in R$, we have $|x(t) - y(t)| \le x(t) + y(t)$ and $|x(t)-y(t)|^2 = x^2(t)+y^2(t)-2x(t)y(t) \le 2(x^2(t)+y^2(t))$. By direct computation, we have

$$\begin{aligned}
|Bx(t) - By(t)| &= |x(t) - \frac{x^4(t)}{4} - y(t) + \frac{y^4(t)}{4}| \\
&= |x(t) - y(t)||1 - \frac{x(t)+y(t)}{4}(x^2(t)+y^2(t))| \\
&\le |x(t) - y(t)|(1 - \frac{|x(t)-y(t)|^3}{8}).
\end{aligned}$$

Let $\varphi(r) = r(1 - \frac{r^3}{8})$. Then $\varphi(\cdot)$ is an increasing function on [0,1]. Thus

$$|Bx(t) - By(t)| \leq \|x - y\|(1 - \frac{\|x - y\|^3}{8}).$$

Then

$$\|Bx - By\| \leq \|x - y\|(1 - \frac{\|x - y\|^3}{8}) = \varphi(\|x - y\|).$$

Take $\psi(r) = r - \varphi(r) = \frac{r^4}{8}$. By Definition 2.5, we see that B is a separate contraction mapping.

On the other hand, we assume B is a contraction mapping with the contractive coefficient $k \in (0,1)$. Let $0 < \varepsilon < 1 - k$ and $E = \{(x,y) \in [0,1] \times [0,1] : (x+y)(x^2+y^2) = 4(1-k-\varepsilon)\}$. Clearly, $E \neq \phi$. Consider two constant value functions x_0 and y_0 satisfying $(x_0, y_0) \in E$. Then

$$\|Bx_0 - By_0\| = (k+\varepsilon)\|x_0 - y_0\| > k\|x_0 - y_0\|.$$

This is a contradiction. Thus B is not a contraction mapping.

Example 4.3. We consider the function $f(x) = \frac{x^2}{1+x}, x \in X = [0, +\infty)$. Then f is a separate contraction mapping, but not a large contraction mapping.

In fact, let

$$\varphi(d(x,y)) = \begin{cases} \frac{d(x,y)[n^2+2n+(n+1)d(x,y)]}{(n+1)^2+(n+1)d(x,y)}, & n-1 < \min\{x,y\} \leq n, \\ \frac{d(x,y)[3+2d(x,y)]}{4+2d(x,y)}, & \min\{x,y\} = 0, \end{cases} \quad n = 1, 2, \cdots.$$

Then

$$\psi(d(x,y)) = \begin{cases} \frac{d(x,y)}{(n+1)^2+(n+1)d(x,y)}, & n-1 < \min\{x,y\} \leq n, \\ \frac{d(x,y)}{4+2d(x,y)}, & \min\{x,y\} = 0, \end{cases} \quad n = 1, 2, \cdots.$$

For any $x, y \in X$ and $y > x$, if $x = 0$, then $d(f(0), f(y)) = f(y) = \frac{y^2}{1+y} \leq \frac{y(3+2y)}{4+2y} = \varphi(d(0,y))$. If $x \neq 0$. Then there exists $n_0 > 0$ such that $n_0 - 1 < x \leq n_0$. Then $y = x + d(x,y)$. By direct computation, we have

$$d(f(x), f(y)) = f(y) - f(x) \leq \frac{d(x,y)[n_0^2 + 2n_0 + (n_0+1)d(x,y)]}{(n_0+1)^2 + (n_0+1)d(x,y)} = \varphi(d(x,y)).$$

By Definition 2.5, we see that f is a separate contraction mapping.

On the other hand, if f is a large contraction mapping, then for any $\delta > 0$, there exists $k \in (0,1)$ such that $d(f(x), f(y)) \leq kd(x,y)$ for all $d(x,y) \geq \delta$. But, for $x > \max\{\delta, \frac{k}{1-k}\}$, we have $d(f(x), f(0)) = \frac{x^2}{1+x} = \frac{x}{1+x}d(x,0) > kd(x,0)$. This implies f is not a large contraction mapping.

Next we show how to construct the separate contraction mapping. First, we give some known preliminaries. Let $P = \{u : u \in X_T, u(t) \geq 0\}$. Then P is a cone in X_T and its internal portion is given as $P^0 = \{u \in X_T :$ there exists $\lambda > 0$ such that $u(t) \geq \lambda$ for all $t \in [0,T]\}$. Clearly, $P^0 \neq \phi$. For any $x, y \in P^0$, we can define

$$m(y/x) = \sup\{\alpha > 0, \alpha x \prec y\}, \quad M(y/x) = \inf\{\beta > 0, y \prec \beta x\},$$

and the Hilbert metric is defined by $d(x,y) = \max\{\ln M(y/x), -\ln m(y/x)\}$. Then (P^0, d) is a complete metric space [15]. Assume the function $f : R \times [0, +\infty) \to [0, +\infty)$ satisfies the conditions:

(C_1) $f(\cdot, x) \in X_T$, and for all $t \in R$, $f(t, \cdot)$ is increasing with $f(t, 0) = 0$.

(C_2) There exists a positive increasing function $\bar{\varphi}:(0,1)\to R^+$ satisfying (i) $f(t,rx)\geq\bar{\varphi}(r)f(t,x)$, (ii) for each $[a,b]\subset(0,1)$, $\inf_{r\in[a,b]}\bar{\varphi}(r)\frac{1}{r}>1$.

(C_3)There exist constants $\rho>0$ and $\delta>0$ such that $\inf_{t\in[0,T]}f(t,\rho)\geq\delta$.

Consider the operator $B:P\to P$ as $(Bx)(t)=f(t,x(t))$. Then we have the following lemmas, which are proved by using the similar methods of Lemma 3.3 and Lemma 3.4 in [12].

Lemma 4.4. *Let the conditions $(C_1)-(C_3)$ be valid. Then B is self-mapping in P^0.*

Lemma 4.5. *Assume (C_1) and (C_2) hold. Then for $[\alpha,\beta]\subset R^+$, we have $\sup_{r\in[\alpha,\beta]}\{\varphi(r)-r\}<0$ and $d(B(x),B(y))\leq\varphi(d(x,y))$, where $\varphi(r)=-\ln(\bar{\varphi}(e^{-r}))$. Furthermore, if $r-\varphi(r)$ is strictly increasing, then B is a separate contraction mapping.*

Example 4.6. Take $a_0=1$, $a_{n+1}=a_n(\frac{2}{\sqrt{n+1}}+1)^{\frac{1}{n+3}-\frac{1}{n+2}}, n=0,1,\ldots$. Then we obtain $\lim_{n\to\infty}a_n\geq 0$. Now, we consider the operator $B:P\to P$ and the function $f:R\times[0,+\infty)\to[0,+\infty)$ defined by $Bx(t)=f(t,x(t))$, $f(t,x)=[\cos(t)+2]^2\sum_{n=0}^{+\infty}F_n(x)$, where $F_n(x)=a_n(1+\sqrt{x})^{\frac{n+1}{n+2}}$ for $x\in[\frac{4}{n+1},\frac{4}{n})$ and $F_n(x)=0$ otherwise $n=0,1,2,\ldots$.

Conclusion 4.7. B is a separate contraction mapping in (P^0,d).

In fact, note that we let $\frac{4}{n}$ denote ∞ for $n=0$. It is easy to see that $f(t,x)$ satisfies the assumptions (C_1) and (C_3). Next, we should verify that $f(t,x)$ satisfies the assumptions (C_2). Let $\bar{\varphi}(r)=\sum_{n=0}^{+\infty}\bar{\varphi}_n(r)$, $r\in(0,1)$, where $\bar{\varphi}_n(r)=r^{\frac{n+1}{n+2}}$ for $r\in[\frac{1}{n+2},\frac{1}{n+1})$ and $\bar{\varphi}_n(r)=0$ otherwise, $n=0,1,2,\ldots$. It is easy to see that $\bar{\varphi}$ is a positive increasing function on $(0,1)$ and $f(t,rx)\geq\bar{\varphi}(r)f(t,x)$.

On the other hand, for any closed interval $[a,b]\subset(0,1)$, we have $\inf_{r\in[a,b]}\bar{\varphi}(r)\frac{1}{r}>1$. Hence $f(t,x)$ satisfies the assumption (C_2). By Lemma 4.4 and Lemma 4.5, we have $d(B(x),B(y))\leq\varphi(d(x,y))$, where $\varphi(r)=-\ln(\bar{\varphi}(e^{-r}))=\sum_{n=0}^{+\infty}\varphi_n(r)$ and

$$\varphi_n(r)=\begin{cases}\frac{n+1}{n+2}r, & r\in[\ln(n+1),\ln(n+2)),\\ 0, & r\in(0,+\infty)\setminus[\ln(n+1),\ln(n+2)).\end{cases}$$

It is easy to see that $\psi(r)=r-\varphi(r)$ is strictly increasing. In view of Definition 2.5 and Lemma 4.5, we conclude that B is a separate contraction mapping.

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