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AMENABILITY OF BANACH AND C*-ALGEBRAS GENERATED BY UNITARY REPRESENTATIONS

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Abstract. We study the amenability of a locally compact group G in relation to the amenability properties of a variety of C^*-algebras and (quantized/dual) Banach algebras naturally associated to a unitary representation of G.

Introduction

Let G be a locally compact group, let $A(G)$ and $B(G)$ respectively denote the Fourier and Fourier-Stieltjes algebras of G [11], and let $\{\pi, \mathcal{H}\}$ be a continuous unitary representation of G. G. Arsac initiated the study of the respective closed and w^*-closed linear subspaces A_π and B_π of $B(G)$ generated by the coefficient functions of π. The C^*-algebra generated by π on $C^*(G)$ (the group C^*-algebra of G), is denoted C^*_π, and the von Neumann algebra generated by π is VN_π.

In [24], we let $A(\pi)$ be the smallest closed subalgebra of $B(G)$ generated by all coefficient functions of π, and denoted the w^*-closure of $A(\pi)$ in $B(G)$ by $B(\pi)$. There is an easily described representation τ_π associated to π such that $A(\pi) = A_{\tau_\pi}$ and $B(\pi) = B_{\tau_\pi}$. We shall let $C^*(\pi) = C^*_{\tau_\pi}$ and $VN(\pi) = VN_{\tau_\pi}$. When our representation π is the left regular representation, λ_G, we have

$$A(G) = A(\lambda_G),\ B_r(G) = B(\lambda_G),\ C^*_r(G) \cong C^*(\lambda_G),\ \text{and } VN(G) \cong VN(\lambda_G)$$

where $\cong$ indicates the existence of a $*$-isomorphism. We thus feel that these various algebras associated to π are very appropriate and natural.

The purpose of this note is to study the amenability properties of the algebras $A(\pi)$, $B(\pi)$, $C^*(\pi)$, $VN(\pi)$ in relation to the amenability of the group G and the amenability (in the sense of M.E.B. Bekka [3]) of the representation π. Our main focus will be on the C^*-algebra $C^*(\pi)$. The following theorem serves as our motivation. Its various parts are due, in general, to E. Kaniuth [16], A. T.-M. Lau and A.L.T. Paterson [20], Z.-J. Ruan [21], and V. Runde and N. Spronk [23]. For discrete groups, the equivalence of parts (i), (ii), and (iii) is due to E.C. Lance [19], and E.G. Effros and E.C. Lance [9].

Theorem 0.1. *The following are equivalent for a locally compact group G:*

(i) *G is amenable.*
(ii) *$C^*_r(G)$ is nuclear and G is inner amenable.*

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(iii) $VN(G)$ *is Connes-amenable and* G *is inner amenable.*
(iv) $A(G)$ *is operator amenable.*
(v) $B_r(G)$ *is operator Connes-amenable.*

This can be phrased entirely in terms of the left regular representation, λ_G, and it is natural to ask if it is possible to replace λ_G with other representations. In this regard, E. Bédos has already shown that G_d is amenable if and only if $C^*_{(\lambda_G)_d}$ is nuclear [2]. Later, E. Kaniuth and A. Markfort showed that G_d is amenable if and only if $C^*_{(\gamma_G)_d}$ is nuclear, where γ_G is the conjugation representation of G [18] (the relevant undefined notation is provided below). Among other things we will show that for a very wide class of representations π, with kernel N in G, G/N is amenable if and only if $C^*(\pi)$ is nuclear and G/N is inner amenable. As a corollary we obtain Bédos's result, and we also show that G is amenable if and only if $C^*(\gamma_G)$ is nuclear and G is inner amenable.

Our approach is to simply use Theorem 0.1 and the fact that all variants of amenability are preserved by homomorphic image in their appropriate category. We shall rely heavily upon the main results from the paper [4] of M.E.B. Bekka, A. T.-M. Lau, and G. Schlichting.

1. Preliminaries

Throughout this paper, G is a locally compact group with fixed left Haar measure dx and modular function Δ_G. For details regarding the material which follows, the reader is referred to [1], [5], [8] and [11].

By a *representation* $\{\pi, \mathcal{H}\}$ of G we will always mean a continuous unitary representation, π, on a Hilbert space $\mathcal{H}$. The group G endowed with the discrete topology is denoted by G_d, and π_d is the representation π viewed as a representation of G_d. If $\xi, \eta \in \mathcal{H}$, then $\xi *_\pi \eta(s) = \langle \pi(s)\xi | \eta\rangle$ $(s \in G)$ is the associated coefficient function. We write $\pi \cong \sigma$ and $\pi \prec \sigma$ to respectively indicate unitary equivalence and weak containment. If π and σ determine the same weak equivalence class we write $\pi \sim \sigma$. When there is a cardinal κ such that $\kappa\pi \cong \kappa\sigma$ then π and σ are said to be *quasi-equivalent,* and we will write $\pi \simeq_Q \sigma$. The notation $\pi \preceq_Q \sigma$ is used to signify that π is quasi-equivalent to a subrepresenation of σ. The kernel of π in $C^*(G)$ is $\ker(\pi)$ and the kernel of π in G is denoted by G-$\ker(\pi)$.

The Fourier-Stieltjes algebra of G is $B(G)$, and is identified with the dual of the group C^*-algebra, $C^*(G)$. We will be concerned with the C^*-algebras associated to $\{\pi, \mathcal{H}\}$, $C^*_\pi = \pi(C^*(G))$ and $C^*_{\pi_d}$. The von Neumann algebra generated by π, VN_π, is the WOT-closure in $B(\mathcal{H})$ of either $\mathrm{span}\{\pi(G)\}$ or C^*_π. We define A_π to be the norm-closure in $B(G)$ of $\mathrm{span}\{\xi *_\pi \eta : \xi, \eta \in \mathcal{H}\}$; B_π is the w^*-closure of A_π in $B(G)$. If $\{\lambda_G, L^2(G)\}$ is the left regular representation of G, $A_{\lambda_G} = A(G)$ is the Fourier algebra of G, and $B_{\lambda_G} = B_r(G)$ is the reduced Fourier-Stieltjes algebra of G. The reduced group C^*-algebra of G is $C^*_r(G) = C^*_{\lambda_G}$, and the group von Neumann algebra of G is $VN(G) = VN_{\lambda_G}$. If ω_G is the universal representation of G, then $B(G) = A_{\omega_G}$ and we will use the standard notation $W^*(G) = VN_{\omega_G}$.

We will call a representation π of G *self-conjugate* if $\{\pi, \mathcal{H}\}$ is quasi-equivalent to its conjugate representation $\{\overline{\pi}, \overline{\mathcal{H}}\}$; if $\{\pi, \mathcal{H}\}$ and $\{\overline{\pi}, \overline{\mathcal{H}}\}$ are weakly equivalent we say that that π is *weakly self-conjugate.* Proposition 3.6 of [1] shows that $A_{\overline{\pi}} = \overline{A_\pi}$ so the w^*-continuity in $B(G)$ of complex conjugation gives $B_{\overline{\pi}} = \overline{B_\pi}$. Thus, Proposition 3.1 of [1] implies that π is self-conjugate exactly when $A_\pi = \overline{A_\pi}$, and π is weakly self-conjugate exactly when $B_\pi = \overline{B_\pi}$.

The *congugation representation* of G, $\{\gamma_G, L^2(G)\}$, is defined by

$$((\gamma_G(x)\xi)(t) = \Delta_G(x)^{1/2}\xi(x^{-1}tx) \quad (\xi \in L^2(G),\ x, t \in G).$$

Let H be a closed subgroup of G, μ a strongly continuous quasi-invariant positive regular Borel measure on the space G/H of left cosets of H in G, and $\sigma(a, xH)$ the Radon-Nikodym derivative $d\mu(axH)/d\mu(xH)$. The *quasi-regular representation,* $\{\lambda_{G/H}, L^2(G/H, \mu)\}$, is defined by

$$\lambda_{G/H}(a)\xi(xH) = \sigma(a, xH)^{1/2}\xi(a^{-1}xH) \quad (a \in G,\ \xi \in L^2(G/H, \mu),\ xH \in G/H).$$

The *core* of H in G is $N_H = \bigcap\{xHx^{-1} : x \in G\}$. It is the largest normal subgroup of G which is contained in H. The centre of G is denoted by $Z(G)$. We will often refer to the following statement of known results.

Lemma 1.1. *The representations λ_G, γ_G, and $\lambda_{G/H}$ are self-conjugate. We have G-$\ker(\lambda_G) = \{e_G\}$, G-$\ker(\gamma_G) = Z(G)$, and G-$\ker(\lambda_{G/H}) = N_H$.*

Proof. In fact, $\lambda_G \cong \overline{\lambda_G}$ and $\gamma_G \cong \overline{\gamma_G}$ as witnessed by the unitary operator $L^2(G) \to \overline{L^2(G)} : \xi \mapsto \overline{\xi}$; similarly $\lambda_{G/H} \cong \overline{\lambda_{G/H}}$ [1, (3.36) Exemple]. The descriptions of the G-kernels of λ_G and γ_G are well-known (and readily verified); G. Arsac computed the G-kernel of $\lambda_{G/H}$ in Lemme 4.11 of his Ph.D. thesis (which contains [1]). □

Undefined notation can be found in the above references and [24]. For definitions of the variants of amenability discussed in this paper we refer to [22] and [23].

2. Some isomorphisms

In this section we provide more definitions and describe some isomorphisms that will be used in the sequel. Throughout, $\{\pi, \mathcal{H}\}$ is a fixed representation of G.

The spaces of operators C^*_π and VN_π are examples of concrete operator spaces. As such, their respective dual and pre-dual spaces B_π and A_π have canonical dual operator space structures [10]. Note further that $B_\pi = A_{\omega_\pi} = (VN_{\omega_\pi})_*$ for a representation ω_π of G [1, Proposition 2.24]. Moreover, we have inclusions $A_\pi \subset B_\pi \subset B(G)$, so that A_π and B_π also have natural subspace operator structures. When $\pi = \lambda_G$, the following lemma is certainly known (although we were not able to find a proof anywhere). We are grateful to the referee for suggesting this short proof.

Lemma 2.1. *The various dual and subspace operator space structures on A_π and B_π all agree.*

Proof. The inclusion $\iota : A_\pi \hookrightarrow B(G)$ has $\pi : W^*(G) \to VN_\pi$, the representation π lifted to $W^*(G)$, as its adjoint. As a surjective $*$-homomorphism of von Neumann algebras, $\iota^* = \pi$ is a complete quotient mapping and therefore ι is a complete isometry [10, Corollary 4.1.9]. We may replace π by ω_π, and conclude that the pre-dual operator space structure (o.s.s.) on A_π ($B_\pi = A_{\omega_\pi}$) agrees with its subspace o.s.s. from $B(G)$. Also, the inclusion $\iota : B_\pi \hookrightarrow B(G)$ is the adjoint of $\pi : C^*(G) \to C^*_\pi$, the representation π on $C^*(G)$, so we can again conclude from [10, Corollary 4.1.9] that ι is a complete isometry. Thus, the dual o.s.s. on B_π agrees with its subspace o.s.s. from $B(G)$. □

Suppose now that G-ker$(\pi) = N$, let $q_N : G \to G/N$ be the canonical map, and let $\{\pi_N, \mathcal{H}\}$ be the (continuous, unitary) representation of G/N defined through $\pi_N \circ q_N = \pi$. We may assume that the Weyl formula

$$\int_G f(x)\, dx = \int_{G/N} \int_N f(xn)\, dn\, d(xN) \quad (f \in L^1(G)) \tag{2.1}$$

holds. Then the map $T : C^*(G) \to C^*(G/N)$ defined through

$$Tf(xN) = \int_N f(xn)\, dn \qquad (f \in L^1(G))$$

is a surjective $*$-homomorphism. We denote the dual map by $k = T^*$ and note that $k : B(G/N) \to B(G) : u \mapsto u \circ q_N$ (for all of this see [11, page 203]). Observe that T is a complete quotient mapping, so by [10, Corollary 4.1.9], k is a complete isometry. By Lemma 2.1, [1, Proposition 2.10] and [11, (2.26) Corollaire 3] we can make the following statement.

Lemma 2.2. *The map k yields completely isometric identifications*

$$A_{\pi_N} \cong A_\pi \quad \text{and} \quad B_{\pi_N} \cong B_\pi.$$

Moreover, as a mapping of B_{π_N} onto B_π, k is $w^* - w^*$ continuous.

Observe that $B_\pi = (C^*_\pi)^* = \ker(\pi)^\perp$, so for $u \in B_\pi$ we have

$$\langle \pi(x), u\rangle = \langle x, u\rangle \qquad (x \in C^*(G)) \tag{2.2}$$

with the (C^*_π, B_π) pairing on the left and the $(C^*(G), B(G))$ pairing on the right.

Lemma 2.3. *The map*

$$T_\pi : C^*_\pi \to C^*_{\pi_N} : \pi(x) \mapsto \pi_N(Tx) \qquad (x \in C^*(G))$$

is a well-defined surjective $*$-isomorphism.

Proof. Letting k_π denote the $w^* - w^*$ continuous isometric isomorphism of B_{π_N} onto B_π from Lemma 2.2, the pre-adjoint, $T_\pi = k^*_\pi\Big|_{C^*_\pi}$, of k_π is a surjective linear isometry of C^*_π onto $C^*_{\pi_N}$. Note that $k_\pi = T^*\Big|_{B_{\pi_N}}$, so for any $u \in B_{\pi_N}$ and any $x \in C^*(G)$ we have

$$\langle T_\pi(\pi(x)), u\rangle = \langle \pi(x), k_\pi(u)\rangle = \langle x, k_\pi(u)\rangle = \langle x, T^*(u)\rangle = \langle Tx, u\rangle = \langle \pi_N(Tx), u\rangle,$$

where we have used equation (2.2). Thus T_π has the desired form, and it is now clear that T_π is a $*$-homomorphism. $\square$

Let $\{\pi, \mathcal{H}\}$ be a representation of G and consider the associated representation

$$\{\tau_\pi, \mathcal{H}_{\tau_\pi}\} = \{\sum_{n=1}^{\infty} \oplus(\pi^{\otimes n}), \sum_{n=1}^{\infty} \oplus(\mathcal{H}^{\otimes n})\}.$$

In [24] it is observed that τ_π is quite natural in the sense that A_{τ_π} is the smallest Banach subalgebra of $B(G)$ which contains all coefficient functions of π. As in [24] we use the notation

$$A(\pi) = A_{\tau_\pi} \quad \text{and} \quad B(\pi) = B_{\tau_\pi}$$

and we will write

$$VN(\pi) = VN_{\tau_\pi} \quad \text{and} \quad C^*(\pi) = C^*_{\tau_\pi}.$$

From [1], we thus we have

$$A(\pi)^* = VN(\pi) \text{ and } C^*(\pi)^* = B(\pi).$$

As subalgebras of the completely contractive Banach algebra $B(G)$ [23, p. 677], $A(\pi)$ and $B(\pi)$ are also completely contractive Banach algebras (see Lemma 2.1).

Observe that for the left regular representation, a well-known result of Fell [5, Corollary E.2.2] shows that $\lambda_G \simeq_Q \tau_{\lambda_G}$, and it follows that we have

$$A(G) = A(\lambda_G),\ B_r(G) = B(\lambda_G),\ C^*_r(G) \cong C^*(\lambda_G), \text{ and } VN(G) \cong VN(\lambda_G)$$

where $\cong$ indicates the existence of a $*$-isomorphism.

It is clear that τ_π is (weakly) self-conjugate whenever the representation π is so. As well, if $N = G\text{-ker}(\pi)$, then it also true that $G\text{-ker}(\tau_\pi) = N$ and $\tau_{(\pi_N)} = (\tau_\pi)_N$. From the above lemmas we thus obtain the following proposition.

Proposition 2.4. *Let π and N be as above. Then $A(\pi) \cong A(\pi_N)$ via a completely isometric Banach algebra isomorphism, $B(\pi) \cong B(\pi_N)$ via a $w^* - w^*$ continuous completely isometric Banach algebra isomorphism, $C^*(\pi) \cong C^*(\pi_N)$ via a $*$-isomorphism, and $VN(\pi) = VN(\pi_N)$.*

3. Nuclearity of $C^*(\pi)$

We first show that for any weakly self-conjugate representation π, the nuclearity (=amenability) of $C^*(\pi)$ reflects the amenability of G as well as one could hope.

Theorem 3.1. *Let π be a weakly self-conjugate representation of G with $N = G\text{-ker}(\pi)$. Then G/N is amenable if and only if $C^*(\pi)$ is nuclear and G/N is inner amenable.*

Proof. If G/N is amenable, then we know from Theorem 0.1 that $C^*_r(G/N) = C^*_{\lambda_{G/N}}$ is amenable and G/N is inner amenable. By the weak containment property of amenable groups [15, Theorem 3.5.2], $(\tau_\pi)_N \prec \lambda_{G/N}$, so $C^*(\pi_N)$ is a quotient of $C^*_r(G/N)$. It follows that $C^*(\pi_N)$ is amenable and therefore, by Proposition 2.4, $C^*(\pi)$ is also amenable.

Suppose conversely that $C^*(\pi)$ is amenable and G/N is inner amenable. Observe that $B(\pi_N)$ is a w^*-closed self-conjugate translation-invariant closed subalgebra of $B(G/N)$ which, because π_N is faithful, separates the points of G/N. By Theorem 1.3 of [4], $B(\pi_N) \supset A(G/N)$ and consequently $B(\pi_N) \supset B_{\lambda_{G/N}}$. It follows from [1, Proposition 3.1] that $(\tau_\pi)_N$ weakly contains $\lambda_{G/N}$, and therefore $C^*_r(G/N) = C^*_{\lambda_{G/N}}$ is a quotient of $C^*(\pi_N)$. But $C^*(\pi_N) \cong C^*(\pi)$, so it follows from our assumption that $C^*_r(G/N)$ is amenable. We are also assuming that G/N is inner amenable, so Theorem 0.1 yields the amenability of G/N. $\square$

As we have already mentioned, $\lambda_G \simeq_Q \tau_{\lambda_G}$ and so $(\lambda_G)_d \simeq_Q (\tau_{\lambda_G})_d$. Consequently $C^*_{(\lambda_G)_d}$ is $*$-isomorphic to $C^*((\lambda_G)_d)$. As λ_G has a trivial kernel in G, an immediate corollary to the above is the following result of E. Bédos (see [2, Theorem 3]). Note that discrete groups are, trivially, inner amenable.

Corollary 3.2 (E. Bédos)**.** *Let G be a locally compact group. Then G_d is amenable if and only if $C^*_{(\lambda_G)_d}$ is nuclear.*

We will need the following proposition which is an improvement of [24, Theorem 3.4] (which generalizes and extends [2, Theorems 1 and 1']). We will use the facts that π is amenable if and only if $1_G \prec \pi \otimes \overline{\pi}$ [3, Theorem 5.1] and $1_G \prec \pi$ if and only if $1_G \in B_\pi$ [1, Proposition 3.1].

Proposition 3.3. *Let π be a weakly self-conjugate representation. Then π is amenable if and only if $C^*(\pi)$ has a non-zero multiplicative linear functional. The statement remains true if $C^*(\pi)$ is replaced by $C^*(\pi_d)$.*

Proof. If π is amenable, then $1_G \prec \pi \otimes \overline{\pi} \sim \pi \otimes \pi \subset \tau_\pi$. Hence $1_G \in B_{\tau_\pi}$. Now

$$\langle \tau_\pi(f), 1_G \rangle = \int_G f(s)\, ds \qquad (f \in L^1(G)),$$

so 1_G is multiplicative on $C^*_{\tau_\pi} = C^*(\pi)$. Conversely, suppose that $u \in B(\pi) = C^*(\pi)^*$ is a non-zero multiplicative linear functional on $C^*(\pi)$. Then u is a state on $C^*(\pi)$, so $u(\tau_\pi(x)^*) = \overline{u(\tau_\pi(x))}$ $(x \in C^*(G))$. But $\langle x, u \rangle = \langle \tau_\pi(x), u \rangle$ $(x \in C^*(G))$, so $u \in B(G)$ is a $*$-representation of $C^*(G)$ on $\mathbb{C}$. That is, u is a character on G and so $1_G = u\overline{u} \in B(\pi) = B_{\tau_\pi}$. Therefore $1_G \prec \tau_\pi$ and so $1_G \prec \tau_\pi \otimes \tau_\pi$. It follows that τ_π is amenable and therefore so is π [24, Lemma 4.2]. For the second statement, note that the amenability of π and π_d coincide [3, Remark 1.2]. $\square$

For any weakly self-conjugate representation π for which amenability is characterized by the condition $1_G \prec \pi$, one can similarly prove that π is amenable if and only if C^*_π (or $C^*_{\pi_d}$) has a non-zero multiplicative linear functional. This is the case for λ_G, γ_G, and $\lambda_{G/H}$. Note that G is amenable, inner amenable, and G/H is G-amenable, precisely when these representations are, respectively, amenable [3]. We now extend [2, Theorem 3].

Proposition 3.4. *Let $\{\pi, \mathcal{H}\}$ be a weakly self-conjugate representation of G and let $N = G\text{-}\ker(\pi)$. Then the following are equivalent:*

(i) *$(G/N)_d$ is amenable.*
(ii) *$C^*(\pi_d)$ is nuclear.*
(iii) *G/N is amenable and $C^*(\pi_d) \cong C^*_r((G/N)_d)$.*

Proof. The equivalence of (i) and (ii) follows from Theorem 3.1. If $(G/N)_d$ is amenable, then G/N is amenable, and $B((G/N)_d) = B_r((G/N)_d)$, so the proof of Theorem 3.1 yields the desired $*$-isomorphism. If we assume condition (iii), then the representation π is amenable [3, Theorem 2.2 and Remark 1.2(ii)]. By Proposition 3.3, $C^*(\pi_d)$ has a non-zero multiplicative linear functional and, therefore, so does $C^*_r((G/N)_d)$. It now follows from [2, Theorem 1] (or Proposition 3.3 and [3, Theorem 2.2]) that $(G/N)_d$ is amenable. $\square$

For the conjugate and quasi-regular representations we now indicate how these results can be improved. We were unable to locate a proof of the following lemma.

Lemma 3.5. *If G is inner amenable, then so is $G/Z(G)$.*

Proof. Let $N = Z(G)$ and consider the map $T : L^1(G) \to L^1(G/N)$ defined as in Section 2. In this case, it is then readily seen that the modular functions satisfy $\Delta_{G/N}(xN) = \Delta_G(x)$, and $T(\delta_x * f * \delta_{x^{-1}}) = \delta_{xN} * Tf * \delta_{x^{-1}N}$ $(x \in G)$. Moreover, from equation (2.1) Tf is non-negative and norm one in $L^1(G/N)$ whenever f is non-negative and norm one in $L^1(G)$. A locally compact group, H, is inner amenable exactly when there is a net $f_\alpha \in L^1(H)$ of norm one non-negative elements satisfying $\|\delta_x * f_\alpha * \delta_{x^{-1}} - f_\alpha\|_1 \to 0$ $(x \in H)$ [22, Ex. 4.4.4], so the lemma follows. $\square$

Because $Z(G)$ is amenable (even as a discrete group), the amenability of G (resp. G_d) coincides with that of $G/Z(G)$ (resp. $G_d/Z(G)$) [22, Section 1.2]. Theorem 3.6 is hence a consequence of Theorem 3.1, Lemma 3.5, and Proposition 3.4.

Theorem 3.6. *Let $\{\gamma_G, L^2(G)\}$ be the conjugation representation of G. Then*

(a) *G is amenable if and only if $C^*(\gamma_G)$ is nuclear and G is inner amenable.*

(b) *Moreover, the following are equivalent:*

  (i) *G_d is amenable.*

  (ii) *$C^*((\gamma_G)_d)$ is amenable.*

  (iii) *G is amenable and $C^*((\gamma_G)_d) \cong C^*_r((G/Z(G))_d)$.*

E. Kaniuth and A. Markfort have proved that G_d is amenable if and only if $C^*_{(\gamma_G)_d}$ is nuclear [18, Theorem 1.2]. As $C^*_{(\gamma_G)_d}$ is a quotient of $C^*((\gamma_G)_d)$, (ii) implies (i) of Theorem 3.6(b) already follows from [18]. However, there seems to be no known analogue of Theorem 3.6(a) for $C^*_{\gamma_G}$ [18, p. 2995] (we should point out that the notation in [18] is different from ours).

We have already noted that G is inner amenable precisely when γ_G is amenable as a representation [3]. By Proposition 3.3 and Theorem 3.6, amenability of G is hence characterized entirely in terms of C*-algebraic (or Banach algebraic) properties of $C^*(\gamma_G)$:

> *G is amenable if and only if $C^*(\gamma_G)$ is nuclear(=amenable) and has a non-zero multiplicative linear functional.*

Recall that a homogeneous space G/H is G-amenable if, with respect to the natural action of G on G/H, $CB(G/H)$, the C^*-algebra of continuous functions on G/H, has a G-invariant mean [12, pp. 28, 29]. We refer the reader to [12] for further details.

Lemma 3.7. *Let H be a closed subgroup of G, and let N be a normal subgroup of G which is contained in H. Then*

(i) *G/H is G-amenable if and only if $(G/N)/(H/N)$ is G/N-amenable; and*

(ii) *G/N is amenable if and only if H/N is amenable and G/H is G-amenable.*

Proof. (i) Let $X = G/H$, $\overline{G} = G/N$, $\overline{X} = (G/N)/(H/N)$, and define $\alpha : X \to \overline{X}$ by putting $\alpha(xH) = xN \cdot (H/N)$. One can then check that α is a well-defined homeomorphism satisfying $q_N(g)\cdot\alpha(x) = \alpha(g\cdot x)$ $(g \in G, x \in X)$. Let $T : CB(\overline{X}) \to CB(X)$ be the map given by $T(f) = f \circ \alpha$, with dual map T^*. Routine calculations then show that $m \in CB(X)^*$ is a G-invariant mean if and only if $T^*m \in CB(\overline{X})^*$ is a $\overline{G}$-invariant mean.

(ii) The backward implication follows from (i) by using [12, page 16 b)] and the converse follows from (i) and [12, page 16 a)]. (Referring to [12], the trivial subgroup of G/N should be used for K). □

Let H be a closed subgroup of G, $\lambda_{G/H}$ the associated quasi-regular representation of G, and $\lambda_{(G/H)_d}$ the associated quasi-regular representation of G_d. (In the notation of induced representations, $\lambda_{G/H} = Ind_H^G 1_H$ and $\lambda_{(G/H)_d} = Ind_{H_d}^{G_d} 1_{H_d}$.)

Theorem 3.8. *The group G_d is amenable if and only if G is amenable, H_d is amenable, and $C^*((\lambda_{G/H})_d) \cong C^*(\lambda_{(G/H)_d})$.*

Proof. Let $\lambda^1 = \lambda_{G/H}$ and $\lambda^2 = \lambda_{(G/H)_d}$. The forward implication follows from Lemma 1.1 and Proposition 3.4. For the converse, by [3, Theorem 2.2], λ^1 is amenable, so by Proposition 3.3, $C^*(\lambda_d^1)$ has a non-zero multiplicative linear functional. Consequently, $C^*(\lambda^2)$ has such a functional, so Proposition 3.3 now implies that λ^2 is amenable. Thus $(G/H)_d$ is G_d-amenable [3, Theorem 2.3]. By Lemma 3.7, we know that $(G/N_H)_d$ is amenable, and the subgroup $(N_H)_d$ of H_d is amenable; therefore G_d is itself amenable. □

4. Amenability of other Banach algebras generated by π

We now discuss the amenability properties of $A(\pi)$, $B(\pi)$, and VN_π with π an unspecified representation. We then focus on the quasi-regular representations.

To save space we let $[AG]$, $[AR]$, $[BAI]$, $[OA]$, and $[OCA]$ respectively denote the classes of amenable locally compact groups, amenable representations, Banach algebras with bounded approximate identities, operator amenable operator Banach algebras, and operator Connes-amenable dual, operator Banach algebras [22], [23].

Proposition 4.1. *Let π be a weakly self-conjugate representation with G-$\ker(\pi) = N$. If $\pi_N \prec \lambda_{G/N}$, then the following are equivalent:*
(i) $\pi \in [AR]$.
(ii) $G/N \in [AG]$.
(iii) $B(\pi) \in [OCA]$.

When π is self-conjugate and $\pi_N \preceq_Q \lambda_{G/N}$, the above statements are equivalent to
(iv) $A(\pi) \in [OA]$.
(v) $A(\pi) \in [BAI]$.

Proof. The representation π is amenable precisely when π_N is amenable, so the equivalence of (i) and (ii) follows from Theorem 2.2 and Corollary 5.3 of [3]. For the equivalence of all remaining statements, we note that the implications

$$\pi_N \preceq_Q \lambda_{G/N} \Longrightarrow A(\pi_N) = A(G/N) \quad \text{and} \quad \pi_N \preceq \lambda_{G/N} \Longrightarrow B(\pi_N) = B_r(G/N)$$

hold. Indeed, if $\pi_N \preceq_Q \lambda_{G/N}$, then $(\tau_\pi)_N \preceq_Q \tau_{\lambda_{G/N}} \simeq_Q \lambda_{G/N}$, so that $A(\pi_N) \subset A(G/N)$ [1, Corollaire 3.14]. As π_N is faithful, $A(\pi_N)$ is point-separating and Theorem 2.1 of [4] gives $A(\pi_N) = A(G/N)$. If $\pi_N \prec \lambda_{G/N}$ we similarly obtain $B(\pi_N) \subset B_r(G/N)$, and $B_r(G/N) \subset B(\pi_N)$ is shown in the proof of Theorem 3.1. The proposition is now a consequence of Proposition 2.4, Theorem 0.1, and Leptin's Theorem. □

Remark 4.2. 1. In Section 3 we obtained quite general results which cannot be expected for $A(\pi)$ and $B(\pi)$. For example, in Proposition 4.1, the implication (ii) $\Rightarrow$ (iv) may fail if $\pi_N \preceq_Q \lambda_{G/N}$ fails: Let G be any non-compact abelian (hence amenable) group, and ω_G its universal representation. Amenability and operator amenability of $A(\omega_G) = B(G)$ are equivalent, because as the dual of an abelian C^*-algebra, $B(G)$ has the MAX operator space structure [10, Proposition 16.1.5 and p. 51]. But, if $B(G) \cong M(\widehat{G})$ is amenable, then $\widehat{G}$ is discrete [7], and G is compact.

2. One can also show that if $\lambda_{G/N} \preceq_Q \pi_N$ (with G-$\ker(\pi) = N$), then G/N is amenable if and only if VN_π is Connes-amenable and G/N is inner amenable. (For this, we do *not* need to assume that $\pi_N \preceq_Q \lambda_{G/N}$.) Under the hypothesis $\lambda_{G/N} \preceq_Q (\tau_\pi)_N$, this becomes the statement that G/N is amenable if and only if $VN(\pi)$ is Connes amenable and G/N is inner amenable. This seems noteworthy because, for self-conjugate representations π, by [4, Corollary 2.3] $\lambda_{G/N} \preceq_Q (\tau_\pi)_N$ holds whenever $A(\pi_N) \cap A(G/N) \neq \{0\}$.

In light of Proposition 4.1 and Remark 4.2, for a fixed representation π it is natural to ask when $\pi_N \preceq \lambda_{G/N}$, $\pi_N \preceq_Q \lambda_{G/N}$, and $\lambda_{G/N} \preceq_Q (\tau_\pi)_N$. We will now examine these questions with regards to quasi-regular representations.

Hereafter, H is a closed subgroup of a locally compact group G, $q_H : G \to G/H$ is the canonical map, and $\{\lambda_{G/H}, L^2(G/H, \mu)\}$ is the associated quasi-regular representation. Recall that G-ker$(\lambda_{G/H}) = N_H$ (Lemma 1.1).

Proposition 4.3. *The following are equivalent for amenable extensions H of N_H:*

(i) *G/H is G-amenable.*
(ii) *$G/N_H \in [AG]$.*
(iii) *$B(\lambda_{G/H}) \in [OCA]$.*

Proof. By Lemma 3.7, statements (i) and (ii) are equivalent. If G/N_H is amenable, then by the weak containment property [15, Theorem 3.5.2], $(\lambda_{G/H})_{N_H} \prec \lambda_{G/N_H}$, so (ii) implies (iii) follows from Proposition 4.1. If (iii) holds, then $B(\lambda_{G/H})$ has an identity [23, proof of Theorem 4.4] so that $\lambda_{G/H}$ is amenable (see the proof of [24, Theorem 4.5]). By [3, Theorem 2.3], statement (i) follows. ☐

Lemma 4.4. *Let N be a normal subgroup of G and suppose that H is a compact extension of N. If L is a compact subset of G/H, then $q_N(q_H^{-1}(L))$ is a compact subset of G/N.*

Proof. We can choose compact subsets H_0 of H and L_0 of G such that $q_N(H_0) = H/N$ and $q_H(L_0) = L$. Now L_0H_0 is a compact subset of G, and one can readily verify that

$$q_N(q_H^{-1}(L)) = \{lhN : l \in L_0, h \in H_0\} = q_N(L_0H_0),$$

which is also compact. ☐

In the second part of the following proposition, we will assume that there is a compact neighbourhood V of e_G such that

$$HV = VH \quad \text{and} \quad \inf_{s\in H} \inf_{x\in V} \sigma(xH, s) > 0. \tag{4.1}$$

We note that H satisfies this weak neutrality condition if *(i)* G is an $[IN]_H$ group, meaning that there is a compact neighbourhood V of e_G such that $sVs^{-1} = V$ $(s \in H)$, or *(ii)* H is a neutral subgroup of G. In either case, the first condition is satisfied. For (i) it is not difficult to verify that the modular functions of G and H agree on H. This is equivalent to the existence of a non-zero G-invariant measure on G/H [13, Corollary 13.16], so $\sigma(xH, a) \equiv 1$. In case (ii) G/H again has a G-invariant measure [17, p. 96, 97].

Proposition 4.5. (i) *Suppose that H is a compact extension of N_H. Then*

$$(\lambda_{G/H})_{N_H} \preceq_Q \lambda_{G/N_H}, \qquad \tau((\lambda_{G/H})_{N_H}) \simeq_Q \lambda_{G/N_H},$$

and the completely contractive Banach algebras $A(\lambda_{G/H})$ and $A(G/N_H)$ are completely isometrically isomorphic.

(ii) *Conversely, suppose that there is a compact neighbourhood V of e_G satisfying equation* (4.1). *If $(\lambda_{G/H})_{N_H} \preceq_Q \lambda_{G/N_H}$, then H is a compact extension of N_H.*

Proof. In this proof let $N = N_H$, $\pi = \lambda_{G/H}$, C_f be the co-zero set of a function f, and $S_f = \overline{C_f}$ be its support. Note that if $k : A_{\pi_N} \to A_\pi$ is the isomorphism from Lemma 2.2, then k^{-1} satisfies $k^{-1}u(xN) = u(x)$, $(x \in G)$.

(i) Let ξ, η be compactly supported functions in $(L^2(G/H), \mu)$ and consider the coefficient function $u = \xi *_\pi \eta$. Observe that $C_u \subset q_H^{-1}(S_\eta)[q_H^{-1}(S_\xi)]^{-1}$. Therefore $C_{k^{-1}(u)} \subset q_N(C_u) \subset q_N(q_H^{-1}(S_\eta))[q_N(q_H^{-1}(S_\xi))]^{-1}$ which, by Lemma 4.4, is compact. Thus $k^{-1}u = \xi *_{\pi_N} \eta \in B(G/N) \cap C_{00}(G/N) \subset A(G/N)$. From the density of the compactly supported functions in $(L^2(G/H), \mu)$, we obtain $A_{\pi_N} \subset A(G/N)$, and hence $\pi_N \preceq_Q \lambda_{G/N}$ [1, Corollaire 3.14]. As shown in the proof of Proposition 4.1, it follows that $A(\pi_N) = A(G/N)$ and part (i) now follows from [1, Proposition 3.1(ii)].

(ii) Let V be a compact neighbourhood of e_G satisfying equation (4.1), let $\xi = 1_{q_H(V)}$, and put $u = \xi *_\pi \xi$. Let $\alpha = \inf_{s \in H} \inf_{x \in V} \sigma(xH, s)$. Fixing $s \in H$, for every $x \in V$ there is some $t \in H$ and $y \in V$ such that $s^{-1}x = yt$; therefore $\xi(s^{-1}xH) = \xi(ytH) = 1$. Consequently,

$$\begin{aligned} u(s) &= \int_{G/H} \sigma(xH, s)^{1/2} \xi(s^{-1}xH)\xi(xH)\, d\mu(xH) \\ &= \int_{q_H(V)} \sigma(xH, s)^{1/2}\, d\mu(xH) \geq \alpha^{1/2}\mu(q_H(V))\ (> 0). \end{aligned}$$

Now, $k^{-1}u \in A(\pi_N) = A(G/N)$, so $k^{-1}u$ vanishes at infinity. But $k^{-1}u(sN) = u(s) \geq \alpha^{1/2}\mu(q(V))$ $(sN \in H/N)$, so H/N is compact. □

The following may be compared with Theorem 4.2 (and Corollary 4.3) of [14]. It is a direct consequence of Propositions 4.5(i), 4.1, and Remark 4.2.

Theorem 4.6. *Suppose that H is a compact extension of N_H. Then G/H is G-amenable if and only if any one of the following conditions holds:*

(i) $G/N_H \in [AG]$.
(ii) $A(\lambda_{G/H}) \in [BAI]$.
(iii) $A(\lambda_{G/H}) \in [OA]$.
(iv) $VN(\lambda_{G/H})$ *is Connes-amenable and* G/N_H *is inner amenable.*
(iv) $\mathcal{M}A(\lambda_{G/H})$, *the multiplier algebra of* $A(\lambda_{G/H})$, *is* $B(\lambda_{G/H})$.

Example 4.7. Let G be a semidirect product group $G = M \rtimes F$ and let $H = M \rtimes K$, where K is a compact subgroup of F. If N_H is the core of H in G and N_K is the core of K in F, then it is easy to see that $N_H = M \rtimes N_K$. Hence, $H/N_H = (M \rtimes K)/(M \rtimes N_K) \cong (K/N_K)$. By choosing M non-compact we thus obtain non-compact examples of H for which H/N_H is compact. In this case, Proposition 4.5 gives $\tau((\lambda_{G/H})_{N_H})) \simeq_Q \lambda_{G/N_H}$ and $A(\lambda_{G/H}) \cong A(F/N_K)$. Also, $B(\lambda_{G/H}) \cong B_r(F/N_K)$, $C^*(\lambda_{G/H}) \cong C_r^*(F/N_K)$, and $VN(\lambda_{G/H}) \cong VN(F/N_K)$. As N_K is compact, G/H is G-amenable if and only if F is amenable (Theorem 4.6).

Acknowledgments

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