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REMARKS ON ELLIPTIC PROBLEMS INVOLVING THE CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

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ABSTRACT. We give some regularity results of the solutions and a Liouville type theorem to singular elliptic equations involving the Caffarelli-Kohn-Nirenberg inequalities.

1. INTRODUCTION

In this paper, we consider the following elliptic problem:

$$\begin{cases} -div(|x|^{-2a}\nabla u) - \mu\dfrac{u}{|x|^{2(1+a)}} = \dfrac{u^{s-1}}{|x|^{bp}} + \lambda\dfrac{u^{q-1}}{|x|^{dD}}, \quad u \geq 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth domain in $\mathbb{R}^N (N \geq 3)$, $0 \in \Omega$, $0 \leq \mu < (\sqrt{\bar{\mu}} - a)^2$, $\bar{\mu} \triangleq (\frac{N-2}{2})^2$, $0 \leq a < \sqrt{\bar{\mu}}$, $a \leq b < a+1$, $a \leq d < a+1$, $\lambda \geq 0$, $1 < s \leq p = p(a,b) \triangleq \frac{2N}{N-2(1+a-b)}$, $D = D(a,d) \triangleq \frac{2N}{N-2(1+a-d)}$, $1 < q < D$. Problem (1.1) is related to the following well known Caffarelli-Kohn-Nirenberg inequalities [6]:

$$\Big(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p\Big)^{\frac{2}{p}} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N), \tag{1.2}$$

where $p = p(a,b)$ is called the critical Sobolev-Hardy exponent. In recent years, much attention has been paid to (1.1); we can refer to [3, 4, 7, 9, 13] and the references therein.

Let $L^q(\Omega, |x|^\beta)$ denote the weighted $L^q(\Omega)$ space with the weight $|x|^\beta$. For $\mu \in [0, (\sqrt{\bar{\mu}} - a)^2)$, define $H = H_0^1(\Omega, |x|^{-2a})$ to be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \|u\|_H \triangleq \Big(\int_\Omega \Big(|x|^{-2a}|\nabla u|^2 - \mu\frac{|u|^2}{|x|^{2(1+a)}}\Big)dx\Big)^{\frac{1}{2}}. \tag{1.3}$$

(1.3) is well defined due to the weighted Hardy inequality (see [7] or [14] for example).

We first study the regularity of the solutions in H for problem (1.1). Set $\beta = \sqrt{(\sqrt{\bar{\mu}} - a)^2 - \mu}$ and $\nu = \sqrt{\bar{\mu}} - a - \beta$. Our results are as follows:

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Theorem 1.1. *If $u(x) \in H$ is a solution of* (1.1) *with Ω bounded, then $v(x) = |x|^\nu u(x)$ is Hölder continuous.*

Theorem 1.2. *Let $\Omega = B_1(0)$. If $u(x) \in H$ is a positive solution of* (1.1)*, then $u(x)$ is radially symmetric and $v(x) = |x|^\nu u(x) \in C^2(B_1(0) \setminus \{0\}) \cap C^1(\overline{B}_1(0))$ if $c^* > N$, and $v(x) = |x|^\nu u(x) \in C^2(B_1(0) \setminus \{0\}) \cap C^\kappa(\overline{B}_1(0))$ for some $\kappa \in (0,1)$ if $c^* \le N$. Moreover,*

$$\begin{cases} v'(0) = 0, & \text{if } c^* > N, \\ v'(0^+) = -\frac{(N-2-2(\nu+a))^\alpha}{1-k_1}, & \text{if } c^* = N, \\ v'(0^+) = -\infty, & \text{if } c^* < N, \end{cases}$$

where $k_1 = \frac{2N-2-2(a+\nu)-(b+\nu)p}{N-2-2(a+\nu)}$, $\alpha = \frac{(b+\nu)p+2-N}{p-2}$ and

$$c^* = 2(N-1-2a-2\nu)(1+a-b).$$

Remark 1.1. From Theorem 1.1, if $\mu = 0$, then $\nu = 0$, hence the solutions of (1.1) are Hölder continuous. In Theorem 1.2, if $\mu = 0$, then the positive solutions of (1.1) in a ball centered at the origin are C^1 smooth provided $(a-b)N + 2(1+2a) < N$.

Remark 1.2. Theorem 1.2 means that if $a = b = 0$, then any positive solution u of (1.1) in a ball centered at the origin has the property that $|x|^\nu u(x)$ is C^1 smooth provided $0 < \mu < 3(N-2)^2/16$. By considering the equation satisfied by $|x|^\nu u(x)$ and using the C^1 smoothness of $|x|^\nu u(x)$ we can partly answer the open problem proposed in [9] affirmatively.

Secondly we give a Liouville type theorem for problem (1.1). To express the main result, we define a strong solution for (1.1) which was first introduced in [8] where $a = b = 0$ as: u is a $C^2(\Omega \setminus \{0\})$ function satisfying problem (1.1) everywhere except possibly at the origin, such that for some $C > 0$ and ρ,

$$0 \le u \le C|x|^{-\nu}, \quad \forall\, x \in B_\rho(0) \subset \Omega.$$

Theorem 1.3. *Let $\max\{1, \frac{N-bp}{N-2-2a-\nu}\} < s < p$, $\lambda = 0$ and $\Omega = \mathbb{R}^N$. u is a strong solution of problem* (1.1)*. If u is radially symmetric, then $u \equiv 0$.*

Remark 1.3. We see from Corollary 3.1 that if $a = 0$ and $2 < s < p$, Theorem 1.3 holds without the radial symmetry assumption. But as for the case $a \ne 0$ or $1 < s \le 2$, whether or not a strong solution of (1.1) is radially symmetric is not clear.

The proof of the theorems are based on the exact estimate of the singularity of the solutions and shooting technique. More precisely, due to the appearance of the Hardy term $\frac{\mu u}{|x|^{2(1+a)}}$, the solution u is singular at the origin. We first prove that the singularity of u at the origin is $|x|^{-\nu}$; then by estimating the L^∞-norm of $v = |x|^\nu u$ with Moser iteration technique we finish the proof of Theorem 1.1. To prove Theorem 1.2, we transform problem (1.1) into a problem of ODE and then employ a shooting technique introduced in [2]. We should point out that, unlike [2], in our case, the singularity at the origin fails to satisfy the conditions of the Gidas-Ni-Nirenberg Theorem [11]. To overcome this difficulty, we first consider the equation satisfied by $v = |x|^\nu u$ and estimate the L^∞-norm of v. Then employing the results in [3], we prove that v and hence u are radial. Moreover, we will apply the shooting argument to the equations satisfied by $v(x) = |x|^\nu u(x)$; hence it is necessary to define the value of v at the origin, which is also needed to estimate the singularity of $u(x)$.

2. Regularity of the solutions

We only consider the critical case $s = p$ since the subcritical case is easier. For the existence of positive solutions, we refer to [14].

Lemma 2.1. *Suppose that $u(x) \in H$ is a solution of* (1.1) *with Ω bounded and $d \leq b$. Then*

(1) there exists $0 < c < \infty$, such that $c|x|^{-\nu} \leq u(x)$, $\forall\, x \in B_\rho(0) \setminus \{0\}$,

(2) for any $k > 0$, $v(x) = |x|^\nu u(x) \in L^k(\Omega, |x|^{-(b+\nu)p})$.

Proof. Let $v(x) = |x|^\nu u(x)$; it is easy to verify that $v(x)$ solves

$$-div(|x|^{-2(a+\nu)}\nabla v) = \frac{v^{p-1}}{|x|^{(b+\nu)p}} + \lambda \frac{v^{q-1}}{|x|^{dD+q\nu}}. \tag{2.1}$$

Since $v(x) \in H_0^1(\Omega, |x|^{-2(a+\nu)})$, the regularity theorem for elliptic equations guarantees that v belongs to $C^2(\Omega \setminus \{0\})$. Hence using Lemma 2.1 in [14], we conclude that for any $B_\rho(0) \subset \Omega$, $v(x) \geq \min_{|x|=\rho} v(x)$ for $\forall x \in B_\rho(0) \setminus \{0\}$, which is exactly part (1) of the lemma.

Part (2) can be proved by using the Moser iteration argument and the standard regularity theorem for elliptic equations. For details, we can refer to [3] or [10]. □

Remark 2.1. From Lemma 2.1, we see that there exist c_1, $c_2 > 0$ such that $c_1 \leq v(x) \leq c_2$, which implies that u is a strong solution for (1.1).

Proof of Theorem 1.1. From (2.1), v satisfies

$$\int_\Omega |x|^{-2(a+\nu)}\nabla v \nabla \varphi = \int_\Omega \frac{f\varphi}{|x|^{(b+\nu)p}}, \quad \forall\, \varphi \in H_0^1(\Omega, |x|^{-(b+\nu)p}),$$

where $f(x) = v^{p-1} + \lambda |x|^{(b+\nu)p-(dD+q\nu)} v^{q-1}$. Since $(b+\nu)p - (dD + q\nu) > 0$ and using Lemma 2.1, we deduce $f \in L^\sigma(\Omega, |x|^{-(b+\nu)p})$, where $\sigma > p/(p-2)$. Hence by Theorem 1.1 in [10], $v(x) \in C^\alpha$ for some $\alpha \in (0,1)$. □

To prove Theorem 1.2, we first give a symmetry result.

Lemma 2.2. *Let $\Omega = B \triangleq B_1(0)$. Suppose that $u(x) \in H$ is positive and satisfies problem* (1.1) *with $d \leq b$; then u is radially symmetric.*

Proof. Since $v(x) = |x|^\nu u(x)$ solves

$$-div(|x|^{-2(a+\nu)}\nabla v) = \frac{v^{p-1}}{|x|^{(b+\nu)p}} + \lambda \frac{v^{q-1}}{|x|^{dD+q\nu}},$$

and $v \in C^2(B \setminus \{0\}) \cap C^1(\overline{B} \setminus \{0\})$ is bounded in B by Lemma 2.1, by Proposition 1.3 in [3] we can prove that $v(x)$ and hence $u(x)$ are radially symmetric. □

Proof of Theorem 1.2. Suppose that $u > 0$ solves (1.1) in $B_1(0)$; then by Lemma 2.2, $u(x)$ must be radially symmetric. Set $v(x) = |x|^\nu u(x)$; then $v(x) = v(|x|)$ solves

$$\begin{cases} v''(\rho) + \dfrac{N-1-2(a+\nu)}{\rho} v'(\rho) + \lambda \dfrac{v^{q-1}(\rho)}{\rho^{dD+q\nu-2(a+\nu)}} + \dfrac{v^{p-1}(\rho)}{\rho^{(b+\nu)p-2(a+\nu)}} = 0, \\ v > 0 \quad \text{for} \quad 0 < \rho < 1, \quad v(1) = 0. \end{cases} \tag{2.2}$$

Set $t = (\frac{N-2-2(\nu+a)}{\rho})^{N-2-2(\nu+a)}$ and $y(t) = (N-2-2(\nu+a))^{-\alpha}v(\rho)$, where $\alpha = \frac{(b+\nu)p+2-N}{p-2}$. Problem (2.11) can be rewritten as

$$\begin{cases} y''(t) = -t^{-k_1}y^{p-1} - \lambda C t^{-k_2}y^{q-1}, \\ y(t) > 0 \text{ for } T < t < \infty, \\ y(T) = 0, \end{cases} \tag{2.3}$$

where $k_1 = \frac{2N-2-2(a+\nu)-(b+\nu)p}{N-2-2(a+\nu)}$, $k_2 = \frac{2N-2-2(a+\nu)-dD-q\nu}{N-2-2(a+\nu)}$,

$$T = (N-2-2(\nu+a))^{N-2-2(\nu+a)}$$

and $C = (N-2-2(a+\nu))^{\alpha(q-1)-dD-q\nu-2(a+\nu)-2+2N} > 0$.

We claim that there exists a positive number $0 < \gamma < +\infty$ such that

$$\lim_{t\to+\infty} y'(t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} y(t) = \gamma.$$

Indeed, $0 < \gamma < +\infty$ follows Remark 2.1. By Lemma 2.1, $y(t)$ is bounded in $[T, +\infty)$. From (2.3) we know $y''(t) < 0$ for all $t > T$, so $y'(t)$ decreases strictly in $t \in (T, +\infty)$. Hence

$$y'(t) \to l \qquad \text{as} \qquad t \to +\infty.$$

In the cases $l > 0$ and $l < 0$, we deduce $y(t) \to +\infty$ and $y(t) \to -\infty$ as $t \to +\infty$, respectively, which contradicts the boundedness of $y(t)$. Therefore $y'(t) \to 0$ and $y(t) \to \gamma < +\infty$ as $t \to +\infty$.

Now define $v(0) = (N-2-2(\nu+a))^{\alpha}\gamma$; then $v(\rho) \in C[0,1]$. Furthermore, $v'(\rho) < 0$ for all $\rho \in (0,1]$. From $y(t) = (N-2-2(\nu+a))^{-\alpha}v(\rho)$, we have

$$v'(\rho) = -(N-2-2(\nu+a))^{\alpha} t^{\frac{N-2(a+\nu)-1}{N-2-2(a+\nu)}} y'(t).$$

Observe that

$$y'(t) \sim \frac{1}{1-k_1} t^{1-k_1} y^{p-1}(t) + \frac{\lambda C}{1-k_2} t^{1-k_2} y^{q-1}(t) \quad \text{as} \quad t \to \infty.$$

Hence,

$$\begin{aligned} v'(\rho) \quad \sim \quad & -\frac{(N-2-2(\nu+a))^{\alpha}}{1-k_1} t^{\frac{N-2(a+\nu)-1}{N-2-2(a+\nu)}+1-k_1} \\ & -\frac{\lambda C(N-2-2(\nu+a))^{\alpha}}{1-k_2} t^{\frac{N-2(a+\nu)-1}{N-2-2(a+\nu)}+1-k_2} \quad \text{as} \quad t \to \infty. \end{aligned}$$

Now set $c^* = 2(N-1-2a-2\nu)(1+a-b)$; then we can verify that

$$\begin{cases} \frac{N-2(a+\nu)-1}{N-2-2(a+\nu)} + 1 - k_2 < \frac{N-2(a+\nu)-1}{N-2-2(a+\nu)} + 1 - k_1 < 0, \text{ if } c^* > N, \\ \frac{N-2(a+\nu)-1}{N-2-2(a+\nu)} + 1 - k_2 < 0 = \frac{N-2(a+\nu)-1}{N-2-2(a+\nu)} + 1 - k_1, \text{ if } c^* = N, \\ \frac{N-2(a+\nu)-1}{N-2-2(a+\nu)} + 1 - k_2 < \frac{N-2(a+\nu)-1}{N-2-2(a+\nu)} + 1 - k_1, \ \frac{N-2(a+\nu)-1}{N-2-2(a+\nu)} + 1 - k_1 > 0, \text{ if } c^* < N. \end{cases}$$

Therefore we see

$$\begin{cases} v'(0^+) = 0, \text{ if } c^* > N, \\ v'(0^+) = -\frac{(N-2-2(\nu+a))^{\alpha}}{1-k_1}, \text{ if } c^* = N, \\ v'(0^+) = -\infty, \text{ if } c^* < N, \end{cases}$$

which means that if $c^* > N$, then $v(x) \in C^1(\overline{B})$. $\square$

3. A theorem of Liouville type

In this section, we intend to give a nonexistence result of strong solutions for the following equation:

$$-div(|x|^{-2a}\nabla u) - \mu\frac{u}{|x|^{2(1+a)}} = \frac{u^{s-1}}{|x|^{bp}}, \quad u \geq 0, \ x \in \mathbb{R}^N, \tag{3.1}$$

where $0 \leq \mu < (\sqrt{\bar{\mu}} - a)^2$, $\bar{\mu} \triangleq (\frac{N-2}{2})^2$, $0 \leq a < \sqrt{\bar{\mu}}$, $a \leq b < a+1$, $1 < s < p$.

Let $u(x) = u(|x|)$ be a strong solution for (3.1). Set $v(x) = |x|^\nu u(x)$; then $v(x) = v(|x|)$ solves

$$\begin{cases} v''(\rho) + \dfrac{N-1-2(a+\nu)}{\rho}v'(\rho) + \dfrac{v^{s-1}(\rho)}{\rho^{bp+s\nu-2(a+\nu)}} = 0, \\ v > 0. \end{cases} \tag{3.2}$$

Define $t = (\frac{N-2-2(\nu+a)}{\rho})^{N-2-2(\nu+a)}$ and $y(t) = (N-2-2(\nu+a))^{-\alpha}v(\rho)$, where $\alpha = \frac{bp+s\nu+2-N}{s-2}$ for $s \neq 2$ and $\alpha = 0$ for $s = 2$. Then, for $s \neq 2$

$$\begin{cases} y''(t) = -t^{-k}y^{s-1}, \\ y(t) > 0 \ \text{ for } \ 0 \leq t < \infty, \end{cases} \tag{3.3}$$

and for $s = 2$

$$\begin{cases} y''(t) = -(N-2-2(\nu+a))^{N-2}t^{-k}y, \\ y(t) > 0 \ \text{ for } \ 0 \leq t < \infty, \end{cases}$$

where $k = \frac{2N-2-2(a+\nu)-(bp+s\nu)}{N-2-2(a+\nu)}$.

Since $u \in C^2(\mathbb{R}^N\setminus\{0\})$, we deduce by using Lemma 2.1 in [14] that for any $B_\rho(0)$, $v(x) \geq \min_{|x|=\rho} v(x)$ for $\forall x \in B_\rho(0) \setminus \{0\}$. Noting that u is a strong solution, we conclude that there exists a number $0 < \gamma < +\infty$ such that

$$\lim_{t\to+\infty} y'(t) = 0 \qquad \lim_{t\to+\infty} y(t) = \gamma \quad \text{and} \quad y'(t) = O(t^{1-k}) \ \text{ as } \ t \to \infty. \tag{3.4}$$

In the sequel we only consider the case $s \neq 2$ since the case $s = 2$ can be dealt with similarly.

Lemma 3.1. *Under our assumptions on a, b, μ and s, we have $k > 2$ and $s < 2k-2$.*

Proof. $k > 2$ is equivalent to

$$2N - 4(1+a)(1+a-b) + 2aN - 2bN > 0. \tag{3.5}$$

Define $f(t) = 2N - 4(1+a)(1+a-t) + 2aN - 2tN$; then $f(a+1) = 0$ and $f'(t) = 4 + 4a - 2N < 0$. Hence $f(t) > 0$ for $a \leq t < a+1$, which is exactly (3.5).

The case $s < 2k-2$ can be verified similarly. □

Consider the equation

$$\begin{cases} y''(t) + t^{-k}y^{s-1} = 0, & t < \infty, \\ \lim\limits_{t\to\infty} y(t) = \gamma, \end{cases} \tag{3.6}$$

where $\gamma > 0$.

Since $k > 2$, it follows from [1] that problem (3.6) has, for every $\gamma > 0$, a unique solution which will be denoted by $y(t,\gamma)$. Define

$$T(\gamma) = \inf\{t > 0 : y(t,\gamma) > 0 \ \text{ on } \ (t,\infty)\}; \tag{3.7}$$

then $T(\gamma) = \gamma^{\frac{s-2}{k-2}}T(1)$.

Now we give an upper bound for $y(t,\gamma)$.

Lemma 3.2.

$$y(t,\gamma) < z(t,\gamma) \ \text{ for } T(\gamma) \le t < \infty, \tag{3.8}$$

where

$$z(t,\gamma) = \gamma t\Big(t^{k-2} + \frac{\gamma^{s-2}}{k-1}\Big)^{-\frac{1}{k-2}},$$

satisfying

$$\begin{cases} z''(t) + t^{-k}\gamma^{s+2-2k}z^{2k-3} = 0, \ \ 0 < t < \infty, \\ \lim\limits_{t\to\infty} z(t,\gamma) = \gamma. \end{cases} \tag{3.9}$$

The proof is similar to [5], and we omit it here.

Now we define the following Pohozaev functional associated with (3.6) which was introduced in [1]:

$$H(t) = ty'^2 - yy' + \frac{t^{1-k}}{k-1}y^s. \tag{3.10}$$

If $y(t)$ solves problem (3.6), from (3.4)

$$\lim_{t\to\infty} H(t) = 0.$$

Moreover

$$H'(t) = \frac{s-(2k-2)}{k-1}t^{1-k}y'y^{s-1} < 0 \ \text{ for } \ t > T(\gamma), \tag{3.11}$$

since $y' > 0$ and Lemma 3.1.

Lemma 3.3. *Let $T(\gamma)$ be defined as (3.7), $s > \frac{N-bp}{N-2-2a-\nu}$, then $T(1) > 0$.*

Proof. By Lemma 3.2, $y(t,1) \le z(t,1)$ for $t \ge T(1)$. Suppose to the contrary that $T(1) = 0$; then

$$y'(0,1) \le z'(0,1) = (k-1)^{\frac{1}{k-1}},$$

since $y(0,1) = z(0,1) = 0$. So

$$y(t,1) \le (k-1)^{\frac{1}{k-1}}t, \quad t \ge 0,$$

which means $H(T(1)) = H(0) = 0$.

On the other hand, $s > \frac{N-bp}{N-2-2a-\nu}$ implies $s > k-1$. So, the combination of (3.11) and the fact that $\lim_{t\to\infty} H(t) = 0$ yields $H(t) > 0$ for $T(1) \le t < \infty$.

Hence, we get a contradiction, and our conclusion follows. □

Proof of Theorem 1.3. Suppose to the contrary that (3.1) possesses a strong solution $u(x) = u(|x|)$ which is nontrivial; then by the strong maximum principle in [14], $u > 0$ in $\mathbb{R}^N$. Hence from the above analysis we see that $y(t) = (N-2-2(\nu+a))^{-\alpha}|x|^{\nu}u(|x|)$ satisfies (3.3) and (3.6).

But, from Lemma 3.3, $T(\gamma) = \gamma^{\frac{s-2}{k-2}}T(1) > 0$, which contradicts (3.3). Therefore we complete the proof. □

Remark 3.1. By Proposition 1.3 in [3], any solution $u \in H$ for (3.1) with $\mathbb{R}^N$ replaced by $B_1(0)$ is radially symmetric. From our proof, we see that (3.1) has a unique nontrivial solution in $H_0^1(B_1(0), |x|^{-2a})$, which is radially symmetric.

To complete this section, we give a corollary which implies that if $a = 0$ and $2 < s < p$, then Theorem 1.3 holds for any strong solution.

Corollary 3.1. *Assume that $a = 0$ and $2 < s < p$, u is a strong solution of* (3.1)*. Then, $u \equiv 0$.*

Proof. It suffices to prove that u is radially symmetric. We use the conformal equivariance of the Laplacian, and we define

$$v(x) = \frac{1}{|x|^{N-2}} u\Big(\frac{x}{|x|^2}\Big).$$

Then $v \in W^{1,2}_{loc}(\mathbb{R}^N \setminus \{0\})$ and, as $\Delta v(x) = |x|^{-N-2}\Delta u(x/|x|^2)$, it solves the equation

$$-\Delta v = \mu \frac{v}{|x|^2} + |x|^{bp+(N-2)(s-1)-(N+2)} v^{s-1}.$$

We will prove that v and hence u is radially symmetric. For this, we employ Theorem 2.1 in [15] and hence we need to verify the assumptions $(f1), (f2)$ and $(u1), (u3)$ in it.

$2 < s < p$ implies that the new nonlinearity $f(x,t) = |x|^{bp+(N-2)(s-1)-(N+2)} t^{s-1}$ satisfies $(f1)$. By convexity of $f(x,t)$,

$$\frac{f(x,h) - f(x,t)}{h-t} \leq (s-1)|x|^{bp+(N-2)(s-1)-(N+2)} t^{s-2}, \quad \forall\ 0 < h < t.$$

Since $u(x) \leq C|x|^{-\nu}$ in $B_1(0)$ and $2 < s < p,\ \nu < (N-2)/2$, we deduce by direct calculations that

$$\|v\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N \setminus B_1(0))} = \|u\|_{L^{\frac{2N}{N-2}}(B_1(0))} < \infty$$

and

$$\big\||x|^{bp+(N-2)(s-1)-(N+2)} v^{s-2}\big\|_{L^{\frac{N}{2}}(\mathbb{R}^N \setminus B_1(0))} = \Big\|\frac{u^{s-2}}{|x|^{bp}}\Big\|_{L^{\frac{N}{2}}(B_1(0))} < \infty.$$

As a result, we conclude that all the assumptions of Theorem 2.1 in [15] are fulfilled and hence v is radially symmetric. $\square$

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