PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 4, April 2008, Pages 1385–1395
S 0002-9939(07)09242-8
Article electronically published on December 18, 2007

RANDOMIZATION OF SHARKOVSKII-TYPE THEOREMS

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(Communicated by Carmen C. Chicone)

Abstract. We formulate an abstract scheme for the randomization of Sharkovskii-type theorems via transformation to the deterministic case. In particular, Sharkovskii-type theorems for scalar differential equations can be randomized in this way. A random version of the standard Sharkovskii theorem is presented explicitly. Many remarks, comments and illustrating examples are supplied.

1. Introduction

The standard Sharkovskii theorem [Sh], which is based on a new ordering of positive integers (Sharkovskii ordering)

$$3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright \cdots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright \cdots \triangleright 2^n \cdot 3 \triangleright 2^n \cdot 5 \triangleright \dots$$
$$\triangleright\, 2^{n+1} \cdot 3 \triangleright 2^{n+1} \cdot 5 \triangleright \cdots \triangleright 2^{n+1} \triangleright 2^n \triangleright 2^{n-1} \triangleright \cdots \triangleright 2^2 \triangleright 2 \triangleright 1,$$

establishes a forcing relationship among the periods that a function can possess. It has been generalized in various directions. In contrast to many deterministic generalizations (see e.g. [ASS, ALM1, ALM2, CL, Kd, Sc, Sz] and the references therein), the random generalizations are so far rare (see [FJJK, Kl]). Hence, our purpose is to at least partly eliminate this handicap.

Following the idea of F. S. DeBlasi, L. Górniewicz and G. Pianigiani to investigate the existence of random fixed points in a deterministic way (see [Go, Chapter III.31]), we develop a simple but powerful method (Proposition 2) for investigating random orbits. This method allows us to present, besides another, a random version of the standard Sharkovskii theorem (Theorem 1).

Instead of further particular variants, we decided to formulate a general abstract scheme for randomization of Sharkovskii-type theorems (Theorem 2). Specifically, full random analogies of Sharkovskii-type theorems for scalar differential equations can be obtained in this way (Corollary 2).

The applicability of our results is documented by several illustrating examples. Since the results in [FJJK, Kl] were obtained in a completely different way, there is no comparable analogy with ours.

Received by the editors February 14, 2007.

2000 *Mathematics Subject Classification.* Primary 37E05, 37E15, 37H10; Secondary 47H04, 47H40.

Key words and phrases. Sharkovskii-type theorems, randomization, transformation to deterministic case, random periodic orbits, transversal maps.

This work was supported by the Council of Czech Government (MSM 6198959214).

2. Preliminaries

In the entire text, all topological spaces are at least metric. For $A \subset X$, the symbol $O_r(A) \subset X$ obviously denotes $O_r(A) := \{x \in X \mid \exists y \in A : d(x, y) < r\}$. Furthermore, all multivalued maps $\varphi : X \multimap Y$ have nonempty values, i.e. $\varphi : X \to 2^Y \setminus \{\emptyset\}$. By a *fixed point* of φ, we mean $x \in X \cap Y \neq \emptyset$ such that $x \in \varphi(x)$. The set of fixed points of φ will be denoted by $\mathrm{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}$.

Rather than by periodic points of φ, we shall deal with periodic orbits of φ. By a *k-orbit* of $\varphi : X \multimap Y$, we shall understand a sequence $\{x_i\}_{i=0}^{k-1}$, where $x_i \in X$, $i = 0, \dots, k-1$, such that

(i) $x_{i+1} \in \varphi(x_i)$, $i = 0, \dots, k-2$, and $x_0 \in \varphi(x_{k-1})$,
(ii) the k-orbit is not a product orbit formed by going p-times around a shorter m-orbit, where $mp = k$.

If still

(iii) $x_i \neq x_j$, $i \neq j$; $i, j = 0, \dots, k-1$, then we speak about a *primary k-orbit.*

By a *measurable space*, we shall mean, as usual, the triple $(\Omega, \mathcal{U}, \mu)$, where a set Ω is equipped with σ-algebra $\mathcal{U}$ of subsets and a countably additive measure μ on $\mathcal{U}$. A typical example is when Ω is a bounded domain in $\mathbb{R}$, equipped with the Lebesgue measure.

Denoting, for $\varphi : X \multimap Y$, by

$$\varphi^{-1}(B) := \{x \in X \mid \varphi(x) \subset B\} \text{ and } \varphi_+^{-1}(B) := \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}$$

the small and large counter-images of $B \subset Y$, we can define (weakly) measurable multivalued maps as follows.

Definition 1. Let $(\Omega, \mathcal{U}, \mu)$ be a measurable space and Y be a separable metric space. A map $\varphi : \Omega \multimap Y$ with closed values is called *measurable* if $\varphi^{-1}(B) \in \mathcal{U}$ for each open $B \subset Y$, or equivalently, if $\varphi_+^{-1}(B) \in \mathcal{U}$ for each closed $B \subset Y$. It is called *weakly measurable* if $\varphi_+^{-1}(B) \in \mathcal{U}$ for each open $B \subset Y$, or equivalently, if $\varphi^{-1}(B) \in \mathcal{U}$ for each closed $B \subset Y$.

It is well known that, for compact-valued maps $\varphi : \Omega \multimap Y$, the notions of measurability and weak measurability coincide. Moreover, if φ and ψ are measurable, then so is their Cartesian product $\varphi \times \psi$. For more properties and details, see Proposition 3.45 in [AG, Chapter I].

As an important tool in our investigations, we shall employ a version of the Aumann selection theorem in [HP, Theorem 2.2.14], which we state here in the form of a proposition.

Proposition 1. *If $\varphi : \Omega \multimap Y$, where Ω is a complete measurable space and Y is a complete separable metric space, is a multivalued map whose graph $\Gamma_\varphi := \{(\omega, y) \in \Omega \times Y \mid y \in \varphi(\omega)\}$ is measurable, i.e. $\Gamma_\varphi \in \mathcal{U} \otimes \mathcal{B}(Y)$, where $\mathcal{U} \otimes \mathcal{B}(Y)$ is a minimal σ-algebra generated by $\mathcal{U} \times \mathcal{B}(Y)$ and $\mathcal{B}(Y)$ stands for the Borel sets of Y, then φ possesses a measurable (single-valued) selection $f \subset \varphi$.*

We shall also consider more regular semicontinuous maps.

Definition 2. A map $\varphi : X \multimap Y$ with closed values is said to be *upper semicontinuous* (u.s.c.) if, for every open $B \subset Y$, the set $\varphi^{-1}(B)$ is open in X, or equivalently, if $\varphi_+^{-1}(B)$ is closed in X. It is said to be *lower semicontinuous* (l.s.c.) if, for every open $B \subset Y$ the set $\varphi_+^{-1}(B)$ is open in X, or equivalently, if $\varphi^{-1}(B)$ is closed in X. If it is both u.s.c. and l.s.c., then it is called *continuous*.

Of course, if φ is u.s.c. or l.s.c., then it is measurable. If a single-valued $f : X \to Y$ is u.s.c. or l.s.c., then it is continuous. If a compact-valued $\varphi : X \multimap Y$ is u.s.c. and $A \subset X$ is a compact subset of X, then $\varphi(A)$ is compact. Moreover, for compact u.s.c. maps φ, $\mathrm{Fix}(\varphi)$ is compact. For more properties and details, see e.g. [AG, Chapter I.3].

The notions of a random operator and a random orbit are essential in this paper. In the sequel, Ω will be a complete measurable space and X a complete separable metric space.

Definition 3. Let $A \subset X$ be a closed subset and $\varphi : \Omega \times A \multimap X$ be a multivalued map with closed values. We say that φ is a *random operator* if it is product-measurable (measurable in the whole), i.e. measurable w.r.t. minimal σ-algebra $\mathcal{U} \otimes \mathcal{B}(X)$, generated by $\mathcal{U} \times \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the Borel sets of X.

Remark 1. For the definition of a random operator, it is usually still required that φ be compact-valued (cf. [Go]), and $\varphi(\omega, \cdot) : A \multimap X$ be u.s.c. (cf. [Go, Chapter III.31]) or h-continuous (cf. [HP, Chapter 5.6]), for almost all $\omega \in \Omega$. Since these restrictions are not necessary for us, we omit them in Definition 3.

Definition 4. Let $A \subset X$ be a closed subset and $\varphi : \Omega \times A \multimap X$ be a random operator. A sequence of measurable maps $\{\xi_i\}_{i=0}^{k-1}$, where $\xi_i : \Omega \to A$, $i = 0, \dots, k-1$, is called a *random k-orbit*, associated to φ, if

(i) $\xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$, $i = 0, \dots, k-2$, and $\xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))$, for almost all $\omega \in \Omega$,
(ii) the random k-orbit is not a random product orbit formed by going p-times around a shorter random m-orbit, where $mp = k$.

If still

(iii) $\xi_i(\omega) \neq \xi_j(\omega)$; $i \neq j$; $i, j = 0, \dots, k-1$, for almost all $\omega \in \Omega$, then we speak about a *random primary k-orbit*.

3. Transformation to the deterministic case

We will show how the study of random periodic orbits can be transformed to the deterministic case. For this transformation, we shall need the following lemma.

Lemma 1. *Let $\varphi : \Omega \times A \multimap A$ be a random operator. Then the function $d_m : \Omega \times A^m \to [0, \infty)$, where $A^m = \underbrace{A \times \cdots \times A}_{m\text{-times}}$, defined by*

$$d_m\left(\omega, \{x_i\}_{i=0}^{m-1}\right) = \operatorname{dist}\big[(x_0, x_1, \dots, x_{m-1}), \{\varphi(\omega, x_{m-1}) \times \varphi(\omega, x_0) \times \cdots \times \varphi(\omega, x_{m-2})\}\big],$$

is product-measurable, for every $m = 1, \dots, k$.

Proof. Since

$$\begin{aligned}&\Big\{\Big(\omega,\{x_i\}_{i=0}^{m-1}\Big)\in\Omega\times A^m \mid d_m\Big(\omega,\{x_i\}_{i=0}^{m-1}\Big)<r\Big\}\\ &\quad=\big\{\big(\omega,(x_0,x_1,\dots,x_{m-1})\big)\\ &\quad\in\Omega\times A^m \mid \big\{\varphi(\omega,x_{m-1})\times\varphi(\omega,x_0)\times\cdots\times\varphi(\omega,x_{m-2})\big\}\\ &\quad\cap O_r\big(\{(x_0,x_1,\dots,x_{m-1})\}\big)\neq\emptyset\big\},\quad m=1,\dots,k-1,\end{aligned}$$

the measurability of these sets turns out to be equivalent with the measurability of

$$\begin{aligned}&\big\{\big(\omega,(x_{m-1},x_0,\dots,x_{m-2})\big)\\ &\quad\in\Omega\times A^m \mid \big\{\varphi(\omega,x_{m-1})\times\varphi(\omega,x_0)\times\cdots\times\varphi(\omega,x_{m-2})\big\}\\ &\quad\cap O_r\big(\{(x_0,x_1,\dots,x_{m-1})\}\big)\neq\emptyset\big\},\quad m=1,\dots,k-1.\end{aligned}$$

This follows, in view of Definition 1, from the measurability of the Cartesian product $\underbrace{\varphi\times\cdots\times\varphi}_{m\text{-times}}$, $m=1,\dots,k-1$, which in turn follows from the product-measurability of φ. □

It will be useful to split Ω in the following way:

$$(*)\qquad\left\{\begin{array}{l}\Omega=\Omega_0\cup\bigcup_{m|k}\Omega_m,\ \text{where } \mu(\Omega_0)=0, \text{ all } \Omega_m\text{'s are measurable,}\\ \text{there are } i_0,i_1,\dots,i_l \text{ such that } \mu(\Omega_{i_j})>0, \text{ for all } i_j\text{'s, and}\\ \text{the least common multiple of } i_j\text{'s is } k.\end{array}\right.$$

Defining the multivalued maps $\mathcal{O}_k:\Omega\multimap A^k$, $k\in\mathbb{N}$, and $\mathcal{O}_m^{k/m}\restriction_{\Omega_m}:\Omega_m\multimap A^k$ by

$$\mathcal{O}_k(\omega):=\Big\{\{x_i\}_{i=0}^{k-1}\in A^k \mid \{x_i\}_{i=0}^{k-1}\ \text{is a } k\text{-orbit of } \varphi(\omega,\,\cdot\,)\Big\}$$

and

$$\begin{aligned}\mathcal{O}_m^{k/m}\restriction_{\Omega_m}(\omega):=\Big\{&\{x_i\}_{i=0}^{k-1}\in A^k \mid \{x_i\}_{i=0}^{m-1}\ \text{is an } m\text{-orbit of } \varphi(\omega,\,\cdot\,)\\ &\text{and } x_{i+tm}=x_i,\ \text{for } t=1,\dots,\frac{k}{m}\Big\},\end{aligned}$$

i.e. $\mathcal{O}_m^{k/m}(\omega)\restriction_{\Omega_m}$, where $\omega\in\Omega_m$ is an m-orbit repeated (k/m)-times, we are ready to give the following crucial statement.

Proposition 2. *Assume that $\varphi:\Omega\times A\multimap X$ is a random operator. Then φ admits a random k-orbit, $k\in\mathbb{N}$, if and only if there is a splitting $(*)$ such that $\mathcal{O}_m(\omega)$ is nonempty for all $\omega\in\Omega_m$, where $m|k$.*

Proof. "If" part. Put

$$\Gamma=\bigcup_{m|k}\Big(\Gamma_{\mathcal{O}_m^{k/m}\restriction_{\Omega_m}}\Big)$$

and observe that since Γ can be extended arbitrarily on Ω_0, we can assume, without any loss of generality, that the domain of Γ is equal to Ω.

At first, let us deduce that the set Γ is measurable. According to Lemma 1, $d_m\restriction_{(\Omega_m\times A^m)}$ is measurable, and so the related $d_m^{-1}(\{0\})\restriction$ are, by Definition 1, measurable, for every $m = 1, \dots, k$. Therefore, since

$$\Gamma_{\mathcal{O}_m^{k/m}\restriction_{\Omega_m}} = \left[d_m^{-1}(\{0\})\restriction\right]^{k/m} \setminus \bigcup_{\substack{p<m\\(m=pq)}} \left[\left[d_p^{-1}(\{0\})\restriction\right]^q\right]^{k/m},$$

where the symbol $[\,\cdot\,]^r$ denotes r-times repetition of each element of $[\,\cdot\,]$, the graph $\Gamma_{\mathcal{O}_m^{k/m}\restriction_{\Omega_m}}$, for every $m = 1, \dots, k$, and subsequently Γ, are also measurable, as claimed.

Since A is, by the hypothesis, a (closed subset of a) complete separable metric space, so is A^k, and the application of Proposition 1 implies the existence of a measurable selection $\{\xi_i\}_{i=0}^{k-1} : \Omega \to A^k$ of $\bigcup_{m|k} \mathcal{O}_m^{k/m} \restriction_{\Omega_m}$. This selection induces a random i_j-orbit of $\varphi \restriction_{(\Omega_{i_j} \times A)}$, for each i_j. Since the least common multiple of i_j's is equal to k, it follows that $\{\xi_i\}_{i=0}^{k-1}$ is a desired random k-orbit of φ.

"Only if" part. Suppose that $\{\xi_i\}_{i=0}^{k-1}$ is a random k-orbit. Then, for almost all $\omega \in \Omega$, $\{\xi_i(\omega)\}_{i=0}^{k-1}$ is an m-orbit repeated (k/m)-times, for some $m|k$. Denote

$$\Omega_m := \left\{\omega \in \Omega \mid \{\xi_i(\omega)\}_{i=0}^{k-1} \text{ is an } m\text{-orbit repeated } (k/m)\text{-times}\right\}.$$

Since Ω_m can be expressed, by means of composed distance functions $d : \Omega \to [0, \infty)$, as

$$\begin{aligned}\Omega_m = &\left\{\omega \in \Omega \mid d\left(\xi_{j+mi}(\omega), \xi_j(\omega)\right) = 0, \text{ for all } i = 1, \dots, \tfrac{k}{m} \text{ and } j = 0, \dots, m-1\right\}\\ &\setminus \bigcup_{\substack{p<m\\(m=pq)}} \left\{\omega \in \Omega \mid d\left(\xi_{j+pki}(\omega), \xi_j(\omega)\right) = 0, \text{ for all } i = 1, \dots, \tfrac{k}{m};\ j = 0, \dots, p-1 \right.\\ &\qquad\qquad \left. \text{and } k = 1, \dots, \tfrac{m}{p}\right\},\end{aligned}$$

the measurability of ξ_u's implies the measurability of composed d's, and subsequently of all Ω_m's.

Hence, still denote

$$\{i_0, \dots, i_l\} := \{i \mid \mu(\Omega_i) > 0\},$$

and put $\Omega_0 = \Omega \setminus \bigcup_{i=0}^{l} \Omega_i$. Since $\{\xi_i\}_{i=0}^{k-1}$ is a random k-orbit, $\mu(\Omega_0) = 0$.

It remains to show that the least common multiple of $i_0, \dots, i_l$ is equal to k. If the least common multiple of $i_0, \dots, i_l$ is equal to a $k' < k$, then $\{\xi_i\}_{i=0}^{k-1}$ would be a random k'-orbit repeated (m/k') times, a contradiction. □

Corollary 1. *If the set $\mathcal{O}_k(\omega)$ of k-orbits of $\varphi(\omega, \cdot\,)$ is nonempty, for almost every $\omega \in \Omega$, then φ admits a random k-orbit.*

Proof. For this particular case, the measurability of the graph $\Gamma_{\mathcal{O}_k}$, where

$$\Gamma_{\mathcal{O}_k} = d_k^{-1}(\{0\}) \setminus \bigcup_{\substack{m<k\\(k=mp)}} \left[d_m^{-1}(\{0\})\right]^p,$$

follows, according to Lemma 1, from the measurability of d_m, $m = 1, \dots, k$. Thus, Proposition 1 implies immediately the existence of a desired measurable selection $\{\xi_i\}_{i=0}^{k-1} : \Omega \to A^k$ of $\mathcal{O}_k$ such that conditions (i), (ii) in Definition 4 are satisfied, i.e. a random k-orbit of φ. □

Remark 2. If $\varphi(\omega, \cdot\,) : A \multimap X$, $\omega \in \Omega$, is additionally u.s.c. and compact, the sets $\mathrm{Fix}(\varphi^n(\omega, \cdot\,))$ of fixed points of $\varphi^n(\omega, \cdot\,) : A \multimap X$ are compact, for all $n \in \mathbb{N}$.

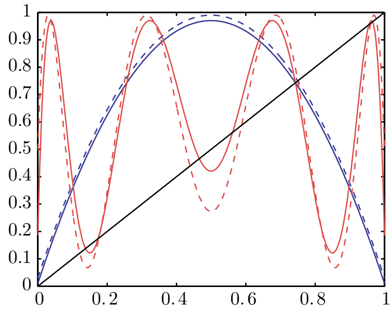

FIGURE 1. Graphs of f_1, f_2 and f_1^3, f_2^3

Therefore, for $k = 1$, Corollary 1 generalizes Proposition 31.3 in [Go, Chapter III.31].

Example 1. Consider the randomly perturbed logistic function $f : \Omega \times [0,1] \to [0,1]$,

$$f(\omega, x) = 3.84x(1-x) + p(\omega),$$

with the additive (measurable) perturbation p satisfying the inequalities $0.01 \leq p(\omega) \leq 0.03$, for almost all $\omega \in \Omega$. In Figure 1, the "boundary" functions

$$\begin{aligned} f_1(x) &= 3.84x(1-x) + 0.01 \quad \text{(continuous)}, \\ f_2(x) &= 3.84x(1-x) + 0.03 \quad \text{(dashed)} \end{aligned}$$

and their third iterates f_1^3, f_2^3 indicate the behaviour of f^n, $n \in \mathbb{N}$, despite the fact that the values of f^n need not be located, for $n > 1$, between those of f_1^n and f_2^n. The application of Corollary 1 so implies the existence of random n-orbits of f, provided $\mathcal{O}_n(\omega) \neq \emptyset$, $\omega \in \Omega$, for every $n \in \mathbb{N}$. For $n = 3$, it is visible in Figure 1.

4. APPLICATION TO SHARKOVSKII-TYPE THEOREMS

Since function $f(\omega, \cdot)$ in Example 1 admits a 3-orbit, it possesses, according to the standard Sharkovskii theorem [Sh], n-orbits, for every $n \in \mathbb{N}$, i.e. $\mathcal{O}_n(\omega) \neq \emptyset$, $\omega \in \Omega$, for every $n \in \mathbb{N}$. Moreover, $\operatorname{card} \mathcal{O}_n(\omega) \leq 2^n$, $\omega \in \Omega$, for every $n \in \mathbb{N}$. Therefore, as pointed out in Example 1, f has by Corollary 1 random n-orbits, for every $n \in \mathbb{N}$.

Another nontrivial example, this time with a multiplicative random perturbation, is as follows.

Example 2. Consider the randomly perturbed tent function $g : \Omega \times [0,1] \to [0,1]$,

$$g(\omega, x) = g_0(x) + \frac{1}{10}|\sin(2\pi x)|p(\omega),$$

where

$$g_0(x) = \begin{cases} 2x, & \text{for } x \in [0, \frac{1}{2}], \\ 2-2x, & \text{for } x \in [\frac{1}{2}, 1], \end{cases}$$

with the multiplicative (measurable) perturbation p satisfying the inequality $|p(\omega)| \leq 1$, for almost all $\omega \in \Omega$. In Figure 2, the "boundary" function

$$g_1(x) = g_0(x) + \frac{1}{10}|\sin(2\pi x)|,$$

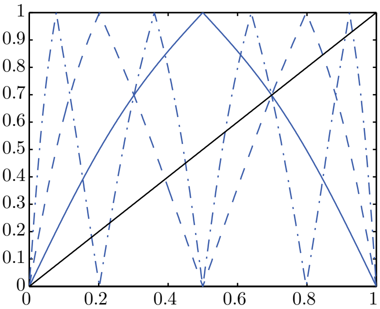

FIGURE 2. Graphs of g_1, g_1^2, g_1^3

jointly with its second and third iterates g_1^2, g_1^3, indicate the behaviour of f^n, $n \in \mathbb{N}$, because p is under the control of the factor $\frac{1}{10}|\sin(2\pi x)|$. The sets $\mathrm{Fix}(g^n(\omega, \cdot))$ are evidently finite, for all $\omega \in \Omega$, because $\left|\frac{\partial g^n(\omega,x)}{\partial x}\,|_{\mathrm{Fix}(g^n(\omega,\,\cdot\,))}\right| > 2^{n-1}$, $\omega \in \Omega$. Corollary 1 again so implies the existence of random n-orbits of g, provided $\mathcal{O}_n(\omega) \neq \emptyset$, $\omega \in \Omega$, for every $n \in \mathbb{N}$.

Since, for $n = 3$, it is visible in Figure 2, the standard Sharkovskii theorem [Sh] yields that $\mathcal{O}_n(\omega) \neq \emptyset$, $\omega \in \Omega$, for every $n \in \mathbb{N}$, and subsequently g has random n-orbits, for every $n \in \mathbb{N}$.

These illustrating examples justify stating the following random version of the standard Sharkovskii theorem.

Theorem 1 (Random Sharkovskii theorem)**.** *Assume that $f(\,\cdot\,, x) : \Omega \to \mathbb{R}$ is measurable, for every $x \in \mathbb{R}$, on a complete measurable space Ω, and $f(\omega, \cdot\,) : \mathbb{R} \to \mathbb{R}$ is continuous, for almost every $\omega \in \Omega$. If, for a given $n \in \mathbb{N}$, f possesses a random n-orbit, then it also admits, for every $k \lhd n$ (in the Sharkovskii ordering of positive integers), a random k-orbit. In particular if, for a given $n \in \mathbb{N}$, the set $\mathcal{O}_n(\omega)$ of n-orbits of $f(\omega, \cdot\,)$ is nonempty, for almost all $\omega \in \Omega$, then f admits a random n-orbit as well as random k-orbits, for every $k \lhd n$.*

Proof. If f has a random n-orbit, then, by Proposition 2, there are measurable sets $\Omega_{i_0}, \Omega_{i_1}, \ldots, \Omega_{i_l}$ such that $\mathcal{O}_{i_j}(\omega)$ is nonempty, for each $\omega \in \Omega_{i_j}$, and the least common multiple of i_j's is n. There are two possible cases:

(i) $n = 2^t$,

(ii) $n = p \cdot 2^t$, for an odd $p > 1$.

ad (i) In the first case, $i_j = 2^t$, for some $j \le l$, and so $\mathcal{O}_k(\omega)$ is nonempty, for every $\omega \in \Omega_{i_j}$. In view of Proposition 2, there is a random k-orbit of $f\restriction_{(\Omega_{i_j} \times A)}$. Let us denote it by $\{\xi_i'\}_{i=0}^{k-1}$ and observe that the domain of each ξ_i' is equal to Ω_{i_j}. For every $m|n$, $m \neq i_j$, $\mathcal{O}_1(\omega)$ is, according to the standard Sharkovskii theorem [Sh], nonempty, for each $\omega \in \Omega_m$. Thus, according to Proposition 2, there is a random fixed point ξ'' of $f\restriction_{((\Omega \setminus \Omega_0 \setminus \Omega_{i_j}) \times A)}$. Observe that the domain of ξ'' is equal to $\Omega \setminus \Omega_0 \setminus \Omega_{i_j}$. Therefore $\{\xi' \cup \xi''\}_{i=0}^{k-1}$, extended arbitrarily on Ω, is a desired random k-orbit of f.

ad (ii) In the second case, there exists $j \le l$ such that $i_j = p' \cdot 2^{t'}$, for an odd $p' \le p$ ($p' > 1$) and $t' \le t$. Since $k \lhd n \lhd i_j$, $f\restriction_{(\Omega_{i_j} \times A)}$ possesses, by the analogous arguments to those used in the first case, a random k-orbit. Thus analogously as in the previous case, f admits a desired random k-orbit.

For the particular case, it is enough to combine the standard Sharkovskii theorem [Sh] and Corollary 1. □

Remark 3. In Theorem 1, $\mathbb{R}$ can be replaced by a complete, separable, metric linear continuum $\mathbb{L}$ (equivalently, by a complete, separable, connected linearly ordered metric space) [Sc] and f by e.g. the derivative $\frac{\partial g}{\partial x} : \Omega \times \mathbb{R} \to \mathbb{R}$ of any function g [Sz] or by a random (multivalued) mapping $\varphi : \Omega \times \mathbb{L} \multimap \mathbb{L}$ such that $\varphi(\omega, \cdot\,) : \mathbb{L} \multimap \mathbb{L}$ is l.s.c. with connected values, for almost every $\omega \in \Omega$, [AFP2] or by a triangular mapping $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ [Kd], etc.

Remark 4. We can formulate more general extensions of Theorem 1, where the assertions concerning the existence of random k-orbits hold with at most two exceptions, for $k \triangleleft n$. For instance, a random operator $\varphi : \Omega \times \mathbb{L} \multimap \mathbb{L}$, where $\mathbb{L}$ is again a complete, separable, metric linear continuum, can be such that $\varphi(\omega, \cdot\,) : \mathbb{L} \multimap \mathbb{L}$ is u.s.c. with connected values, for almost every $\omega \in \Omega$, [APS] (for $\mathbb{L} = \mathbb{R}$, cf. [AP]).

Another posibility is that a random operator $\varphi : \Omega \times \mathbb{R} \multimap \mathbb{R}$ can be such that the graph $\Gamma_{\varphi(\omega,\,\cdot\,)}$ of $\varphi(\omega, \cdot\,) : \mathbb{R} \multimap \mathbb{R}$ is a finite composition of maps with connected intersection of open subsets of $\mathbb{R}^2$, for almost every $\omega \in \Omega$. The values of all iterates of $\varphi(\omega, \cdot\,)$ are still assumed to be closed [ASS].

Since one can even replace the space $\mathbb{R}$, resp. $\mathbb{L}$, with the above properties by other suitable (separable, complete, metric) spaces, jointly with the Sharkovskii ordering by the combination of related orderings (see e.g. [ALM1, CL], and the references therein), rather than formulating further particular variants, we shall present an abstract scheme for the randomization of Sharkovskii-type theorems.

Hence, denoting

DA := {assumptions of the given Sharkovskii-type theorem,
for $\varphi(\omega, \cdot\,) : A \multimap A$, $\omega \in \Omega$} (deterministic assumptions),
DC := {conclusions of the given Sharkovskii-type theorem,
for $\varphi(\omega, \cdot\,) : A \multimap A$, $\omega \in \Omega$} (deterministic conclusions),
RA := {conclusions of Proposition 2, for φ, w.r.t. DA} (random assumptions),
RC := {conclusions of Proposition 2, for φ, w.r.t. DC} (random conclusions),

we are ready to give the following main result.

Theorem 2 (Randomization scheme). *Assume that $\varphi : \Omega \times A \multimap A$ is a random operator (in the sense of Definition* 3*). Then the following commutative diagram takes place:*

$$\begin{tikzcd}
DA \arrow[r, shift left] \arrow[d] & RA \arrow[l, shift left] \arrow[d] \\
DC \arrow[r, shift left] & RC \arrow[l, shift left]
\end{tikzcd}$$

Proof. The commutativity of the diagram is a direct consequence of the implications:
DA→DC: (by a given Sharkovskii-type theorem), DA→RA: (by Proposition 2), RA→DA: (by Proposition 2), DC→RC: (by Proposition 2), RC→DC: (by Proposition 2), RA→RC: (by equivalence of DA and RA, DC and RC, and by implication DA→DC). □

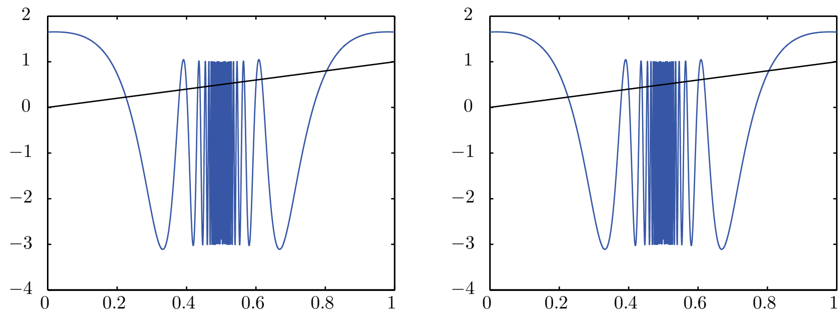

FIGURE 3. Graphs of f_2' and $f_2'^{\,3}$

Example 3. Unlike the function

$$f_2(x) = \begin{cases} 1 - x + 2(x-\frac{1}{2})^2 \sin\left(\frac{1}{x-\frac{1}{2}}\right), & \text{for } x \in \mathbb{R} \setminus \left\{\frac{1}{2}\right\}, \\ \frac{1}{2}, & \text{for } x = \frac{1}{2}, \end{cases}$$

which seems to have only 2-orbits and fixed points, its discontinuous (at $x = \frac{1}{2}$) derivative

$$f_2'(x) = \begin{cases} -1 + 4(x-\frac{1}{2}) \sin\left(\frac{1}{x-\frac{1}{2}}\right) - 2\cos\left(\frac{1}{x-\frac{1}{2}}\right), & \text{for } x \in \mathbb{R} \setminus \left\{\frac{1}{2}\right\}, \\ -1, & \text{for } x = \frac{1}{2}, \end{cases}$$

plotted together with its third iterate $f_2'^{\,3}$ in Figure 3, evidently admits (by comparison of the graphs of f_2' and $f_2'^{\,3}$) 3-periodic orbits. Thus, according to the Sharkovskii-type theorem for derivatives [Sz], f_2' possesses k-orbits, for every $k \in \mathbb{N}$.

Hence, additively perturbing f_2' by $p : \Omega \to \mathbb{R}$, where p is a measurable function such that $|p(\omega)| \leq 0.1$, for almost all $\omega \in \Omega$, we obtain the random map $f : \Omega \times \mathbb{R} \to \mathbb{R}$, defined as $f(\omega, x) := f_2'(x) + p(\omega)$, for all $(\omega, x) \in \Omega \times \mathbb{R}$.

Since the behaviour of the discontinuous (at $x = \frac{1}{2}$) function $f(\omega, \cdot\,) = f_2'(\,\cdot\,) + p(\omega)$, $\omega \in \Omega$, and its third iterate $f^3(\omega, \cdot\,) : \mathbb{R} \to \mathbb{R}$, $\omega \in \Omega$, does not qualitatively differ much from the unperturbed functions f_2' and $f_2'^3$, Theorem 2 (cf. Remark 3) implies the existence of random n-orbits of f, for every $n \in \mathbb{N}$.

Because of the correspondence of periodic solutions of random differential equations and periodic orbits of the associated random Poincaré translation operators, we can also randomize the Sharkovskii-type theorems for differential equations.

Corollary 2. *Assume that $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a random operator (in the sense of Definition* 3*) such that $\varphi(\omega, \cdot\,) : \mathbb{R} \to \mathbb{R}$, $\omega \in \Omega$, is a multivalued map with connected values whose margins are either both nondecreasing or both nonincreasing. Let φ have a random n-orbit, where $n > 1$ is an arbitrary positive integer. Then φ admits a random k-orbit, for every $k \in \mathbb{N}$.*

Consider still the random scalar Carathéodory differential equation or, more generally, upper-Carathéodory differential inclusion (for the definition, see [AG, Chapter III.4.E]*)*

$$x'(\omega, t) \in F(\omega, t, x(\omega, t)), \quad \omega \in \Omega,\ t \in \mathbb{R},\ x \in \mathbb{R},$$

where $F(\omega, t, y) \equiv F(\omega, t+1, y)$. If it possesses a random n-periodic solution, for some $n > 1$, then it also admits a random k-periodic solution, for every $k \in \mathbb{N}$.

Proof. For multivalued maps $\varphi(\omega,\cdot):\mathbb{R}\multimap\mathbb{R}$, $\omega\in\Omega$, with nonempty connected values whose margins $\sup\{y \mid y\in\varphi(\omega,\cdot)\}$, $\inf\{y \mid y\in\varphi(\omega,\cdot)\}$ are either both nondecreasing or both nonincreasing, the existence of an n-orbit, with $n>1$, of $\varphi(\omega,\cdot)$ implies the existence of a (primary) k-orbit of $\varphi(\omega,\cdot)$, for every $k\in\mathbb{N}$ (see [AFP1, AFP2]). For the first part, it is therefore enough to combine, in an analogous way as in the proof of Theorem 1, this theorem in [AFP1, AFP2] with Proposition 2 via Theorem 2.

Since the Poincaré translation operators associated with the scalar ordinary differential equations or inclusions always enjoy the properties of $\varphi(\omega,\cdot)$, $\omega\in\Omega$, from the first part (see [AFP1, AFP2]), the same conclusion is true for the random Poincaré translation operators associated with the given random inclusions. Since, moreover, every random k-orbit can be shown to determine the existence of a random k-periodic solution, the second part is a direct consequence of the first one. □

Acknowledgments

The author is indebted to Professor Russell Johnson (Florence, Italy) for the stimulating discussion on this subject and to Dr. Tomáš Fürst (Olomouc, Czech Republic) and Dr. Piotr Szuca (Gdańsk, Poland) for their critical reading of the manuscript and valuable suggestions.

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