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THE 3-PRIMARY CLASSIFYING SPACE OF THE FIBER OF THE DOUBLE SUSPENSION

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Abstract. Gray showed that the homotopy fiber W_n of the double suspension $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$ has an integral classifying space BW_n, which fits in a homotopy fibration $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$. In addition, after localizing at an odd prime p, BW_n is an H-space and if $p \geq 5$, then BW_n is homotopy associative and homotopy commutative, and ν is an H-map. We positively resolve a conjecture of Gray's that the same multiplicative properties hold for $p = 3$ as well. We go on to give some exponent consequences.

1. Introduction

Let $E^2 : S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ be the double suspension. Let W_n be its homotopy fiber. Gray [G1] showed that W_n has an integral classifying space BW_n and there exists an integral homotopy fibration

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n.$$

Further, after localizing at an odd prime p, BW_n is an H-space and if $p \geq 5$, then ν is an H-map, BW_n is homotopy associative, and [G2] showed that BW_n is also homotopy commutative. Gray conjectured that these stronger multiplicative properties also hold at the prime 3. The purpose of this paper is to positively resolve this conjecture and to discuss some consequences related to exponents.

Theorem 1.1. *Let $p \geq 3$. Then there exists a unique multiplication on BW_n such that $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ is an H-map. Further, this multiplication is both homotopy associative and homotopy commutative.*

Note that the uniqueness statement in Theorem 1.1 is new, regardless of the prime. The idea behind proving Theorem 1.1 is to alter Gray's H-structure on BW_n to obtain a new H-structure for which ν is an H-map. The uniqueness statement assures that the new H-structure is homotopic to Gray's original H-structure when $p \geq 5$. The homotopy associativity and homotopy commutativity statements are then shown to be consequences of ν being an H-map, which is a different approach than that of [G1, G2].

The fact that ν is an H-map can be used to show an important and useful exponent property of BW_n. In [CMN1] it was shown that $p \cdot \pi_*(W_n) = 0$ by

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showing that the p^{th}-power map on W_n is null homotopic. Theorem 1.2 shows that this statement can be delooped. Note that this statement is new for any $p \geq 3$, although its proof for $p \geq 5$ would certainly have been known to Gray.

Theorem 1.2. *Let $p \geq 3$. Then the p^{th}-power map on BW_n is null homotopic.*

There are additional consequences of Theorem 1.1 for the prime 3 which were already known for $p \geq 5$. Gray [G1] showed that at all primes $p \geq 2$ there is a homotopy fibration

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1}$$

with the property that the composite $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n \xrightarrow{j} \Omega^2 S^{2np+1}$ is homotopic to ΩH, where H is the p^{th} James-Hopf invariant. Further, the fact that ΩH has order p was used to show that j also has order p. Thus j lifts to a map $j' : BW_n \longrightarrow \Omega^2 S^{2np+1}\{p\}$, where $\Omega^2 S^{2n+1}\{p\}$ is the homotopy fiber of the p^{th}-power map on $\Omega^2 S^{2n+1}$. In [G1] it was shown that if ν is an H-map, then so is j, and this was refined in [T] to show that if ν is an H-map and $p \geq 3$, then the lift j' can be chosen so that it too is an H-map. Previously, it was only known that ν, and therefore j and j', were H-maps for $p \geq 5$. Theorem 1.1 now implies that these are all H-maps for $p = 3$ as well.

The multiplicativity of j' was used in [T] to prove a factorization of the p^{th}-power map on $\Omega^2 S^{2np+1}$ through the double suspension, delooping a result of Harper [H]. This proof is now also valid for $p = 3$. Stated in full, it is as follows.

Theorem 1.3. *Let $p \geq 3$. Then there is a self-equivalence $\epsilon : \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2n+1}$ such that $\phi' = \phi \circ \epsilon$ has the following properties:*

(a) *there is a homotopy fibration*

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi'} S^{2np-1};$$

(b) *there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^2 S^{2np+1} & \xrightarrow{\phi'} & S^{2np-1} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle E^2} \\ \Omega^2 S^{2np+1} & = & \Omega^2 S^{2np+1}. \end{array}$$

□

2. Some properties of BW_n

This section first records some properties of BW_n proved by Gray and then builds on them in Lemmas 2.2, 2.3, and 2.5. Beginning integrally, define the space Y and the map ∂ by the homotopy fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} Y \longrightarrow S^{2n} \xrightarrow{E} \Omega S^{2n+1}$$

where E is the single suspension. In [G1] it was shown that there is a map $Y \xrightarrow{q} S^{4n-1}$ with a right homotopy inverse such that the composite $\Omega^2 S^{2n+1} \xrightarrow{\partial} Y \xrightarrow{q} S^{4n-1}$ is null homotopic. The space BW_n and the map i are defined by the homotopy fibration

$$BW_n \xrightarrow{i} Y \xrightarrow{q} S^{4n-1}.$$

The null homotopy for $q \circ \partial$ implies that there is a factorization

(1)
$$\begin{tikzcd} \Omega^2 S^{2n+1} \arrow[r, "\nu"] \arrow[d, equal] & BW_n \arrow[d, "i"] \\ \Omega^2 S^{2n+1} \arrow[r, "\partial"] & Y \end{tikzcd}$$

for some map ν. The homotopy fiber of ν is S^{2n-1} and it maps into $\Omega^2 S^{2n+1}$ by E^2. The following theorem proved in [G1] will be pivotal.

Theorem 2.1. *The map $\Sigma^2\Omega^2 S^{2n+1} \xrightarrow{\Sigma^2\nu} \Sigma^2 BW_n$ has a right homotopy inverse, and there is a homotopy equivalence $e : \Sigma^2\Omega^2 S^{2n+1} \longrightarrow \Sigma^2(S^{2n-1} \times BW_n)$ such that $(\Sigma^2\nu) \circ e$ is homotopic to $\Sigma^2\pi_2$, where π_2 is the projection onto the second factor.* □

Next, using the loop structure μ on $\Omega^2 S^{2n+1}$, we can multiply to define the map

$$E^2 \cdot 1 : S^{2n-1} \times \Omega^2 S^{2n+1} \xrightarrow{E^2 \times 1} \Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} \xrightarrow{\mu} \Omega^2 S^{2n+1}.$$

Applying [G1, Proposition 1] to the homotopy fibration sequence $\Omega^2 S^{2n+1} \xrightarrow{\partial} Y \longrightarrow S^{2n} \xrightarrow{E} \Omega S^{2n+1}$ shows that there is a homotopy pushout

(2)
$$\begin{tikzcd} S^{2n-1} \times \Omega^2 S^{2n+1} \arrow[r, "E^2 \cdot 1"] \arrow[d, "\pi_2"] & \Omega^2 S^{2n+1} \arrow[d, "\partial"] \\ \Omega^2 S^{2n+1} \arrow[r, "\partial"] & Y \end{tikzcd}$$

where π_2 is the projection onto the second factor.

Now localize at an odd prime. In [G1] it was shown that there exists a homotopy decomposition $Y \simeq S^{4n-1} \times BW_n$. Thus ν is homotopic to the composite $\Omega^2 S^{2n+1} \xrightarrow{\partial} Y \xrightarrow{\pi} BW_n$, where π is the projection. Composing (2) with this projection immediately gives the following.

Lemma 2.2. *Let $p \geq 3$. Then there is a homotopy commutative square*

$$\begin{tikzcd} S^{2n-1} \times \Omega^2 S^{2n+1} \arrow[r, "E^2 \cdot 1"] \arrow[d, "\pi_2"] & \Omega^2 S^{2n+1} \arrow[d, "\nu"] \\ \Omega^2 S^{2n+1} \arrow[r, "\nu"] & BW_n. \end{tikzcd}$$

□

We now turn to new material. The connecting map ∂ in the homotopy fibration sequence defining Y determines a canonical homotopy action $\theta : \Omega^2 S^{2n+1} \times Y \longrightarrow Y$ with two properties. First, the restrictions of θ to $\Omega^2 S^{2n+1}$ and Y respectively are ∂ and the identity map, and second, there is a homotopy commutative square

$$\begin{tikzcd} \Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} \arrow[r, "\mu"] \arrow[d, "1 \times \partial"] & \Omega^2 S^{2n+1} \arrow[d, "\partial"] \\ \Omega^2 S^{2n+1} \times Y \arrow[r, "\theta"] & Y. \end{tikzcd}$$

We use this to define an action of $\Omega^2 S^{2n+1}$ on BW_n. The decomposition of Y gives a composite

$$\overline{\theta} : \Omega^2 S^{2n+1} \times BW_n \xrightarrow{1\times i} \Omega^2 S^{2n+1} \times Y \xrightarrow{\theta} Y \xrightarrow{\pi} BW_n.$$

Lemma 2.3. *Let $p \geq 3$. Then there is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} \arrow[r, "\mu"] \arrow[d, "1\times\nu"] & \Omega^2 S^{2n+1} \arrow[d, "\nu"] \\
\Omega^2 S^{2n+1} \times BW_n \arrow[r, "\overline{\theta}"] & BW_n.
\end{tikzcd}$$

Further, the restrictions of $\overline{\theta}$ to $\Omega^2 S^{2n+1}$ and BW_n respectively are ν and the identity map.

Proof. The action of $\Omega^2 S^{2n+1}$ on Y gives a homotopy $\theta \circ (1 \times \partial) \simeq \partial \circ \mu$. Compose this with $Y \xrightarrow{\pi} BW_n$. On the one hand, as $\nu \simeq \pi \circ \partial$, we have $\pi \circ \partial \circ \mu \simeq \nu \circ \mu$. On the other hand, as $\partial \simeq i \circ \nu$ by (1), the definition of $\overline{\theta}$ shows that $\pi \circ \theta \circ (1 \times \partial) \simeq \pi \circ \theta \circ (1 \times i) \circ (1 \times \nu) \simeq \overline{\theta} \circ (1 \times \nu)$. Thus $\nu \circ \mu \simeq \overline{\theta} \circ (1 \times \nu)$, proving the homotopy commutativity of the asserted diagram.

As the restrictions of θ to $\Omega^2 S^{2n+1}$ and Y respectively are ∂ and the identity map, the definition of $\overline{\theta}$ implies that its restrictions to $\Omega^2 S^{2n+1}$ and BW_n respectively are ν and the identity map. $\square$

Next, Theorem 2.1 implies that there is a homotopy cofibration

$$\Sigma^2(S^{2n-1} \vee (S^{2n-1} \wedge BW_n)) \xrightarrow{\alpha} \Sigma^2 \Omega^2 S^{2n+1} \xrightarrow{\Sigma^2 \nu} \Sigma^2 BW_n$$

for some map α. In Lemma 2.5 we desuspend in the sense that we construct an explicit map $\Sigma(S^{2n-1} \vee (S^{2n-1} \wedge BW_n)) \longrightarrow \Sigma\Omega^2 S^{2n+1}$ whose homotopy cofiber is given by $\Sigma\Omega^2 S^{2n+1} \xrightarrow{\Sigma\nu} \Sigma BW_n$.

In general, there is a natural homotopy decomposition $\Sigma A \vee \Sigma B \vee (\Sigma A \wedge B) \xrightarrow{e} \Sigma(A \times B)$ where

$$t : \Sigma A \wedge B \hookrightarrow \Sigma A \vee \Sigma B \vee (\Sigma A \wedge B) \xrightarrow{e} \Sigma(A \times B)$$

composes trivially with each of the suspended projections $\Sigma\pi_1$ and $\Sigma\pi_2$. In our case, t can be used to define a Hopf construction on $\Sigma\Omega^2 S^{2n+1}$ by the composite

$$\mu^* : \Sigma\Omega^2 S^{2n+1} \wedge \Omega^2 S^{2n+1} \xrightarrow{t} \Sigma(\Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1}) \xrightarrow{\Sigma\mu} \Sigma\Omega^2 S^{2n+1}.$$

Lemma 2.2 then has the following useful refinement.

Lemma 2.4. *Let $p \geq 3$. Then the composite*

$$\Sigma S^{2n-1} \wedge \Omega^2 S^{2n+1} \xrightarrow{\Sigma E^2 \wedge 1} \Sigma\Omega^2 S^{2n+1} \wedge \Omega^2 S^{2n+1} \xrightarrow{\mu^*} \Sigma\Omega^2 S^{2n+1} \xrightarrow{\Sigma\nu} \Sigma BW_n$$

is null homotopic.

Proof. The naturality of t and the definition of $E^2 \cdot 1$ give a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma S^{2n-1} \wedge \Omega^2 S^{2n+1} \arrow[r, "t"] \arrow[d, "\Sigma E^2 \wedge 1"] & \Sigma(S^{2n-1} \times \Omega^2 S^{n+1}) \arrow[r, "\Sigma(E^2 \cdot 1)"] \arrow[d, "\Sigma(E^2 \times 1)"] & \Sigma\Omega^2 S^{2n+1} \arrow[d, equal] \\
\Sigma\Omega^2 S^{2n+1} \wedge \Omega^2 S^{2n+1} \arrow[r, "t"] & \Sigma(\Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1}) \arrow[r, "\Sigma\mu"] & \Sigma\Omega^2 S^{2n+1}.
\end{tikzcd}$$

Note that the bottom row is the definition of μ^*. Now compose the diagram with the map $\Sigma\Omega^2 S^{2n+1} \xrightarrow{\Sigma\nu} \Sigma BW_n$. We obtain $\Sigma\nu\circ\mu^*\circ(\Sigma E^2\wedge 1)\simeq\Sigma\nu\circ\Sigma(E^2\cdot 1)\circ t$. By Lemma 2.2, $\nu\circ(E^2\cdot 1)\simeq\nu\circ\pi_2$, where π_2 is the projection onto the second factor. As t has the property that $\Sigma\pi_2\circ t$ is null homotopic, $\Sigma\nu\circ\Sigma(E^2\cdot 1)\circ t$ is null homotopic. Thus $\Sigma\nu\circ\mu^*\circ(\Sigma E^2\wedge 1)$ is null homotopic. □

By Theorem 2.1, there is a map $\Sigma^2 BW_n \longrightarrow \Sigma^2\Omega^2 S^{2n+1}$ which is a right homotopy inverse of $\Sigma^2\nu$. Shifting suspension coordinates, we then obtain a composite

$$f : \Sigma S^{2n-1}\wedge BW_n \xrightarrow{\simeq} S^{2n-2}\wedge\Sigma^2 BW_n \longrightarrow S^{2n-1}\wedge\Sigma^2\Omega^2 S^{2n+1} \xrightarrow{\simeq} \Sigma S^{2n-1}\wedge\Omega^2 S^{2n+1}.$$

Incorporating this, let g be the composite

$$g : \Sigma S^{2n-1}\wedge BW_n \xrightarrow{f} \Sigma S^{2n-1}\wedge\Omega^2 S^{2n+1} \xrightarrow{\Sigma E^2\wedge 1} \Sigma\Omega^2 S^{2n+1}\wedge\Omega^2 S^{2n+1} \xrightarrow{\mu^*} \Sigma\Omega^2 S^{2n+1}.$$

Denote the wedge sum operation by "$\perp$".

Lemma 2.5. *Let $p \geq 3$. Then there is a homotopy cofibration*

$$\Sigma S^{2n-1}\vee(\Sigma S^{2n-1}\wedge BW_n) \xrightarrow{\Sigma E^2\perp g} \Sigma\Omega^2 S^{2n+1} \xrightarrow{\Sigma\nu} \Sigma BW_n.$$

Proof. We first observe that $\Sigma\nu\circ(\Sigma E^2\perp g)$ is null homotopic. A map out of a wedge is determined by its restriction to each wedge summand, so it is equivalent to show that both $\Sigma\nu\circ\Sigma E^2$ and $\Sigma\nu\circ g$ are null homotopic. The former is null homotopic because ν and E^2 are consecutive maps in a homotopy fibration. The latter is null homotopic because g is defined as $\mu^*\circ\Sigma(E^2\wedge 1)\circ f$ and Lemma 2.2 says that the composite $\Sigma\nu\circ\mu^*\circ(\Sigma E^2\wedge 1)$ is null homotopic.

Let X be the homotopy cofiber of $\Sigma E^2\perp g$. The null homotopy for the composite $\Sigma\nu\circ(\Sigma E^2\perp g)$ results in an extension

$$\begin{tikzcd}
\Sigma S^{2n-1}\vee(\Sigma S^{2n-1}\wedge BW_n) \arrow[r, "\Sigma E^2\perp g"] & \Sigma\Omega^2 S^{2n+1} \arrow[r] \arrow[d, "\Sigma\nu"] & X \arrow[dl, dashed, "\lambda"] \\
& \Sigma BW_n &
\end{tikzcd}$$

for some map λ. We claim that λ is an isomorphism in homology. If so, then it is a homotopy equivalence and the lemma follows.

By Theorem 2.1, there is a homotopy equivalence

$$\Sigma^2\Omega^2 S^{2n+1}\simeq\Sigma^2(S^{2n-1}\times BW_n)$$

in which $\Sigma^2\nu$ becomes the suspended projection $\Sigma^2\pi_2$. In mod-p homology this implies that $H_*(\Omega^2 S^{2n+1})\cong H_*(S^{2n-1}\times BW_n)\cong H_*(S^{2n-1})\otimes H_*(BW_n)$ and ν_* is the projection onto $H_*(BW_n)$. Further, $(E^2)_*$ is the inclusion of $H_*(S^{2n-1})$. Therefore, as a module map, ν_* has a right inverse $s : H_*(BW_n)\longrightarrow H_*(\Omega^2 S^{2n+1})$ and the product map $H_*(S^{2n-1})\otimes H_*(BW_n) \xrightarrow{(E^2)_*\cdot s} H_*(\Omega^2 S^{2n+1})$ is an isomorphism. Let M be the submodule of $\Sigma(H_*(S^{2n-1})\otimes H_*(BW_n))$ consisting of elements of the form $\sigma(x\otimes y)$ where both x and y have degree ≥ 1. The right homotopy inverse for $\Sigma^2\nu$ geometrically realizes $\Sigma^2 s$ and so g geometrically realizes the restriction of $\Sigma((E^2)_*\cdot s)$ to the submodule M. Thus $(\Sigma E^2\perp g)$ geometrically realizes the restriction of $\Sigma((E^2)_*\cdot s)$ to $H_*(S^{2n-1})\oplus M$. The cokernel of $(\Sigma E^2\perp g)_*$ is therefore given by the map $\Sigma H_*(\Omega^2 S^{2n+1}) \xrightarrow{\Sigma\nu_*} \Sigma H_*(BW_n)$. On the other hand, the homotopy cofibration defining X says that cokernel of $(\Sigma E^2 \perp g)_*$ is given by the map $H_*(\Sigma\Omega^2 S^{2n+1}) \longrightarrow H_*(X)$. Thus λ_* is an isomorphism. $\square$

3. An H-structure on BW_n

In Lemma 2.3 it was shown that there is a homotopy action $\Omega^2 S^{2n+1} \times BW_n \xrightarrow{\overline{\theta}} BW_n$. In Proposition 3.2 we will show that this action factors through a multiplication on BW_n. The bulk of the work comes in a preliminary lemma, which proves a suspended factorization.

Lemma 3.1. *Let $p \geq 3$. Then there is a homotopy commutative diagram*

$$\begin{tikzcd} \Sigma(\Omega^2 S^{2n+1} \times BW_n) \arrow[r, "\Sigma\overline{\theta}"] \arrow[d, "\Sigma(\nu\times 1)"] & \Sigma BW_n \arrow[d, equal] \\ \Sigma(BW_n \times BW_n) \arrow[r, "\psi"] & \Sigma BW_n \end{tikzcd}$$

for some map ψ with the property that its restriction to $\Sigma(BW_n \vee BW_n)$ is homotopic to the suspension of the folding map.

Proof. As the proof is lengthy, we give an outline. The idea is to decompose $\Sigma(\Omega^2 S^{2n+1} \times BW_n)$ as $\Sigma\Omega^2 S^{2n+1} \vee \Sigma BW_n \vee (\Sigma\Omega^2 S^{2n+1} \wedge BW_n)$ and show the corresponding factorization exists for each wedge summand. The factorizations for the first two summands will easily follow from the properties of $\overline{\theta}$. The factorization for the smash product is harder. We use the homotopy cofibration in Lemma 2.5, smashed with BW_n. This means we need certain null homotopies for the two-term smash product $S^{2n-1} \wedge BW_n$ and the three-term smash product $S^{2n-1} \wedge BW_n \wedge BW_n$. The latter is made more complicated as we have to control the map g in Lemma 2.5, which is defined using a Hopf construction, and this will involve some reassociating.

We now begin the proof. Consider the natural homotopy decomposition

$$e : \Sigma\Omega^2 S^{2n+1} \vee \Sigma BW_n \vee \Sigma(\Omega^2 S^{2n+1} \wedge BW_n) \longrightarrow \Sigma(\Omega^2 S^{2n+1} \times BW_n).$$

The composition $\Sigma(\nu \times 1) \circ e$ is homotopic to the wedge sum $\Sigma\nu \vee \Sigma 1 \vee \Sigma(\nu \wedge 1)$. So to prove the homotopy commutativity of the diagram asserted by the lemma, it is equivalent to show that $\Sigma\overline{\theta} \circ e$ factors through $\Sigma\nu \vee \Sigma 1 \vee \Sigma(\nu \wedge 1)$. Such a factorization produces a map

$$\psi' : \Sigma BW_n \vee \Sigma BW_n \vee \Sigma(BW_n \wedge BW_n) \longrightarrow \Sigma BW_n,$$

which lets us define ψ as the composite

$$\psi : \Sigma(BW_n \times BW_n) \xrightarrow{\simeq} \Sigma BW_n \vee \Sigma BW_n \vee \Sigma(BW_n \wedge BW_n) \xrightarrow{\psi'} \Sigma BW_n.$$

The additional assertion that ψ can be chosen so its restriction to $\Sigma(BW_n \vee BW_n)$ is homotopic to the suspension of the fold map is equivalent to the assertion that ψ' can be chosen so that its restriction to $\Sigma BW_n \vee \Sigma BW_n$ is homotopic to the fold map.

By Lemma 2.3, the restrictions of $\overline{\theta}$ to $\Omega^2 S^{2n+1}$ and BW_n respectively are ν and 1. Thus the restrictions of $\Sigma\overline{\theta} \circ e$ to $\Sigma\Omega^2 S^{2n+1}$ and ΣBW_n respectively are $\Sigma\nu$ and $\Sigma 1$. Therefore we can define ψ' on the wedge summands $\Sigma BW_n \vee \Sigma BW_n$ as the folding map.

It remains to show the factorization of $\Sigma(\Omega^2 S^{2n+1} \wedge BW_n)$ through $\Sigma(\nu \wedge 1)$. We break this into three steps.

Step 1: Setting up. In all that follows, a map labelled with a "t" will be an instance of the natural inclusion $\Sigma A \wedge B \xrightarrow{t} \Sigma(A \times B)$ which composes trivially with $\Sigma\pi_1$ and $\Sigma\pi_2$. In particular, let t be the map

$$t : \Sigma\Omega^2 S^{2n+1} \wedge BW_n \longrightarrow \Sigma(\Omega^2 S^{2n+1} \times BW_n).$$

We need to show that $(\Sigma\overline{\theta}) \circ t$ factors through $\Sigma(\nu \wedge 1)$. Smashing the homotopy cofibration in Lemma 2.5 and distributing the smash product across the wedge, we obtain a homotopy cofibration

$$(\Sigma S^{2n-1} \wedge BW_n) \vee (\Sigma S^{2n-1} \wedge BW_n \wedge BW_n) \xrightarrow{\gamma} \Sigma\Omega^2 S^{2n+1} \wedge BW_n \xrightarrow{\Sigma\nu\wedge 1} \Sigma BW_n \wedge BW_n$$

where $\gamma = (\Sigma E^2 \wedge 1) \vee (g \wedge 1)$. So to show that $(\Sigma\overline{\theta}) \circ t$ factors through $\Sigma(\nu \wedge 1)$ it is equivalent to show that both $(\Sigma\overline{\theta}) \circ t \circ (\Sigma E^2 \wedge 1)$ and $(\Sigma\overline{\theta}) \circ t \circ (g \wedge 1)$ are null homotopic.

Step 2: The null homotopy for $(\Sigma\overline{\theta}) \circ t \circ (\Sigma E^2 \wedge 1)$. Consider the diagram

$$\begin{tikzcd}
\Sigma S^{2n-1} \wedge \Omega^2 S^{2n+1} \arrow[r, "t_1"] \arrow[d, "\Sigma(1\wedge\nu)"] & \Sigma(S^{2n-1} \times \Omega^2 S^{2n-1}) \arrow[r, "\Sigma(E^2\times 1)"] \arrow[d, "\Sigma(1\times\nu)"] & \Sigma(\Omega^2 S^{2n+1} \wedge \Omega^2 S^{2n+1}) \arrow[r, "\Sigma\mu"] \arrow[d, "\Sigma(1\times\nu)"] & \Sigma\Omega^2 S^{2n+1} \arrow[d, "\Sigma\nu"] \\
\Sigma S^{2n-1} \wedge BW_n \arrow[r, "t_2"] & \Sigma(S^{2n-1} \times BW_n) \arrow[r, "\Sigma(E^2\times 1)"] & \Sigma(\Omega^2 S^{2n+1} \times BW_n) \arrow[r, "\Sigma\overline{\theta}"] & \Sigma BW_n.
\end{tikzcd}$$

The left square homotopy commutes by naturality, the middle square clearly homotopy commutes, and the right square homotopy commutes by Lemma 2.3. The upper direction around the diagram is $\Sigma\nu \circ \Sigma\mu \circ \Sigma(E^2 \times 1) \circ t_1$. By definition, $E^2 \cdot 1 = \mu \circ (E^2 \times 1)$, so by Lemma 2.2, $\nu \circ \mu \circ (E^2 \times 1) \simeq \nu \circ \pi_2$. As $(\Sigma\pi_2) \circ t_1$ is null homotopic, we therefore have $\Sigma\nu \circ \Sigma\mu \circ \Sigma(E^2 \times 1) \circ t_1 \simeq \Sigma\nu \circ \Sigma\pi_2 \circ t_1$ null homotopic. Now consider the lower direction around the diagram. The naturality of the "t"-maps implies that $\Sigma(E^2 \times 1) \circ t_2 \simeq t \circ (\Sigma E^2 \wedge 1)$. Thus $\Sigma\overline{\theta} \circ \Sigma(E^2 \times 1) \circ t_2 \circ \Sigma(1 \wedge \nu) \simeq \Sigma\overline{\theta} \circ t \circ (\Sigma E^2 \wedge 1) \circ \Sigma(1 \wedge \nu)$. The homotopy between the upper and lower directions around the diagram then implies that $\Sigma\overline{\theta} \circ t \circ (\Sigma E^2 \wedge 1) \circ \Sigma(1 \wedge \nu)$ is null homotopic. By Theorem 2.1, $1 \wedge \nu$ has a right homotopy inverse and so $\Sigma\overline{\theta} \circ t \circ (\Sigma E^2 \wedge 1)$ is null homotopic.

Step 3: The null homotopy for $(\Sigma\overline{\theta}) \circ t \circ (g \wedge 1)$. To compress notation, let $M = \Omega^2 S^{2n+1}$. Consider the diagram

$$\begin{tikzcd}
(\Sigma S^{2n-1} \wedge M) \wedge M \arrow[r, "t_1\wedge 1"] \arrow[dr, "h\wedge\nu"'] & \Sigma(S^{2n-1} \times M) \wedge M \arrow[r, "t_3"] \arrow[d, "\Sigma(E^2\cdot 1)\wedge\nu"] & \Sigma((S^{2n-1} \times M) \times M) \arrow[d, "\Sigma((E^2\cdot 1)\times\nu)"] \\
& \Sigma M \wedge BW_n \arrow[r, "t"] & \Sigma(M \times BW_n)
\end{tikzcd}$$

where h is defined as the composite $\Sigma S^{2n-1} \wedge M \xrightarrow{\Sigma E^2 \wedge 1} \Sigma M \wedge M \xrightarrow{\mu^*} \Sigma M$. By definition, $E^2 \cdot 1 = \mu \circ (E^2 \times 1)$ and $\mu^* = \Sigma\mu \circ t_4$, for $t_4 : \Sigma M \wedge M \longrightarrow \Sigma(M \times M)$. The naturality of the "t"-maps implies that $\Sigma(E^2 \times 1) \circ t_1 \simeq \mu^* \circ (\Sigma E^2 \wedge 1) = h$ and so the left triangle homotopy commutes. The right square homotopy commutes by the naturality of the "t"-maps. Let $s = t_3 \circ (t_1 \wedge 1)$ be the composite along the top row. Observe that s is trivial when composed with $\Sigma(\pi_2 \times \pi_3)$, the suspension of the product of the projections onto the second and third factors. Now consider the diagram

$$\begin{tikzcd}
\Sigma S^{2n-1}\wedge M\wedge M \arrow[r, "s"] \arrow[d, "h\wedge\nu"] & \Sigma(S^{2n-1}\times M\times M) \arrow[r, "\Sigma(E^2\times 1\times 1)"] \arrow[d, "\Sigma((E^2\cdot 1)\times\nu)"] & \Sigma(M\times M\times M) \arrow[r, "\Sigma(\mu\circ(1\times\mu))"] \arrow[d, "\Sigma(\mu\times\nu)"] & \Sigma M \arrow[d, "\Sigma\nu"] \\
\Sigma M\wedge BW_n \arrow[r, "t"] & \Sigma(M\times BW_n) \arrow[r, equal] & \Sigma(M\times BW_n) \arrow[r, "\Sigma\overline{\theta}"] & \Sigma BW_n .
\end{tikzcd}$$

The left square homotopy commutes by the previous diagram, and the middle square homotopy commutes as $E^2\cdot 1$ is defined as $\mu\circ(E^2\times 1)$. The right square homotopy commutes because Lemma 2.3 shows that $\overline{\theta}\circ(1\times\nu)\simeq\nu\circ\mu$ and so, using the homotopy associativity of M, we obtain a string of homotopies $\overline{\theta}\circ(\mu\times\nu) = \overline{\theta}\circ(1\times\nu)\circ(\mu\times 1)\simeq\nu\circ\mu\circ(\mu\times 1)\simeq\nu\circ\mu\circ(1\times\mu)$. Consider the upper direction around the diagram. To distinguish maps, let 1_S and 1_M be the identity maps on S^{2n-1} and M respectively. In the following,

$$\begin{array}{rcl}
\nu\circ\mu\circ(1_M\times\mu)\circ(E^2\times 1_M\times 1_M) & \simeq & \nu\circ\mu\circ(E^2\times\mu) \\
 & \simeq & \nu\circ\mu\circ(E^2\times 1_M)\circ(1_S\times\mu) \\
 & \simeq & \nu\circ\pi_2\circ(1_S\times\mu) \\
 & \simeq & \nu\circ\mu\circ(\pi_2\times\pi_3),
\end{array}$$

the first two homotopies reorganize the composites, the third is due to Lemma 2.2, and the fourth uses the naturality of projections. Now suspend and precompose with s. As $\Sigma(\pi_2\times\pi_3)\circ s$ is null homotopic, the upper direction around the diagram, $\Sigma\nu\circ\Sigma(\mu\circ(1_M\times\mu))\circ\Sigma(E^2\times 1_M\times 1_M)\circ s$, is therefore null homotopic. Hence the lower direction around the diagram, $\Sigma\overline{\theta}\circ t\circ(h\wedge\nu)$, is null homotopic.

Just before Lemma 2.5 we defined a map $\Sigma S^{2n-1}\wedge BW_n\xrightarrow{f}S^{2n-1}\wedge M$ which is a left homotopy inverse of $\Sigma(1_S\wedge\nu)$. Also recall that the map $\Sigma S^{2n-1}\wedge BW_n\xrightarrow{g}\Sigma M$ was defined as $g=\mu^*\circ(\Sigma E^2\wedge 1)\circ f$, and so $g=h\circ f$. Let k be the composite

$$\begin{aligned}
k:\Sigma S^{2n-1}\wedge BW_n\wedge BW_n &\xrightarrow{T\wedge 1} BW_n\wedge\Sigma S^{2n-1}\wedge BW_n\xrightarrow{1\wedge f} \\
&BW_n\wedge\Sigma S^{2n-1}\wedge M\xrightarrow{T\wedge 1}\Sigma S^{2n-1}\wedge BW_n\wedge M,
\end{aligned}$$

where T is the interchange map. Consider the diagram

$$\begin{tikzcd}
\Sigma S^{2n-1}\wedge BW_n\wedge BW_n \arrow[r, "k"] & (\Sigma S^{2n-1}\wedge BW_n)\wedge M \arrow[r, "f\wedge 1"] \arrow[d, "1\wedge\nu"] & (\Sigma S^{2n-1}\wedge M)\wedge M \arrow[d, "h\wedge\nu"] \\
 & (\Sigma S^{2n-1}\wedge BW_n)\wedge BW_n \arrow[r, "g\wedge 1"] & \Sigma M\wedge BW_n .
\end{tikzcd}$$

The square homotopy commutes as $g=h\circ f$. Since f is a left homotopy inverse of $\Sigma(1_S\wedge\nu)$, the composite $(1\wedge\nu)\circ k$ in the lower direction around the diagram is homotopic to the identity map. Thus $(h\wedge\nu)\circ(f\wedge 1)\circ k\simeq g\wedge 1$. Therefore the null homotopy for $\Sigma\overline{\theta}\circ t\circ(h\wedge\nu)$ in the previous paragraph implies that $\Sigma\overline{\theta}\circ t\circ(g\wedge 1)$ is null homotopic, as required. $\square$

Proposition 3.2. *Let $p \geq 3$. Then there is an H-structure $m : BW_n \times BW_n \longrightarrow BW_n$ which satisfies a homotopy commutative diagram*

$$\begin{tikzcd} \Omega^2 S^{2n+1} \times BW_n \arrow[r, "\overline{\theta}"] \arrow[d, "\nu\times 1"] & BW_n \arrow[d, equal] \\ BW_n \times BW_n \arrow[r, "m"] & BW_n. \end{tikzcd}$$

Proof. Since BW_n is an H-space, the canonical suspension $BW_n \xrightarrow{E} \Omega\Sigma BW_n$ has a left homotopy inverse $r : \Omega\Sigma BW_n \longrightarrow BW_n$. Consider the diagram

$$\begin{tikzcd} \Omega^2 S^{2n+1} \times BW_n \arrow[r, "E"] \arrow[d, "\nu\times 1"] & \Omega\Sigma(\Omega^2 S^{2n+1} \times BW_n) \arrow[r, "\Omega\Sigma\overline{\theta}"] \arrow[d, "\Omega\Sigma(\nu\times 1)"] & \Omega\Sigma BW_n \arrow[d, equal] & \\ BW_n \times BW_n \arrow[r, "E"] & \Omega\Sigma(BW_n \times BW_n) \arrow[r, "\Omega\psi"] & \Omega\Sigma BW_n \arrow[r, "r"] & BW_n. \end{tikzcd}$$

The left square homotopy commutes by the naturality of E, and the right square homotopy commutes by Lemma 3.1. Along the top row, the naturality of E implies that $\Omega\Sigma\overline{\theta} \circ E \simeq E \circ \overline{\theta}$, where the suspension E on the right side of the homotopy is $BW_n \xrightarrow{E} \Omega\Sigma BW_n$. Thus the upper direction around the diagram satisfies

$$r \circ \Omega\Sigma\overline{\theta} \circ E \simeq r \circ E \circ \overline{\theta} \simeq \overline{\theta}.$$

Let $m : BW_n \times BW_n \longrightarrow BW_n$ be the composite $r \circ \Omega\psi \circ E$ along the bottom row. The homotopy commutativity of the diagram implies that $\overline{\theta} \simeq m \circ (\nu \times 1)$, as asserted by the proposition. Finally, since the restriction of ψ to $\Sigma(BW_n \vee BW_n)$ is homotopic to the suspension of the folding map, it follows that the restriction of m to $BW_n \vee BW_n$ is homotopic to the folding map, and so m is an H-structure. $\square$

Juxtaposing the homotopy commutative diagrams in Lemma 2.3 and Proposition 3.2 gives a homotopy commutative diagram

$$\begin{tikzcd} \Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} \arrow[r, "\mu"] \arrow[d, "\nu\times\nu"] & \Omega^2 S^{2n+1} \arrow[d, "\nu"] \\ BW_n \times BW_n \arrow[r, "m"] & BW_n, \end{tikzcd}$$

which immediately proves the following.

Corollary 3.3. *Let $p \geq 3$. Then the map $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ is an H-map with respect to the loop multiplication on $\Omega^2 S^{2n+1}$ and the multiplication m on BW_n.* $\square$

4. Properties of the H-structure m on BW_n

In this section we use the multiplication m on BW_n in Proposition 3.2 to prove Theorems 1.1 and 1.2. We begin with a general lemma, which will get used repeatedly.

Lemma 4.1. *Suppose there exist maps $f : X \longrightarrow Y$ and $g, h : Y \longrightarrow Z$ such that $g \circ f \simeq h \circ f$, $\Sigma^2 f$ has a right homotopy inverse, and Z is an H-space. Then $g \simeq h$.*

Proof. As Z is an H-space, there is a map $r : \Omega\Sigma Z \longrightarrow Z$ which is a left homotopy inverse of the suspension $E : Z \longrightarrow \Omega\Sigma Z$. Thus it suffices to show that $\Sigma g \simeq \Sigma h$, for then the naturality of E with respect to loop suspensions implies that there is a string of homotopies $g \simeq r \circ E \circ g \simeq r \circ \Omega\Sigma g \circ E \simeq r \circ \Omega\Sigma h \circ E \simeq r \circ E \circ h \simeq h$.

Define the space C and the map t by the homotopy cofibration

$$X \xrightarrow{f} Y \xrightarrow{t} C.$$

Since $\Sigma^2 f$ has a left homotopy inverse, there is a homotopy equivalence $\Sigma^2 X \simeq \Sigma C \vee \Sigma^2 Y$. This implies that Σt is null homotopic. The hypothesis $g \circ f \simeq h \circ f$ implies that $\Sigma g \circ \Sigma f \simeq \Sigma h \circ \Sigma f$, and so, using the co-$H$ structure to subtract and distribute, we have $(\Sigma g - \Sigma h) \circ \Sigma f$ null homotopic. Thus there is an extension

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma t"] \arrow[d, "d"] & C \arrow[dl, dashed] \\ & \Sigma Z & \end{tikzcd}$$

where $d = \Sigma g - \Sigma h$. The null homotopy for Σt therefore implies that $\Sigma g - \Sigma h$ is null homotopic, and so $\Sigma g \simeq \Sigma h$, as required. □

We now move on to the assertions of Theorem 1.1, taken one at a time.

Lemma 4.2. *There is a unique multiplication on BW_n for which $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ is an H-map.*

Proof. Let μ be the loop multiplication on $\Omega^2 S^{2n+1}$. Suppose

$$\mu_1, \mu_2 : BW_n \times BW_n \longrightarrow BW_n$$

are both multiplications for which ν is an H-map. Then $\mu_1 \circ (\nu \times \nu) \simeq \nu \circ \mu \simeq \mu_2 \circ (\nu \times \nu)$. By Theorem 2.1, $\Sigma^2 \nu$ has a right homotopy inverse, and therefore so does $\Sigma^2(\nu \times \nu)$. Summarizing, we have maps $\Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} \xrightarrow{\nu\times\nu} BW_n \times BW_n$ and $BW_n \times BW_n \xrightarrow{\mu_1,\mu_2} BW_n$ such that $\mu_1 \circ (\nu \times \nu) \simeq \mu_2 \circ (\nu \times \nu)$, $\Sigma^2(\nu \times \nu)$ has a right homotopy inverse, and BW_n is an H-space. So by Lemma 4.1, $\mu_1 \simeq \mu_2$. □

Lemma 4.3. *Suppose BW_n has an H-structure for which $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ is an H-map. Then this H-structure is both homotopy associative and homotopy commutative.*

Proof. Let μ be the loop structure on $\Omega^2 S^{2n+1}$, and let m be the H-structure on BW_n for which ν is an H-map. First consider homotopy associativity. Let $g = m \circ (m \times 1)$ and $h = m \circ (1 \times m)$. Since $\Omega^2 S^{2n+1}$ is homotopy associative and ν is an H-map, we have $g \circ (\nu \times \nu \times \nu) \simeq h \circ (\nu \times \nu \times \nu)$. By Theorem 2.1, $\Sigma^2 \nu$ has a right homotopy inverse, and therefore so does $\Sigma^2(\nu \times \nu \times \nu)$. Summarizing, let $X = \Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1}$, $Y = BW_n \times BW_n \times BW_n$, $Z = BW_n$, and $f = \nu \times \nu \times \nu$. Then we have maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g,h} Z$ such that $g \circ f \simeq h \circ f$, $\Sigma^2 f$ has a right homotopy inverse, and Z is an H-space. So by Lemma 4.1, $g \simeq h$. That is, $m \circ (m \times 1) \simeq m \circ (1 \times m)$ and so m is homotopy associative.

A similar argument holds for homotopy commutativity by taking $f = \nu \times \nu$, $g = m$, and $h = m \circ T$ where T is the map which interchanges factors. □

Proof of Theorem 1.1. Combine the statements of Proposition 3.2, Corollary 3.3 and Lemmas 4.2 and 4.3. □

Finally, we prove the exponent result that the p^{th}-power map on BW_n is null homotopic.

Proof of Theorem 1.2. Since $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ is an H-map, $p \circ \nu \simeq \nu \circ p$. By [CMN2] for $p \geq 5$ and [N] for $p = 3$, the p^{th}-power map on $\Omega^2 S^{2n+1}$ factors through $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$. Thus $\nu \circ p$ factors through $\nu \circ E^2$, which is null homotopic as E^2 and ν are consecutive maps in a homotopy fibration. Hence $p \circ \nu$ is null homotopic. Summarizing, we have maps $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ and $BW_n \xrightarrow{p,*} BW_n$ such that $p \circ \nu \simeq * \circ \nu$, $\Sigma^2 \nu$ has a right homotopy inverse (by Theorem 2.1), and BW_n is an H-space. So by Lemma 4.1, $p \simeq *$. $\square$

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