ON GENERIC DIFFERENTIAL $\text{SO}_n$-EXTENSIONS

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Abstract. Let $\mathcal{C}$ be an algebraically closed field with trivial derivation and let $\mathcal{F}$ denote the differential rational field $\mathcal{C}(Y_{ij})$, with $Y_{ij}, 1 \leq i \leq n - 1, 1 \leq j \leq n, i \leq j$, differentially independent indeterminates over $\mathcal{C}$. We show that there is a Picard-Vessiot extension $\mathcal{E} \supset \mathcal{F}$ for a matrix equation $X' = X A(Y_{ij})$, with differential Galois group $\text{SO}_n$, with the property that if $F$ is any differential field with field of constants $\mathcal{C}$, then there is a Picard-Vessiot extension $E \supset F$ with differential Galois group $H \leq \text{SO}_n$ if and only if there are $f_{ij} \in F$ with $A(f_{ij})$ well defined and the equation $X' = X A(f_{ij})$ giving rise to the extension $E \supset F$.

1. Introduction

Let $\mathcal{C}$ denote an algebraically closed field with trivial derivation, $G$ a linear algebraic group over $\mathcal{C}$, and $\mathfrak{gl}_m(\cdot)$ the Lie algebra of $m \times m$ matrices with coefficients in some specified field. The short form ‘Picard-Vessiot $G$-extension’ (or sometimes ‘PVE with group $G$’) will be used for ‘Picard-Vessiot extension (PVE) with differential Galois group isomorphic to $G$’. We consider the differential rational field $\mathcal{F} = \mathcal{C}(Z_1, \ldots, Z_k)$, where $Z_1, \ldots, Z_k$ are differentially independent indeterminates over $\mathcal{C}$.

Definition 1. A Picard-Vessiot $G$-extension $\mathcal{E} \supset \mathcal{F}$ for the equation $X' = X A(Z_1, \ldots, Z_k)$, with $A(Z_1, \ldots, Z_k) \in \mathfrak{gl}_m(\mathcal{F})$ for some $m$, is said to be a generic extension for $G$ if for every Picard-Vessiot $G$-extension $E \supset F$ there is a specialization $Z_i \rightarrow f_i \in F$, such that the equation $X' = X A(f_1, \ldots, f_k)$ gives rise to $E \supset F$ and any fundamental solution matrix maps to one for the specialized equation.

Note that by making the assumption that $G = G(\mathcal{C})$, we are also assuming that the base field of a Picard-Vessiot $G$-extension and the extension itself have field of constants $\mathcal{C}$.

In this paper we produce generic extensions for the special orthogonal groups $\text{SO}_n, n \geq 3$. For $n = 2$ the group is isomorphic to the (cohomologically trivial) multiplicative group, a case already studied in [5].

The construction that we provide is based on Kolchin’s Structure Theorem, which describes the possible Picard-Vessiot $G$-extensions of a differential field $F$ as function fields of $F$-irreducible $G$-torsors [11, Theorem 5.12], [12, Theorem 1.28]. The isomorphism classes of $G$-torsors, in turn, are in bijective correspondence with the elements of the first Galois cohomology set $H^1(F, G)$ [13, 15]. The latter is
a particularly convenient feature since for the special orthogonal groups the first cohomology can be described in terms of regular quadratic forms of discriminant 1 (cf. [7]).

In previous work the first author has studied generic extensions in two special situations. The first was when $G$ is connected and the extension is the function field of the trivial $G$-torsor (cf. [5]). The second was when $G$ is the semidirect product $H \rtimes G^0$ of its connected component by a finite group $H$ and the extensions are the function fields of $F$-irreducible $G$-torsors of the form $W \times G^0$, where $W$ is an $F$-irreducible $H$-torsor (cf. [6]).

In the present paper we turn our attention to the general case, that is, when $H^1(F, G)$ is not necessarily trivial. In [7] we showed that in such a situation, it might be possible to find a Picard-Vessiot $G$-extension of $F$ that is the function field of a non-trivial torsor. We will use the machinery developed there and a version of a method to construct generic extensions from [5] to attack this general situation when $G$ is the special orthogonal group $SO_n$, $n \geq 3$. With the description of the $SO_n$-torsors in terms of regular quadratic forms of discriminant 1 at our disposal we can provide a good description of the twisted Lie algebras associated with the torsors [7], a key ingredient of our construction.

Having a good grasp of the torsors also allows us to show that this extension fully descends to subgroups of $SO_n$, that is, there is a specialization of the parameters over the base field $F$ yielding a Picard-Vessiot $H$-extension if and only if $H \leq SO_n$.

Finally, we discuss how to proceed with connected groups in general, when a good description of the torsors is not available. In this case a generic extension relative to the trivial torsor along with the Trivialization Lemma from Section 3 allows a (not so explicit but quite similar) construction in which the specialization of the parameters takes place over a finite extension of $F$ instead of $F$.

All the differential fields that we consider are of characteristic zero and have an algebraically closed field of constants. We keep the notations $C$ and $F$ introduced above.

2. Generic extension vs. generic equation

The $SO_n$ case is included among the groups studied by Goldman [3] and Bhandari and Sankaran [1].

Definition 2 (Goldman [3]). Let $G$ be a linear algebraic group over $C$ and assume that a faithful representation in $GL_n(C)$ is given. Let $L(t, y) = Q_0(t_1, \ldots, t_r)y^{(n)} + \cdots + Q_n(t_1, \ldots, t_r)y \in C\{t_1, \ldots, t_r, y\}$ and write $(\pi_1, \ldots, \pi_n)$ for a fundamental system of zeros of $L(t, y)$ such that $C\{t_1, \ldots, t_r, \pi_1, \ldots, \pi_n\}$ is a PVE of $C\{t_1, \ldots, t_r\}$ with group $G$. Then $L(t, y) = 0$ will be called a generic equation with group $G$ if:

1. $t_1, \ldots, t_r$ are differentially independent over $C$, and $C\{t_1, \ldots, t_r\} \subset C\langle \pi_1, \ldots, \pi_n \rangle$.
2. For every specialization $(t_1, \ldots, t_r, \pi_1, \ldots, \pi_n) \rightarrow (\bar{t}_1, \ldots, \bar{t}_r, \bar{\pi}_1, \ldots, \bar{\pi}_n)$ over $C$ such that $C\langle \bar{t}_1, \ldots, \bar{t}_r, \bar{\pi}_1, \ldots, \bar{\pi}_n \rangle$ is a PVE of $C\langle \bar{t}_1, \ldots, \bar{t}_r \rangle$ and the field of constants of the latter is $C$, the differential Galois group of this extension is a subgroup of $G$.
3. If $(\omega_1, \ldots, \omega_n)$ is a fundamental system of zeros of $L(y) = y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny \in F\{y\}$, where $F$ is any differential field with field of constants $C$, and $F\langle \omega_1, \ldots, \omega_n \rangle$ is a PVE of $F$ with differential Galois group $H \leq G$,
then there exists a specialization \((t_1, \ldots, t_r) \rightarrow (\bar{t}_1, \ldots, \bar{t}_r)\) over \(F\) with \(t_i \in F\) such that \(Q_\sigma(t_1, \ldots, t_r) \neq 0\) and 
\[
a_i = Q_i(\bar{t}_1, \ldots, \bar{t}_r)Q_\sigma^{-1}(\bar{t}_1, \ldots, \bar{t}_r).
\]

Goldman shows that a necessary condition for such an equation to exist is that the number of parameters \(r\) equals the order \(n\) of the equation \([3, \text{Lemma 1, p. 343}]\). The groups studied in that paper include \(\text{GL}_n\), \(\text{SL}_n\) as well as the orthogonal and symplectic groups.

Now, let \(G\) act on \(C\langle y_1, \ldots, y_n \rangle\), where \(y_1, \ldots, y_n\) are differentially independent indeterminates over \(C\), by \(\sigma(y_i) = \sum_{j=1}^{n} c_{ij} y_j\) for \(\sigma = (c_{ij}) \in G(C) \subset \text{GL}_n(C)\). Then 
\[
P_i = \frac{W_i(y_1, \ldots, y_n)}{W_0(y_1, \ldots, y_n)} \quad (i = 1, \ldots, n),
\]
where 
\[
W_i = (-1)^i \begin{vmatrix}
y_1 & \cdots & y_n \\
\vdots & \ddots & \vdots \\
y_1^{(n-i-1)} & \vdots & y_n^{(n-i-1)} \\
y_1^{(n-i+1)} & \vdots & y_n^{(n-i+1)} \\
\vdots & \ddots & \vdots \\
y_1^{(n)} & \cdots & y_n^{(n)}
\end{vmatrix}
\]
are invariant under the \(G\) action.

The procedure used by Goldman for the groups above first finds \(n\) differentially independent generators \(t_1, \ldots, t_n\) over \(C\) of the fixed field \(C\langle y_1, \ldots, y_n \rangle^G\) and \(n + 1\) differential polynomials \(Q_0(t_1, \ldots, t_n), \ldots, Q_n(t_1, \ldots, t_n) \in C\{t_1, \ldots, t_n\}\) with 
\[
P_i = \frac{Q_i(t_1, \ldots, t_n)}{Q_0(t_1, \ldots, t_n)} \quad (i = 1, \ldots, n).
\]
He then shows that a generic equation with group \(G\) is given by 
\[
(2.1) \quad L(t, y) = Q_0(t_1, \ldots, t_n)y^{(n)} + \cdots + Q_n(t_1, \ldots, t_n)y = 0.
\]

This method, however, fails to produce a generic equation for \(G = \text{SO}_3\) as \([3, \text{Example 3, p. 355}]\) illustrates.

Bhandari and Sankaran \([1]\) proved that \((2.1)\) is generic for the special orthogonal groups in a weaker sense, that is, replacing \((3)\) in Goldman’s definition with the following:

\((3')\) If \(F\) is a differential field with field of constants \(C\) and \(E\) is a PVE of \(F\) with differential Galois group \(H \leq G\), then there exists a linear differential equation 
\[
L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0, \quad a_i \in F
\]
such that \(Q_0(\bar{t}_1, \ldots, \bar{t}_r) \neq 0\), \(a_i = Q_i(\bar{t}_1, \ldots, \bar{t}_r)Q_\sigma^{-1}(\bar{t}_1, \ldots, \bar{t}_r)\), \(i = 1, \ldots, n\), for suitable \(\bar{t}_i \in F\) and \(E = F(\omega_1, \ldots, \omega_n)\) for a fundamental system of zeros of \(L(y)\).

There are, however, some key differences in our approaches. In constructing their equations, both Goldman and Bhandari-Sankaran start with the differential rational field \(\mathcal{F} = C\langle y_1, \ldots, y_n \rangle\), where \(n\) is the order of the equation, and find the differential fixed field \(C\langle y_1, \ldots, y_n \rangle^G\). We start instead with \(\mathcal{F}\) as our base field and show that \(\mathcal{F}(Y)\), where \(Y\) is a generic point of a “general” \(G\)-torsor, is a generic PVE in the sense of Definition 1. Furthermore, it satisfies descent conditions analogous to \((2)\) and \((3')\) above. In our case, the number \(n\) of parameters is given by the
dimension of the group and the description of the torsors, so it is independent of the representation of $G$ in a $GL_n$. By using a general derivation in the function field of our special $G$-torsor (that is, a typical element in the twisted Lie algebra) the specialization of our parameters comes in a very natural and painless fashion, whereas in the case of the generic equations in [1, 3], showing that $Q_0(t_1, \ldots, t_n) \neq 0$ is quite involved.

In connection with the previous notions of generic equation [1, 3], Juan and Magid [8] study the ring of generic solutions for a linear monic order $n$ equation, that is, $R = \mathbb{C}\{P_1, \ldots, P_n\} \otimes_{\mathbb{C}} \mathbb{C}[y^{(i)}_j, 1 \leq i \leq n, 0 \leq j \leq n - 1][w_0^{-1}]$, where $P_i, y_i, 1 \leq i \leq n$, and $w_0$, are as above, with the $GL_n(\mathbb{C})$ action extended from the linear action on $V = \mathbb{C}y_1 + \cdots + \mathbb{C}y_n$ using the $\mathbb{C}$-basis $y_1, \ldots, y_n$. The ring $R$ has the following properties:

Assume that $E \supset F$ is a Picard-Vessiot $G$-extension and that $G$ has a faithful representation $\rho$ in $GL_n$. Then there is a differential homomorphism $\Psi : R \to F$ such that

1. $E$ is the quotient field of $F\Psi(R)$;
2. $E \supset F$ is a PVE for $L(Y) = Y^{(n)} + \rho(P_1)Y^{(n-1)} + \cdots + \rho(P_n)Y^{(0)}$;
3. $\Psi$ is $G$-equivariant, so $\Psi(R^G) \subset E^G = F$.

Conversely, assume that $G$ is an observable subgroup of $GL_n$ and let $\phi : R^G \to F$ be a differential $F$-algebra homomorphism with restriction $\alpha$ to $R^{GL_n}$. Let $P$ be a maximal differential ideal of $R = F \otimes_{\alpha} R$ whose inverse image in $R$ contains the kernel of $\phi$, and let $E$ be the fraction field of $R/P$. Then $E$ is a PVE of $F$ with differential Galois group contained in $G$.

The special orthogonal groups are observable (see [4]) and therefore satisfy the above conditions. We point out that in our construction the coordinate ring $\mathbb{C}\{Y_{ij}\}[Y, 1/\det(Y)]$, where $Y$ is a generic point of a general $SO_n$-torsor, has properties similar to that of the ring $R$.

The work in [1, 3, 8] describes equations given by linear differential operators attached to a representation of the differential Galois group $G$ in $GL_n$. Our work describes matrix equations with group $G$ in connection with the structure of the Picard-Vessiot $G$-extensions.

3. $SO_n$-extensions

In [7] we saw that every $F$-irreducible $SO_n$-torsor has a generic point of the form $Y = XP$, where $X$ is a generic point for $SO_n$ and

$$P = \begin{pmatrix} \sqrt{a_1} & \sqrt{a_2} & \cdots & \sqrt{a_n} \end{pmatrix},$$

for $a_i \in F^*$ with $a_1 \cdots a_n = 1$ and the roots chosen to have product 1 as well. A PVE of $F$ with group $SO_n$ corresponding to this torsor, if any, equals the function field $F(Y)$ of the torsor and has derivation given by $Y' = YB$, where the matrix $B$
is of the form

\[
\begin{pmatrix}
\frac{a'_1}{2a_1} & b_{12} & b_{13} & \ldots & b_{1n} \\
-\frac{a_1}{a_2} b_{12} & \frac{a'_2}{2a_2} & b_{23} & \ldots & b_{2n} \\
-\frac{a_1}{a_3} b_{13} & -\frac{a_2}{a_3} b_{23} & \frac{a'_3}{2a_3} & \ldots & b_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{a_1}{a_n} b_{1n} & -\frac{a_2}{a_n} b_{2n} & -\frac{a_3}{a_n} b_{3n} & \ldots & \frac{a'_n}{2a_n}
\end{pmatrix}
\]

for \( b_{ij} \in F, 1 \leq i \leq n-1, 2 \leq j \leq n \) and \( a_i \in F^* \) as before. An explicit example was given there, with \( Y \) corresponding to a non-trivial torsor, by making the simplifying assumption that \( b_{i,i+1} = a_i \). We point out that with that assumption, the number of parameters used in [7] to produce a PVE associated with a non-trivial torsor is \( \frac{1}{2}n(n-1) \), the dimension of \( SO_n \).

Since our goal here is to produce a generic extension we need to modify that example in order to retain the \( \frac{1}{2}(n-1)(n+2) \) parameters in the matrix \( B \).

We assume that \( a_1, \ldots, a_{n-1}, b_{12}, \ldots, b_{n-1,n} \) are differentially independent indeterminates over \( C \) and let \( \mathcal{F} = C(a_1, \ldots, a_{n-1}, b_{12}, \ldots, b_{n-1,n}) \). We first show that the equation \( \eta' = \eta A \) over the algebraic closure \( \bar{\mathcal{F}} \) of \( \mathcal{F} \), with coefficient matrix

\[
A = \begin{pmatrix}
0 & \sqrt{a_2} b_{12} & \sqrt{a_2} b_{13} & \ldots & \sqrt{a_2} b_{1n} \\
-\sqrt{a_1} b_{12} & 0 & \sqrt{a_3} b_{23} & \ldots & \sqrt{a_3} b_{2n} \\
-\sqrt{a_1} b_{13} & -\sqrt{a_2} b_{23} & 0 & \ldots & \sqrt{a_3} b_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\sqrt{a_1} b_{1n} & -\sqrt{a_2} b_{2n} & -\sqrt{a_3} b_{3n} & \ldots & 0
\end{pmatrix}
\]

has differential Galois group \( SO_n \). From this it will follow that the corresponding equation \( \eta' = \eta B \over\mathcal{F} \) has the same group.

Let \( Z_{ij} = \sqrt{a_i}/\sqrt{a_j} b_{ij}, 1 \leq i \leq n-1, 2 \leq j \leq n, i < j \). Clearly the \( Z_{ij} \) are differentially independent over \( C \) since all the \( a_i \) and \( b_{ij}^2 \) are in the differential field \( \mathcal{L} = C(a_1, \ldots, a_{n-1}, Z_{12}, \ldots, Z_{n-1,n}) \), which forces the differential transcendence degree [10, Definition 3.2.33 and Theorem 5.4.12] of \( \mathcal{L} \) over \( C \) to be \( \frac{1}{2}(n-1)(n+2) \).

Now, since \( A = \sum_{i=1}^{n-1} \sum_{j=i+2}^{n} Z_{ij} A_{ij} \), where \( \{ A_{ij} \} \) is the basis of \( \text{Lie}(SO_n) \) consisting of the antisymmetric matrices with 1 in the \((i,j)\)-entry, \(-1\) in the \((j,i)\)-entry and 0 otherwise, by [5, Theorem 4.1.2] it then follows that \( \mathcal{L}(SO_n) \supset \mathcal{L} \) is a PVE with group \( SO_n \) for the equation \( X' = XA \).

Since \( a_i, b_{ij}^2 \in \mathcal{L} \) we have that \( a_i, b_{ij} \in \mathcal{L} \) and thus \( \bar{\mathcal{F}} = \bar{\mathcal{L}} \). Therefore, \( \bar{\mathcal{F}}(SO_n) \supset \bar{\mathcal{L}}(SO_n) \) is an algebraic extension. Since the field of constants of \( \mathcal{L}(SO_n) \) is the algebraically closed field \( \mathcal{C} \), \( \bar{\mathcal{F}}(SO_n) \) must have no new constants and \( \bar{\mathcal{F}}(SO_n) \supset \bar{\mathcal{F}} \) is a PVE with group \( SO_n \).
The discussion in [7, Section 4] implies that the matrix
\[
B = \begin{pmatrix}
\frac{a_1}{2a_1} & b_{12} & b_{13} & \ldots & b_{1n} \\
-\frac{a_2}{a_2} b_{12} & \frac{a_2}{2a_2} & b_{23} & \ldots & b_{2n} \\
-\frac{a_3}{a_3} b_{13} & -\frac{a_3}{2a_3} b_{23} & \frac{a_3}{2a_3} & \ldots & b_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{a_n}{a_n} b_{1n} & -\frac{a_n}{a_n} b_{2n} & -\frac{a_n}{a_n} b_{3n} & \ldots & \frac{a_n}{2a_n}
\end{pmatrix}
\]
defines a derivation on the coordinate ring \( T = \mathcal{F}[Y] \) of the SO\(_n\)-torsor corresponding to the quadratic form given by the matrix
\[
Q = \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix}
\]
which by [7, Lemma 1] is non-trivial.

Since \( \bar{\mathcal{F}}(Y) = \mathcal{F}(X) \), as a differential field it will be isomorphic to \( \bar{\mathcal{F}}(\mathrm{SO}_n) \). Therefore, the field of constants of \( \mathcal{F}(Y) \) is \( \mathbb{C} \). In particular, this implies that \( \mathcal{F}(Y) \supset \mathcal{F} \) is a no new constant extension. This shows that the function field of the (non-trivial) SO\(_n\)-torsor corresponding to \( Y \) is a PVE of \( \mathcal{F} \) with group SO\(_n\).

We point out for later use that the previous argument can be shown in a more general setting:

**Trivialization Lemma.** Let \( E \supset F \) be a Picard-Vessiot \( G \)-extension with \( G \) connected. Then there are a finite extension \( k \supset F \) and a Picard-Vessiot \( G \)-extension \( K = kE \) of \( k \) such that \( K = k(G) \).

In other words, if there is a PVE of \( F \) with group \( G \), then the trivial \( G \)-torsor can be realized over a finite extension of \( F \). Although this is a known result (see [14, p. 142, Corollary]), for the convenience of the reader we include a short proof using the tools that we develop here.

**Proof.** Let \( X \) be a generic point of \( G \). Then \( E = F(Y) \) where \( Y = XP \), for a matrix \( P \) with coefficients in \( \bar{F} \) [7, Section 3]. Let \( k \) denote the field generated over \( F \) by the entries of \( P \). Then \( k(X) = k(Y) \supset F(Y) \) is an algebraic extension. Therefore, \( k(G) = k(X) \supset k \) is a no new constant extension and thus a Picard-Vessiot \( G \)-extension. Clearly, \( K = k(X) = kE \). \( \square \)

4. **Generic extensions**

First we introduce the following notion, analogous to one for generic polynomial equations (see Kemper [9]).

**Definition 3.** A generic extension \( E \supset \mathcal{F} \) for \( G \) is called **descent generic** when the following condition holds: for any differential field \( F \) with field of constants \( C \) there is a PVE \( E \supset F \) with group \( H \leq G \) if and only if there are \( f_i \in F \) such that the matrix \( A(f_1, \ldots, f_k) \) is well defined and the equation \( X' = XA(f_1, \ldots, f_k) \) gives rise to the extension \( E \supset F \).

**Theorem 1.** The extension \( \mathcal{F}(Y) \supset \mathcal{F} \) is a generic PVE for SO\(_n\). Furthermore, it is descent generic.
Proof. For convenience, we will use the double subscript notation $Y_{ii}$ for $a_i$, $i = 1, \ldots, n - 1$, and put $Y_{ij} = b_{ij}$, $i < j$. We then let $A(Y_{ij}) = B$.

Suppose that $E \supset F$ is a PVE with group $H \leq \text{SO}_n$. Let $X, X_H$ respectively denote generic points of $\text{SO}_n$ and $H$. Then $E = F(Y)$, where $Y = X_H P$ for some invertible matrix $P$ with coefficients in $F$. Moreover, there is an $F$-algebra homomorphism of coordinate rings

$$F[XP, \det(XP)^{-1}] ightarrow F[X_H P, \det(X_H P)^{-1}].$$

Since $X_H P$ is a generic point for an $H$-torsor we have that $XP$ is a generic point for an $\text{SO}_n$-torsor, and therefore the (twisted) Lie algebra associated with the $H$-torsor is contained in that for the $\text{SO}_n$-torsor. In turn, this implies that the generic point $Y$ satisfies an equation with matrix $B = A(f_{ij})$ for some $f_{ij} \in F$.

Likewise, a specialization $A(f_{ij})$ of $A(Y_{ij})$ with $f_{ij} \in F$ gives a derivation on the coordinate ring $F[XP, \det(XP)^{-1}]$ of an $\text{SO}_n$-torsor. When extended to the quotient field this derivation may have new constants. We get a PVE of $F$ by taking the quotient field of the factor ring

$$F[XP, \det(XP)^{-1}]/M,$$

where $M$ is a maximal differential ideal. The differential Galois group in this case is the closed subgroup of $\text{SO}_n$ consisting of those elements that stabilize $M$.

Finally, it is clear that a fundamental matrix for the equation $\eta' = \eta A(Y_{ij})$ specializes to one for $\eta' = \eta A(f_{ij})$ since, on the one hand, a solution of $\eta' = \eta A(Y_{ij})$ is given by a generic point $XP$ of the $\text{SO}_n$-torsor corresponding to the quadratic form

$$Q = \begin{pmatrix} Y_{11} & Y_{22} & \cdots & 1/Y_{11} \cdots Y_{n-1,n-1} \\ \sqrt{Y_{11}} & \sqrt{Y_{22}} & \cdots & \sqrt{1/Y_{11} \cdots Y_{n-1,n-1}} \end{pmatrix}$$

with

$$P = \begin{pmatrix} \sqrt{Y_{11}} \\ \sqrt{Y_{22}} \\ \cdots \\ \sqrt{1/Y_{11} \cdots Y_{n-1,n-1}} \end{pmatrix}$$

and $X$ a generic point of $\text{SO}_n$.

On the other hand, a solution of $\eta' = \eta A(f_{ij})$ is given by a generic point $XP(f_{ij})$ of the $\text{SO}_n$-torsor corresponding to the quadratic form

$$Q(f_{ij}) = \begin{pmatrix} f_{11} & f_{22} & \cdots & 1/f_{11} \cdots f_{n-1,n-1} \\ \sqrt{f_{11}} & \sqrt{f_{22}} & \cdots & \sqrt{1/f_{11} \cdots f_{n-1,n-1}} \end{pmatrix}$$

with

$$P(f_{ij}) = \begin{pmatrix} \sqrt{f_{11}} \\ \sqrt{f_{22}} \\ \cdots \\ \sqrt{1/f_{11} \cdots f_{n-1,n-1}} \end{pmatrix}.$$
Note. In the case of $SO_3$ we can exhibit a generic point using the classical Euler parametrization:

$$X = \frac{1}{x^2 + y^2 + z^2 + w^2} \begin{pmatrix} x^2 + y^2 - z^2 - w^2 & 2xw + 2yz & 2yw - 2xz \\ 2yz - 2xw & x^2 - y^2 + z^2 - w^2 & 2xy + 2zw \\ 2xw + 2yw & 2zw - 2xy & x^2 - y^2 - z^2 + w^2 \end{pmatrix},$$

obtained by interpreting the quaternion $x + yi + zj + wk$ as an isometry by conjugation on the quadratic space with basis $i, j, k$, where $x, y, z$ and $w$ are indeterminates [2, Theorem 3, Chapter 3]. A generic point for the torsor is then

$$Y = XP = \frac{1}{x^2 + y^2 + z^2 + w^2} \begin{pmatrix} (x^2 + y^2 - z^2 - w^2)\sqrt{a} & 2(xw + yz)\sqrt{b} & 2(yw - xz)/\sqrt{ab} \\ 2(yz - xw)\sqrt{a} & (x^2 - y^2 + z^2 - w^2)\sqrt{b} & 2(xy + zw)/\sqrt{ab} \\ 2(xz + yw)\sqrt{a} & 2(zw - xy)\sqrt{b} & (x^2 - y^2 - z^2 + w^2)/\sqrt{ab} \end{pmatrix}.$$

Clearly, this matrix permits specialization of $a$ and $b$ to any non-zero values.

Remark. Observe that when the $f_{ii}$ are all 1, the matrix $\mathcal{A}(f_{ij})$ then has the form

$$\begin{pmatrix} 0 & f_{12} & f_{13} & \cdots & f_{1n} \\ -f_{12} & 0 & f_{23} & \cdots & f_{2n} \\ -f_{13} & -f_{23} & 0 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_{1n} & -f_{2n} & -f_{3n} & \cdots & 0 \end{pmatrix} \in \text{Lie}(SO_n).$$

Therefore this situation corresponds to the trivial torsor case. In general, if the $f_{ii}$ are (not all equal) constants, the torsor associated to the quadratic form will still be trivial and the specialized matrix will be in a Lie algebra isomorphic to $\text{Lie}(SO_n)$.

5. Remarks on the general case

In general, when the matrices $P$ parametrizing the $G$-torsors are not known, it will not be possible to carry out the same kind of explicit construction done here for $SO_n$. In such a situation we can use the generic extension relative to the trivial torsor [6, Definition 3.1, Theorem 3.3] and obtain the extensions corresponding to non-trivial $G$-torsors indirectly:

Assume that $G$ is connected and let $E \supset F$ be a generic extension for $G$ relative to the trivial $G$-torsor, with equation $Z' = \mathcal{A}(Y_i)Z$. 

Theorem 2. Let $F$ be a differential field with field of constants $C$. There is a PVE $E \supset F$ with differential Galois group $H \leq G$ if and only if there are a finite extension $k \supset F$, a matrix $P$ with coefficients in $k$ and a specialization $Y_i \mapsto f_i \in k$, such that the equation $Z' = Z(P^{-1}A(Y_i)P + P^{-1}P')$ gives rise to the extension $E \supset F$.

Proof. As before, we let $X$ denote a generic point for $G$ and write $Y = XP$ for a generic point of the $G$-torsor with $E = F(Y)$. The proof then follows from the description of the twisted Lie algebras [7, Section 3] and the Trivialization Lemma shown in Section 3 of this paper. \qed
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REFERENCES