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# ON GENERIC DIFFERENTIAL $\mathrm{SO}_n$-EXTENSIONS

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ABSTRACT. Let $\mathcal{C}$ be an algebraically closed field with trivial derivation and let $\mathcal{F}$ denote the differential rational field $\mathcal{C}\langle Y_{ij}\rangle$, with $Y_{ij}$, $1 \leq i \leq n-1$, $1 \leq j \leq n$, $i \leq j$, differentially independent indeterminates over $\mathcal{C}$. We show that there is a Picard-Vessiot extension $\mathcal{E} \supset \mathcal{F}$ for a matrix equation $X' = X\mathcal{A}(Y_{ij})$, with differential Galois group $\mathrm{SO}_n$, with the property that if $F$ is any differential field with field of constants $\mathcal{C}$, then there is a Picard-Vessiot extension $E \supset F$ with differential Galois group $H \leq \mathrm{SO}_n$ if and only if there are $f_{ij} \in F$ with $\mathcal{A}(f_{ij})$ well defined and the equation $X' = X\mathcal{A}(f_{ij})$ giving rise to the extension $E \supset F$.

## 1. INTRODUCTION

Let $\mathcal{C}$ denote an algebraically closed field with trivial derivation, $G$ a linear algebraic group over $\mathcal{C}$, and $\mathfrak{gl}_m(\cdot)$ the Lie algebra of $m \times m$ matrices with coefficients in some specified field. The short form 'Picard-Vessiot $G$-extension' (or sometimes 'PVE with group $G$') will be used for 'Picard-Vessiot extension (PVE) with differential Galois group isomorphic to $G$'. We consider the differential rational field $\mathcal{F} = \mathcal{C}\langle Z_1, \dots, Z_k\rangle$, where $Z_1, \dots, Z_k$ are differentially independent indeterminates over $\mathcal{C}$.

**Definition 1.** A Picard-Vessiot $G$-extension $\mathcal{E} \supset \mathcal{F}$ for the equation $X' = X\mathcal{A}(Z_1, \dots, Z_k)$, with $\mathcal{A}(Z_1, \dots, Z_k) \in \mathfrak{gl}_m(\mathcal{F})$ for some $m$, is said to be *a generic extension for $G$* if for every Picard-Vessiot $G$-extension $E \supset F$ there is a specialization $Z_i \to f_i \in F$, such that the equation $X' = X\mathcal{A}(f_1, \dots, f_k)$ gives rise to $E \supset F$ and any fundamental solution matrix maps to one for the specialized equation.

Note that by making the assumption that $G = G(\mathcal{C})$, we are also assuming that the base field of a Picard-Vessiot $G$-extension and the extension itself have field of constants $\mathcal{C}$.

In this paper we produce generic extensions for the special orthogonal groups $\mathrm{SO}_n$, $n \geq 3$. For $n = 2$ the group is isomorphic to the (cohomologically trivial) multiplicative group, a case already studied in [5].

The construction that we provide is based on Kolchin's Structure Theorem, which describes the possible Picard-Vessiot $G$-extensions of a differential field $F$ as function fields of $F$-irreducible $G$-torsors [11, Theorem 5.12], [12, Theorem 1.28]. The isomorphism classes of $G$-torsors, in turn, are in bijective correspondence with the elements of the first Galois cohomology set $H^1(F, G)$ [13, 15]. The latter is

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a particularly convenient feature since for the special orthogonal groups the first cohomology can be described in terms of regular quadratic forms of discriminant 1 (cf. [7]).

In previous work the first author has studied generic extensions in two special situations. The first was when $G$ is connected and the extension is the function field of the trivial $G$-torsor (cf. [5]). The second was when $G$ is the semidirect product $H \ltimes G^0$ of its connected component by a finite group $H$ and the extensions are the function fields of $F$-irreducible $G$-torsors of the form $W \times G^0$, where $W$ is an $F$-irreducible $H$-torsor (cf. [6]).

In the present paper we turn our attention to the general case, that is, when $H^1(F, G)$ is not necessarily trivial. In [7] we showed that in such a situation, it might be possible to find a Picard-Vessiot $G$-extension of $F$ that is the function field of a non-trivial torsor. We will use the machinery developed there and a version of a method to construct generic extensions from [5] to attack this general situation when $G$ is the special orthogonal group $\mathrm{SO}_n$, $n \geq 3$. With the description of the $\mathrm{SO}_n$-torsors in terms of regular quadratic forms of discriminant 1 at our disposal we can provide a good description of the twisted Lie algebras associated with the torsors [7], a key ingredient of our construction.

Having a good grasp of the torsors also allows us to show that this extension fully descends to subgroups of $\mathrm{SO}_n$, that is, *there is a specialization of the parameters over the base field $F$ yielding a Picard-Vessiot $H$-extension if and only if $H \leq \mathrm{SO}_n$.*

Finally, we discuss how to proceed with connected groups in general, when a good description of the torsors is not available. In this case a generic extension relative to the trivial torsor along with the Trivialization Lemma from Section 3 allows a (not so explicit but quite similar) construction in which the specialization of the parameters takes place over a finite extension of $F$ instead of $F$.

All the differential fields that we consider are of characteristic zero and have an algebraically closed field of constants. We keep the notations $\mathcal{C}$ and $F$ introduced above.

## 2. Generic extension vs. generic equation

The $\mathrm{SO}_n$ case is included among the groups studied by Goldman [3] and Bhandari and Sankaran [1].

**Definition 2** (Goldman [3])**.** Let $G$ be a linear algebraic group over $\mathcal{C}$ and assume that a faithful representation in $\mathrm{GL}_n(\mathcal{C})$ is given. Let $L(t, y) = Q_0(t_1, \ldots, t_r)y^{(n)} + \cdots + Q_n(t_1, \ldots, t_r)y \in \mathcal{C}\{t_1, \ldots, t_r, y\}$ and write $(\pi_1, \ldots, \pi_n)$ for a fundamental system of zeros of $L(t, y)$ such that $\mathcal{C}\langle t_1, \ldots, t_r, \pi_1, \ldots, \pi_n\rangle$ is a PVE of $\mathcal{C}\langle t_1, \ldots, t_r\rangle$ with group $G$. Then $L(t, y) = 0$ will be called a *generic equation with group $G$* if:

(1) $t_1, \ldots, t_r$ are differentially independent over $\mathcal{C}$, and $\mathcal{C}\langle t_1, \ldots, t_r\rangle \subset \mathcal{C}\langle \pi_1, \ldots, \pi_n\rangle$.

(2) For every specialization $(t_1, \ldots, t_r, \pi_1, \ldots, \pi_n) \to (\bar{t}_1, \ldots, \bar{t}_r, \bar{\pi}_1, \ldots, \bar{\pi}_n)$ over $\mathcal{C}$ such that $\mathcal{C}\langle \bar{t}_1, \ldots, \bar{t}_r, \bar{\pi}_1, \ldots, \bar{\pi}_n\rangle$ is a PVE of $\mathcal{C}\langle \bar{t}_1, \ldots, \bar{t}_r\rangle$ and the field of constants of the latter is $\mathcal{C}$, the differential Galois group of this extension is a subgroup of $G$.

(3) If $(\omega_1, \ldots, \omega_n)$ is a fundamental system of zeros of $L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y \in F\{y\}$, where $F$ is any differential field with field of constants $\mathcal{C}$, and $F\langle \omega_1, \ldots, \omega_n\rangle$ is a PVE of $F$ with differential Galois group $H \leq G$,

then there exists a specialization $(t_1, \ldots, t_r) \to (\bar{t}_1, \ldots, \bar{t}_r)$ over $F$ with $\bar{t}_i \in F$ such that $Q_o(\bar{t}_1, \ldots, \bar{t}_r) \neq 0$ and

$$a_i = Q_i(\bar{t}_1, \ldots, \bar{t}_r) Q_o^{-1}(\bar{t}_1, \ldots, \bar{t}_r).$$

Goldman shows that a necessary condition for such an equation to exist is that the number of parameters $r$ equals the order $n$ of the equation [3, Lemma 1, p. 343]. The groups studied in that paper include $\mathrm{GL}_n$, $\mathrm{SL}_n$ as well as the orthogonal and symplectic groups.

Now, let $G$ act on $\mathcal{C}\langle y_1, \ldots, y_n\rangle$, where $y_1, \ldots, y_n$ are differentially independent indeterminates over $\mathcal{C}$, by $\sigma(y_i) = \sum_{i=1}^n c_{ij} y_j$ for $\sigma = (c_{ij}) \in G(\mathcal{C}) \subset \mathrm{GL}_n(\mathcal{C})$. Then

$$P_i = \frac{W_i(y_1, \ldots, y_n)}{W_0(y_1, \ldots, y_n)} \qquad (i = 1, \ldots, n),$$

where

$$W_i = (-1)^i \begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-i-1)} & & y_n^{(n-i-1)} \\ y_1^{(n-i+1)} & & y_n^{(n-i+1)} \\ \vdots & & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

are invariant under the $G$ action.

The procedure used by Goldman for the groups above first finds $n$ differentially independent generators $t_1, \ldots, t_n$ over $\mathcal{C}$ of the fixed field $\mathcal{C}\langle y_1, \ldots, y_n\rangle^G$ and $n+1$ differential polynomials $Q_0(t_1, \ldots, t_n), \ldots, Q_n(t_1, \ldots, t_n) \in \mathcal{C}\{t_1, \ldots, t_n\}$ with

$$P_i = \frac{Q_i(t_1, \ldots, t_n)}{Q_0(t_1, \ldots, t_n)} \qquad (i = 1, \ldots, n).$$

He then shows that a generic equation with group $G$ is given by

$$L(t, y) = Q_0(t_1, \ldots, t_n) y^{(n)} + \cdots + Q_n(t_1, \ldots, t_n) y = 0. \tag{2.1}$$

This method, however, fails to produce a generic equation for $G = \mathrm{SO}_3$ as [3, Example 3, p. 355] illustrates.

Bhandari and Sankaran [1] proved that (2.1) is generic for the special orthogonal groups in a weaker sense, that is, replacing (3) in Goldman's definition with the following:

(3′) If $F$ is a differential field with field of constants $\mathcal{C}$ and $E$ is a PVE of $F$ with differential Galois group $H \leq G$, then there exists a linear differential equation

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0, \qquad a_i \in F$$

such that $Q_o(\bar{t}_1, \ldots, \bar{t}_r) \neq 0$, $a_i = Q_i(\bar{t}_1, \ldots, \bar{t}_r) Q_o^{-1}(\bar{t}_1, \ldots, \bar{t}_r)$, $i = 1, \ldots, n$, for suitable $\bar{t}_i \in F$ and $E = F\langle \omega_1, \ldots, \omega_n\rangle$ for a fundamental system of zeros of $L(y)$.

There are, however, some key differences in our approaches. In constructing their equations, both Goldman and Bhandari-Sankaran start with the differential rational field $\mathcal{F} = \mathcal{C}\langle y_1, \ldots, y_n\rangle$, where $n$ is the order of the equation, and find the differential fixed field $\mathcal{C}\langle y_1, \ldots, y_n\rangle^G$. We start instead with $\mathcal{F}$ as our base field and show that $\mathcal{F}\langle Y\rangle$, where $Y$ is a generic point of a "general" $G$-torsor, is a generic PVE in the sense of Definition 1. Furthermore, it satisfies descent conditions analogous to (2) and (3′) above. In our case, the number $n$ of parameters is given by the dimension of the group and the description of the torsors, so it is independent of the representation of $G$ in a $\mathrm{GL}_m$. By using a *general derivation* in the function field of our special $G$-torsor (that is, a typical element in the twisted Lie algebra) the specialization of our parameters comes in a very natural and painless fashion, whereas in the case of the generic equations in [1, 3], showing that $Q_0(t_1, \dots, t_n) \neq 0$ is quite involved.

In connection with the previous notions of generic equation [1, 3], Juan and Magid [8] study the *ring of generic solutions for a linear monic order n equation*, that is, $\mathcal{R} = \mathcal{C}\{P_1, \dots, P_n\} \otimes_{\mathcal{C}} \mathcal{C}[y_i^{(j)}, 1 \leq i \leq n, 0 \leq j \leq n-1][w_0^{-1}]$, where $P_i$, $y_i$, $1 \leq i \leq n$, and $w_0$, are as above, with the $\mathrm{GL}_n(\mathcal{C})$ action extended from the linear action on $V = \mathcal{C}y_1 + \cdots + \mathcal{C}y_n$ using the $\mathcal{C}$-basis $y_1, \dots, y_n$. The ring $\mathcal{R}$ has the following properties:

Assume that $E \supset F$ is a Picard-Vessiot $G$-extension and that $G$ has a faithful representation $\rho$ in $\mathrm{GL}_n$. Then there is a differential homomorphism $\Psi : R \to F$ such that

1. $E$ is the quotient field of $F\Psi(\mathcal{R})$;
2. $E \supset F$ is a PVE for
$$L(Y) = Y^{(n)} + \Psi(P_1)Y^{(n-1)} + \cdots + \Psi(P_n)Y^{(0)};$$
3. $\Psi$ is $G$-equivariant, so $\Psi(\mathcal{R}^G) \subset E^G = F$.

Conversely, assume that $G$ is an observable subgroup of $\mathrm{GL}_n$ and let $\phi : \mathcal{R}^G \to F$ be a differential $F$-algebra homomorphism with restriction $\alpha$ to $\mathcal{R}^{\mathrm{GL}_n}$. Let $P$ be a maximal differential ideal of $R = F \otimes_\alpha \mathcal{R}$ whose inverse image in $\mathcal{R}$ contains the kernel of $\phi$, and let $E$ be the fraction field of $R/P$. Then $E$ is a PVE of $F$ with differential Galois group contained in $G$.

The special orthogonal groups are observable (see [4]) and therefore satisfy the above conditions. We point out that in our construction the coordinate ring $\mathcal{C}\{Y_{ij}\}[Y, 1/\det(Y)]$, where $Y$ is a generic point of a general $\mathrm{SO}_n$-torsor, has properties similar to that of the ring $\mathcal{R}$.

The work in [1, 3, 8] describes equations given by linear differential operators attached to a representation of the differential Galois group $G$ in $\mathrm{GL}_n$. Our work describes matrix equations with group $G$ in connection with the structure of the Picard-Vessiot $G$-extensions.

## 3. $\mathrm{SO}_n$-extensions

In [7] we saw that every $F$-irreducible $\mathrm{SO}_n$-torsor has a generic point of the form $Y = XP$, where $X$ is a generic point for $\mathrm{SO}_n$ and

$$P = \begin{pmatrix} \sqrt{a_1} & & & \\ & \sqrt{a_2} & & \\ & & \ddots & \\ & & & \sqrt{a_n} \end{pmatrix},$$

for $a_i \in F^*$ with $a_1 \cdots a_n = 1$ and the roots chosen to have product 1 as well. A PVE of $F$ with group $\mathrm{SO}_n$ corresponding to this torsor, if any, equals the function field $F(Y)$ of the torsor and has derivation given by $Y' = YB$, where the matrix $B$

is of the form

$$\begin{pmatrix} \frac{a_1'}{2a_1} & b_{12} & b_{13} & \cdots & b_{1n} \\ -\frac{a_1}{a_2}b_{12} & \frac{a_2'}{2a_2} & b_{23} & \cdots & b_{2n} \\ -\frac{a_1}{a_3}b_{13} & -\frac{a_2}{a_3}b_{23} & \frac{a_3'}{2a_3} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_1}{a_n}b_{1n} & -\frac{a_2}{a_n}b_{2n} & -\frac{a_3}{a_n}b_{3n} & \cdots & \frac{a_n'}{2a_n} \end{pmatrix}$$

for $b_{ij} \in F$, $1 \le i \le n-1$, $2 \le j \le n$ and $a_i \in F^*$ as before. An explicit example was given there, with $Y$ corresponding to a non-trivial torsor, by making the simplifying assumption that $b_{i,i+1} = a_i$. We point out that with that assumption, the number of parameters used in [7] to produce a PVE associated with a non-trivial torsor is $\frac{1}{2}n(n-1)$, the dimension of $\mathrm{SO}_n$.

Since our goal here is to produce a generic extension we need to modify that example in order to retain the $\frac{1}{2}(n-1)(n+2)$ parameters in the matrix $B$.

We assume that $a_1, \ldots, a_{n-1}, b_{12}, \ldots, b_{n-1,n}$ are differentially independent indeterminates over $\mathcal{C}$ and let $\mathcal{F} = \mathcal{C}\langle a_1, \ldots, a_{n-1}, b_{12}, \ldots, b_{n-1,n}\rangle$. We first show that the equation $\eta' = \eta A$ over the algebraic closure $\bar{\mathcal{F}}$ of $\mathcal{F}$, with coefficient matrix

$$A = \begin{pmatrix} 0 & \frac{\sqrt{a_1}}{\sqrt{a_2}}b_{12} & \frac{\sqrt{a_1}}{\sqrt{a_3}}b_{13} & \cdots & \frac{\sqrt{a_1}}{\sqrt{a_n}}b_{1n} \\ -\frac{\sqrt{a_1}}{\sqrt{a_2}}b_{12} & 0 & \frac{\sqrt{a_2}}{\sqrt{a_3}}b_{23} & \cdots & \frac{\sqrt{a_2}}{\sqrt{a_n}}b_{2n} \\ -\frac{\sqrt{a_1}}{\sqrt{a_3}}b_{13} & -\frac{\sqrt{a_2}}{\sqrt{a_3}}b_{23} & 0 & \cdots & \frac{\sqrt{a_3}}{\sqrt{a_n}}b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\sqrt{a_1}}{\sqrt{a_n}}b_{1n} & -\frac{\sqrt{a_2}}{\sqrt{a_n}}b_{2n} & -\frac{\sqrt{a_3}}{\sqrt{a_n}}b_{3n} & \cdots & 0 \end{pmatrix}$$

has differential Galois group $\mathrm{SO}_n$. From this it will follow that the corresponding equation $\eta' = \eta B$ over $\mathcal{F}$ has the same group.

Let $Z_{ij} = \sqrt{a_i}/\sqrt{a_j}b_{ij}$, $1 \le i \le n-1$, $2 \le j \le n$, $i < j$. Clearly the $Z_{ij}$ are differentially independent over $\mathcal{C}$ since all the $a_i$ and $b_{ij}^2$ are in the differential field $\mathcal{L} = \mathcal{C}\langle a_1, \ldots, a_{n-1}, Z_{12}, \ldots, Z_{n-1,n}\rangle$, which forces the differential transcendence degree [10, Definition 3.2.33 and Theorem 5.4.12] of $\mathcal{L}$ over $\mathcal{C}$ to be $\frac{1}{2}(n-1)(n+2)$.

Now, since $A = \sum_{i=1}^{n-1}\sum_{j=i+2}^{n} Z_{ij}A_{ij}$, where $\{A_{ij}\}$ is the basis of $\mathrm{Lie}(\mathrm{SO}_n)$ consisting of the antisymmetric matrices with 1 in the $(i,j)$-entry, $-1$ in the $(j,i)$-entry and 0 otherwise, by [5, Theorem 4.1.2] it then follows that $\mathcal{L}(\mathrm{SO}_n) \supset \mathcal{L}$ is a PVE with group $\mathrm{SO}_n$ for the equation $X' = XA$.

Since $a_i, b_{ij}^2 \in \mathcal{L}$ we have that $a_i, b_{ij} \in \bar{\mathcal{L}}$ and thus $\bar{\mathcal{F}} = \bar{\mathcal{L}}$. Therefore, $\bar{\mathcal{F}}(\mathrm{SO}_n) \supset \mathcal{L}(\mathrm{SO}_n)$ is an algebraic extension. Since the field of constants of $\mathcal{L}(\mathrm{SO}_n)$ is the algebraically closed field $\mathcal{C}$, $\bar{\mathcal{F}}(\mathrm{SO}_n)$ must have no new constants and $\bar{\mathcal{F}}(\mathrm{SO}_n) \supset \bar{\mathcal{F}}$ is a PVE with group $\mathrm{SO}_n$.

The discussion in [7, Section 4] implies that the matrix

$$B = \begin{pmatrix} \frac{a_1'}{2a_1} & b_{12} & b_{13} & \dots & b_{1n} \\ -\frac{a_1}{a_2}b_{12} & \frac{a_2'}{2a_2} & b_{23} & \dots & b_{2n} \\ -\frac{a_1}{a_3}b_{13} & -\frac{a_2}{a_3}b_{23} & \frac{a_3'}{2a_3} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_1}{a_n}b_{1n} & -\frac{a_2}{a_n}b_{2n} & -\frac{a_3}{a_n}b_{3n} & \dots & \frac{a_n'}{2a_n} \end{pmatrix}$$

defines a derivation on the coordinate ring $T = \mathcal{F}[Y]$ of the $\mathrm{SO}_n$-torsor corresponding to the quadratic form given by the matrix

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

which by [7, Lemma 1] is non-trivial.

Since $\bar{\mathcal{F}}(Y) = \bar{\mathcal{F}}(X)$, as a differential field it will be isomorphic to $\bar{F}(\mathrm{SO}_n)$. Therefore, the field of constants of $\bar{\mathcal{F}}(Y)$ is $\mathcal{C}$. In particular, this implies that $\mathcal{F}(Y) \supset \mathcal{F}$ is a no new constant extension. This shows that the function field of the (non-trivial) $\mathrm{SO}_n$-torsor corresponding to $Y$ is a PVE of $\mathcal{F}$ with group $\mathrm{SO}_n$.

We point out for later use that the previous argument can be shown in a more general setting:

**Trivialization Lemma.** *Let $E \supset F$ be a Picard-Vessiot $G$-extension with $G$ connected. Then there are a finite extension $k \supset F$ and a Picard-Vessiot $G$-extension $K = kE$ of $k$ such that $K = k(G)$.*

In other words, if there is a PVE of $F$ with group $G$, then the trivial $G$-torsor can be realized over a finite extension of $F$. Although this is a known result (see [14, p. 142, Corollary]), for the convenience of the reader we include a short proof using the tools that we develop here.

*Proof.* Let $X$ be a generic point of $G$. Then $E = F(Y)$ where $Y = XP$, for a matrix $P$ with coefficients in $\bar{F}$ [7, Section 3]. Let $k$ denote the field generated over $F$ by the entries of $P$. Then $k(X) = k(Y) \supset F(Y)$ is an algebraic extension. Therefore, $k(G) = k(X) \supset k$ is a no new constant extension and thus a Picard-Vessiot $G$-extension. Clearly, $K = k(X) = kE$. □

## 4. Generic extensions

First we introduce the following notion, analogous to one for generic polynomial equations (see Kemper [9]).

**Definition 3.** A generic extension $\mathcal{E} \supset \mathcal{F}$ for $G$ is called *descent generic* when the following condition holds: for any differential field $F$ with field of constants $\mathcal{C}$ there is a PVE $E \supset F$ with group $H \leq G$ if and only if there are $f_i \in F$ such that the matrix $\mathcal{A}(f_1, \dots, f_k)$ is well defined and the equation $X' = X\mathcal{A}(f_1, \dots, f_k)$ gives rise to the extension $E \supset F$.

**Theorem 1.** *The extension $\mathcal{F}(Y) \supset \mathcal{F}$ is a generic PVE for* $\mathrm{SO}_n$. *Furthermore, it is descent generic.*

*Proof.* For convenience, we will use the double subscript notation $Y_{ii}$ for $a_i$, $i = 1, \ldots, n-1$, and put $Y_{ij} = b_{ij}$, $i < j$. We then let $\mathcal{A}(Y_{ij}) = B$.

Suppose that $E \supset F$ is a PVE with group $H \leq \mathrm{SO}_n$. Let $X$, $X_H$ respectively denote generic points of $\mathrm{SO}_n$ and $H$. Then $E = F(Y)$, where $Y = X_H P$ for some invertible matrix $P$ with coefficients in $\bar{F}$. Moreover, there is an $F$-algebra homomorphism of coordinate rings

$$F[XP, \det(XP)^{-1}] \twoheadrightarrow F[X_H P, \det(X_H P)^{-1}].$$

Since $X_H P$ is a generic point for an $H$-torsor we have that $XP$ is a generic point for an $\mathrm{SO}_n$-torsor, and therefore the (twisted) Lie algebra associated with the $H$-torsor is contained in that for the $\mathrm{SO}_n$-torsor. In turn, this implies that the generic point $Y$ satisfies an equation with matrix $\tilde{B} = \mathcal{A}(f_{ij})$ for some $f_{ij} \in F$.

Likewise, a specialization $\mathcal{A}(f_{ij})$ of $\mathcal{A}(Y_{ij})$ with $f_{ij} \in F$ gives a derivation on the coordinate ring $F[XP, \det(XP)^{-1}]$ of an $\mathrm{SO}_n$-torsor. When extended to the quotient field this derivation may have new constants. We get a PVE of $F$ by taking the quotient field of the factor ring

$$F[XP, \det(XP)^{-1}]/M,$$

where $M$ is a maximal differential ideal. The differential Galois group in this case is the closed subgroup of $\mathrm{SO}_n$ consisting of those elements that stabilize $M$.

Finally, it is clear that a fundamental matrix for the equation $\eta' = \eta\mathcal{A}(Y_{ij})$ specializes to one for $\eta' = \eta\mathcal{A}(f_{ij})$ since, on the one hand, a solution of $\eta' = \eta\mathcal{A}(Y_{ij})$ is given by a generic point $XP$ of the $\mathrm{SO}_n$-torsor corresponding to the quadratic form

$$Q = \begin{pmatrix} Y_{11} & & & \\ & Y_{22} & & \\ & & \ddots & \\ & & & 1/Y_{11}\ldots Y_{n-1,n-1} \end{pmatrix}$$

with

$$P = \begin{pmatrix} \sqrt{Y_{11}} & & & \\ & \sqrt{Y_{22}} & & \\ & & \ddots & \\ & & & \sqrt{1/Y_{11}\ldots Y_{n-1,n-1}} \end{pmatrix}$$

and $X$ a generic point of $\mathrm{SO}_n$.

On the other hand, a solution of $\eta' = \eta\mathcal{A}(f_{ij})$ is given by a generic point $XP(f_{ij})$ of the $\mathrm{SO}_n$-torsor corresponding to the quadratic form

$$Q(f_{ij}) = \begin{pmatrix} f_{11} & & & \\ & f_{22} & & \\ & & \ddots & \\ & & & 1/f_{11}\cdots f_{n-1,n-1} \end{pmatrix}$$

with

$$P(f_{ij}) = \begin{pmatrix} \sqrt{f_{11}} & & & \\ & \sqrt{f_{22}} & & \\ & & \ddots & \\ & & & \sqrt{1/f_{11}\cdots f_{n-1,n-1}} \end{pmatrix}. \qquad \square$$

**Note.** In the case of $\mathrm{SO}_3$ we can exhibit a generic point using the classical *Euler parametrization*:

$$X = \frac{1}{x^2+y^2+z^2+w^2}\begin{pmatrix} x^2+y^2-z^2-w^2 & 2xw+2yz & 2yw-2xz \\ 2yz-2xw & x^2-y^2+z^2-w^2 & 2xy+2zw \\ 2xz+2yw & 2zw-2xy & x^2-y^2-z^2+w^2 \end{pmatrix},$$

obtained by interpreting the quaternion $x+yi+zj+wk$ as an isometry by conjugation on the quadratic space with basis $i, j, k$, where $x$, $y$, $z$ and $w$ are indeterminates [2, Theorem 3, Chapter 3]. A generic point for the torsor is then

$$Y = XP = \frac{1}{x^2+y^2+z^2+w^2} \times$$
$$\begin{pmatrix} (x^2+y^2-z^2-w^2)\sqrt{a} & 2(xw+yz)\sqrt{b} & 2(yw-xz)/\sqrt{ab} \\ 2(yz-xw)\sqrt{a} & (x^2-y^2+z^2-w^2)\sqrt{b} & 2(xy+zw)/\sqrt{ab} \\ 2(xz+yw)\sqrt{a} & 2(zw-xy)\sqrt{b} & (x^2-y^2-z^2+w^2)/\sqrt{ab} \end{pmatrix}.$$

Clearly, this matrix permits specialization of $a$ and $b$ to any non-zero values.

*Remark.* Observe that when the $f_{ii}$ are all 1, the matrix $\mathcal{A}(f_{ij})$ then has the form

$$\begin{pmatrix} 0 & f_{12} & f_{13} & \cdots & f_{1n} \\ -f_{12} & 0 & f_{23} & \cdots & f_{2n} \\ -f_{13} & -f_{23} & 0 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_{1n} & -f_{2n} & -f_{3n} & \cdots & 0 \end{pmatrix} \in \mathrm{Lie}(\mathrm{SO}_n).$$

Therefore this situation corresponds to the trivial torsor case. In general, if the $f_{ii}$ are (not all equal) constants, the torsor associated to the quadratic form will still be trivial and the specialized matrix will be in a Lie algebra isomorphic to $\mathrm{Lie}(\mathrm{SO}_n)$.

## 5. Remarks on the general case

In general, when the matrices $P$ parametrizing the $G$-torsors are not known, it will not be possible to carry out the same kind of explicit construction done here for $\mathrm{SO}_n$. In such a situation we can use the generic extension relative to the trivial torsor [6, Definition 3.1, Theorem 3.3] and obtain the extensions corresponding to non-trivial $G$-torsors indirectly:

Assume that $G$ is connected and let $\mathcal{E} \supset \mathcal{F}$ be a generic extension for $G$ relative to the trivial $G$-torsor, with equation $Z' = \mathcal{A}(Y_i)Z$.

**Theorem 2.** Let $F$ be a differential field with field of constants $\mathcal{C}$. There is a PVE $E \supset F$ with differential Galois group $H \leq G$ if and only if there are a finite extension $k \supset F$, a matrix $P$ with coefficients in $k$ and a specialization $Y_i \mapsto f_i \in k$, such that the equation $Z' = Z(P^{-1}\mathcal{A}(f_i)P + P^{-1}P')$ gives rise to the extension $E \supset F$.

*Proof.* As before, we let $X$ denote a generic point for $G$ and write $Y = XP$ for a generic point of the $G$-torsor with $E = F(Y)$. The proof then follows from the description of the twisted Lie algebras [7, Section 3] and the Trivialization Lemma shown in Section 3 of this paper. ∎

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