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EVERY m-PERMUTABLE VARIETY SATISFIES THE CONGRUENCE IDENTITY $\alpha\beta_h = \alpha\gamma_h$

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Abstract. It is known that congruence lattices of algebras in m-permutable varieties satisfy non-trivial identities; however, the identities discovered so far are rather artificial and seem to have little intrinsic interest.

We show here that every m-permutable variety satisfies the well-known and well-studied congruence identity $\alpha\beta_h = \alpha\gamma_h$. We also get a new condition equivalent to m-permutability.

1. Introduction

This paper is part of a program, initiated in the 19th century by Richard Dedekind in the case of groups, to study the arithmetic of congruences of algebraic structures. Our main result, Theorem 3.5, is a variation on Dedekind's well-known result that the congruence lattices of groups satisfy the modular law $\alpha\gamma_2 = \alpha\beta_2$.

Results on congruence identities satisfied by m-permutable varieties date back to the early 1980s. A. Day and J. B. Nation (see [J, Lemma 3.10]) showed that if some algebra $\mathbf{A}$ is $2m$-permutable and has a semilattice operation, then $\operatorname{Con} \mathbf{A}$ satisfies the identity $\alpha(\beta + \gamma) \leq \alpha\beta_{2m} + \alpha\gamma_{2m}$. As usual, β_n and γ_n are defined as follows:

$$\beta_0 = \gamma_0 = 0,$$
$$\beta_{n+1} = \beta + \alpha\gamma_n, \quad \gamma_{n+1} = \gamma + \alpha\beta_n.$$

G. Czédli [C] weakened to meet semidistributivity the assumption of the existence of a semilattice operation: he showed that an m-permutable variety $\mathcal{V}$ is congruence meet semidistributive if and only if for some n, $\mathcal{V}$ satisfies the congruence identity $\alpha(\beta + \gamma) = \alpha\beta_n$. He also proved the dual result.

D. Hobby and R. McKenzie [HMK, Theorem 9.19] showed that for every locally finite m-permutable variety $\mathcal{V}$ there is a non-trivial lattice identity satisfied by all congruence lattices of algebras in $\mathcal{V}$. Finally, the assumption that $\mathcal{V}$ is locally finite has been removed in [L1]; moreover, the identity found in [L1] depends only on m and does not depend on $\mathcal{V}$. More identities have been found in [L2]. However, the identities obtained in [L1, L2] are ad hoc and rather weak. For the most part,

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such identities simply state that a certain small interval in the congruence lattice is modular; they say almost nothing about the global shape of the congruence lattice.

In the present paper we show that every m-permutable variety satisfies a congruence identity similar to the identities found by A. Day, J. B. Nation and G. Czédli and mentioned at the beginning. It is not the case that all m-permutable varieties satisfy, say, the congruence identity $\alpha(\beta+\gamma)=\alpha\beta_n$, since there are non-semidistributive m-permutable varieties. However, we show here that every m-permutable variety satisfies the related congruence identity $\alpha\gamma_h=\alpha\beta_h$, for an appropriate h depending only on m (Theorem 3.5). Notice that the three identities $\alpha\gamma_h\leq\beta_h$, $\alpha\gamma_h=\alpha\beta_h$ and $\beta_h=\beta_{h+1}$ are lattice-theoretically equivalent (for $h>0$).

The terms β_n and γ_n are well known and have been frequently used in lattice theory and universal algebra (some authors use different indices, starting with $\beta_0=\beta$, $\gamma_0=\gamma$). B. Jónsson and I. Rival [JR] proved that a variety of lattices is (both meet and join) semidistributive if and only if for some n, it satisfies $\alpha(\beta+\gamma)=\alpha\beta_n=\alpha\gamma_n$, as well as the dual identity. The term β_n played a fundamental role in D. Hobby and R. McKenzie's deep analysis of finite algebras and locally finite varieties (see [HMK, Chapter 9]). For example, they proved that a locally finite variety of algebras is congruence meet semidistributive if and only if it satisfies the congruence inclusion $\alpha(\beta\circ\gamma)\subseteq\beta_n$ for some n.

In [L3] we showed that every congruence variety satisfying $\alpha\gamma_h=\alpha\beta_h$ satisfies more identities, which do not follow from it lattice-theoretically. This appears to be the first example of a non-trivial congruence implication involving identities weaker than modularity, and, together with the results presented here, confirms the importance of the identity $\alpha\gamma_h=\alpha\beta_h$. Furthermore, we have results suggesting that varieties satisfying the congruence identity $\alpha\gamma_h=\alpha\beta_h$ satisfy many of the good properties of m-permutable varieties (see also Problem 3.6). Notice that $\alpha\gamma_2=\alpha\beta_2$ is an identity equivalent to modularity; thus $\alpha\gamma_h=\alpha\beta_h$ can be seen as a generalization of modularity.

Let us mention that the dual identity $\alpha+\gamma^h=\alpha+\beta^h$, too, has proven particularly important. K. Kearnes [K1] showed that a locally finite variety $\mathcal{V}$ satisfies some non-trivial congruence identity expressible in the language of lattices if and only if there is k such that $\mathcal{V}$ satisfies the congruence identity $\alpha+\gamma^k=\alpha+\beta^k$. Thus, locally finite m-permutable varieties satisfy also the congruence identity $\alpha+\gamma^k=\alpha+\beta^k$, for some k, by the results proved here, or simply by [HMK, Theorem 9.19]. The k given by the proof seems to depend on $\mathcal{V}$, not just on m. The result we prove is stronger, since there are varieties which for some k satisfy the congruence identity $\alpha+\gamma^k=\alpha+\beta^k$, but for no n satisfy the congruence identity $\alpha\gamma_n=\alpha\beta_n$ (see [K1, p. 385], [L3, p. 606]). Let us also recall that, in the meantime, many results proved under the assumption of local finiteness have been proved without such an assumption.

Our proof of Theorem 3.5 splits into two neatly separated parts. In the first step, in Section 2, we get an identity similar to $\alpha\beta_h=\alpha\gamma_h$, except that the lattice operation $+$ is replaced by $\circ_3$ (see below for definitions). Theorem 3.5 is then obtained by a quite straightforward application of the commutator theory developed in [L1, L4]. Notice that Section 2 is commutator-free. At the end of each section some problems are stated.

Here is the notation we use. α, β denote *congruences* on some *algebra* $\mathbf{A}$. Join and meet in the lattice $\operatorname{Con}\mathbf{A}$ of all congruences of $\mathbf{A}$ are denoted, respectively, by $+$ and juxtaposition. We use juxtaposition also to denote intersection. The relational product is denoted by $\circ$, and $\alpha \circ_n \beta$ is a shorthand for $\alpha \circ \beta \circ \alpha \circ \beta \circ \dots$, with $n-1$ occurrences of $\circ$. Two congruences α, β are said to *m-permute* if and only if $\alpha \circ_m \beta = \beta \circ_m \alpha$ (thus, in particular, $\alpha + \beta = \alpha \circ_m \beta$). An algebra $\mathbf{A}$ is *m-permutable* if and only if every pair of congruences in $\mathbf{A}$ m-permutes. A variety $\mathcal{V}$ is *m-permutable* if and only if every algebra in $\mathcal{V}$ is m-permutable. 2-permutability is simply called *permutability.*

2. A nice property of m-permutable varieties

In this section we shall prove that every m-permutable variety satisfies the identity introduced in the following definition.

Definition 2.1. If $\alpha, \beta, \gamma, \delta$ are congruences on some algebra, and m is a natural number, we shall denote by (X_m) the following identity:

$$\begin{aligned} &\alpha(\beta \circ \alpha(\gamma \circ \alpha(\beta \circ \dots \alpha(\gamma^\bullet \circ \alpha(\beta^\bullet \circ \alpha\delta \circ \beta^\bullet) \circ \gamma^\bullet) \dots \circ \beta) \circ \gamma) \circ \beta) \\ &\qquad = \alpha(\gamma \circ \alpha(\beta \circ \alpha(\gamma \circ \dots \alpha(\beta^\bullet \circ \alpha(\gamma^\bullet \circ \alpha\delta \circ \gamma^\bullet) \circ \beta^\bullet) \dots \circ \gamma) \circ \beta) \circ \gamma) \end{aligned}$$

with exactly m open brackets (and exactly m closed brackets) on each side, where

$$\beta^\bullet = \beta, \gamma^\bullet = \gamma$$

if m is odd, and

$$\beta^\bullet = \gamma, \gamma^\bullet = \beta$$

if m is even.

If (a_0, b_0) belongs to the left-hand side of (X_m), then $a_0 \alpha b_0$, and there are elements a_1, b_1 such that $a_0 \beta a_1$, $b_1 \beta b_0$ and

$$(a_1, b_1) \in \alpha(\gamma \circ \alpha(\beta \circ \dots \alpha(\gamma^\bullet \circ \alpha(\beta^\bullet \circ \alpha\delta \circ \beta^\bullet) \circ \gamma^\bullet) \dots \circ \beta) \circ \gamma),$$

with $m-1$ open brackets. Repeating this argument m times, we get that (a_0, b_0) belongs to the left-hand side of (X_m) if and only if there are further elements $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_m$ such that

$$\begin{array}{lll} & a_i \alpha b_i, & \text{for } i = 0, \dots, m, \\ & a_m \delta b_m, & \\ a_i \beta a_{i+1}, & b_i \beta b_{i+1}, & \text{for } i \text{ even}, 0 \le i \le m-1, \\ a_i \gamma a_{i+1}, & b_i \gamma b_{i+1}, & \text{for } i \text{ odd}, 0 \le i \le m-1. \end{array}$$

The conditions asserting that (a_0, b_0) belongs to the right-hand side of (X_m) are similar, with β and γ interchanged.

The situation is better represented by a diagram:

$$
\begin{array}{ccc}
a_0 & \overset{\alpha}{\text{———}} & b_0 \\
\beta \big| & & \big| \beta \\
a_1 & \overset{\alpha}{\text{———}} & b_1 \\
\gamma \big| & & \big| \gamma \\
a_2 & \overset{\alpha}{\text{———}} & b_2 \\
\beta \big| & & \big| \beta \\
& \dots & \\
\gamma^\bullet \big| & & \big| \gamma^\bullet \\
a_{m-1} & \overset{\alpha}{\text{———}} & b_{m-1} \\
\beta^\bullet \big| & & \big| \beta^\bullet \\
a_m & \overset{\alpha\delta}{\text{———}} & b_m
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccc}
a_0 & \overset{\alpha}{\text{———}} & b_0 \\
\gamma \big| & & \big| \gamma \\
c_1 & \overset{\alpha}{\text{———}} & d_1 \\
\beta \big| & & \big| \beta \\
c_2 & \overset{\alpha}{\text{———}} & d_2 \\
\gamma \big| & & \big| \gamma \\
& \dots & \\
\beta^\bullet \big| & & \big| \beta^\bullet \\
c_{m-1} & \overset{\alpha}{\text{———}} & d_{m-1} \\
\gamma^\bullet \big| & & \big| \gamma^\bullet \\
c_m & \overset{\alpha\delta}{\text{———}} & d_m
\end{array}
$$

where, as above, $\beta^\bullet = \beta$, $\gamma^\bullet = \gamma$ if m is odd, and $\beta^\bullet = \gamma$, $\gamma^\bullet = \beta$ if m is even. (X_m) asserts that the pair (a_0, b_0) can be extended to a sequence (a_i, b_i) $(0 \leq i \leq m)$ satisfying the conditions represented in the left-hand side of the above diagram if and only if (a_0, b_0) can be extended to a sequence as in the right-hand side.

We say that the algebra $\mathbf{A}$ satisfies (X_m) if and only if (X_m) holds for every congruence $\alpha, \beta, \gamma, \delta$ of $\mathbf{A}$, and we say that a variety $\mathcal{V}$ satisfies (X_m) if and only if every algebra in $\mathcal{V}$ satisfies (X_m). Notice that (X_m) is not a *lattice* identity, due to the occurrence of composition in it.

Theorem 2.2. *If every subalgebra of* $\mathbf{A}^2$ *is* m*-permutable, then* $\mathbf{A}$ *satisfies* (X_m).

Proof. Suppose that every subalgebra of $\mathbf{A}^2$ is m-permutable, and $\alpha, \beta, \gamma, \delta \in \operatorname{Con} \mathbf{A}$.

Suppose that $a_0, b_0 \in \mathbf{A}$, and that (a_0, b_0) belongs to the left-hand side of (X_m). It is enough to show that (a_0, b_0) belongs to the right-hand side. The reverse inclusion is obtained by symmetry.

Since (a_0, b_0) belongs to the left-hand side of (X_m) we have elements $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_m \in \mathbf{A}$ as on the left-hand side of the diagram in Definition 2.1. We want to obtain elements $c_1, c_2, \dots, c_m$ and $d_1, d_2, \dots, d_m$ as on the right-hand side.

Let $\mathbf{B}$ be the congruence α, considered as a subalgebra of $\mathbf{A}^2$, that is, $\mathbf{B} = \{(a, b) | a, b \in A, \text{ and } a \alpha b\}$. Notice that the pairs $(a_0, b_0), (a_1, b_1), \dots, (a_m, b_m)$ belong to $\mathbf{B}$. Moreover, working in $\mathbf{B}$, $\big((a_i, b_i), (a_{i+1}, b_{i+1})\big) \in (\beta \times \beta)_{|\mathbf{B}}$ for i even, $0 \leq i < m$, and $\big((a_i, b_i), (a_{i+1}, b_{i+1})\big) \in (\gamma \times \gamma)_{|\mathbf{B}}$ for i odd, $0 \leq i < m$. Thus, $\big((a_0, b_0), (a_m, b_m)\big) \in (\beta \times \beta)_{|\mathbf{B}} \circ_m (\gamma \times \gamma)_{|\mathbf{B}}$. Since, by the assumption, $\mathbf{B}$ is m-permutable, we have $\big((a_0, b_0), (a_m, b_m)\big) \in (\gamma \times \gamma)_{|\mathbf{B}} \circ_m (\beta \times \beta)_{|\mathbf{B}}$. This means that in $\mathbf{B}$ there are pairs $(c_0, d_0) = (a_0, b_0)$, (c_1, d_1), (c_2, d_2), $\dots$, (c_{m-1}, d_{m-1}), $(c_m, d_m) = (a_m, b_m)$, such that $\big((c_i, d_i), (c_{i+1}, d_{i+1})\big) \in (\gamma \times \gamma)_{|\mathbf{B}}$ for i even, and $\big((c_i, d_i), (c_{i+1}, d_{i+1})\big) \in (\beta \times \beta)_{|\mathbf{B}}$ for i odd, $0 \leq i < m$.

Translating the above relations in the algebra $\mathbf{A}$ we get that $c_i \gamma c_{i+1}$ and $d_i \gamma d_{i+1}$ for i even, as well as $c_i \beta c_{i+1}$ and $d_i \beta d_{i+1}$ for i odd. Moreover, $c_i \alpha d_i$ for $0 \leq i \leq m$, by the definition of $\mathbf{B}$, and since $(c_i, d_i) \in \mathbf{B}$. Finally, $c_0 = a_0$, $d_0 = b_0$, and $c_m = a_m \delta b_m = d_m$; thus the elements c_i, d_i satisfy the desired relations. □

See [L5] for variations on Theorem 2.2.

If we do not care about the value assumed by the index, Property (X_m) characterizes m-permutability.

Proposition 2.3. *For every variety $\mathcal{V}$, the following are equivalent:*

(i) $\mathcal{V}$ is n-permutable for some n, and

(ii) $\mathcal{V}$ satisfies (X_m) for some m.

Proof. By Theorem 2.2, if $\mathcal{V}$ is n-permutable, then $\mathcal{V}$ satisfies (X_n); thus (i)⇒(ii) is proved.

For (ii)⇒(i), notice that every algebra satisfying (X_m) is $(2m-1)$-permutable: just take $\alpha = 1$ and $\delta = 0$ in (X_m). □

Problem 2.4. Is the relationship between n and m given by the above proof optimal?

As far as small values of m and n are concerned, we know that for no m is (X_m) equivalent to permutability. Moreover, (X_2) is equivalent to 3-permutability.

3. Applying commutator theory

We first recall the definitions of some commutators from [L1]. The actual definitions shall not be used in the present paper: we shall use only the properties stated in Theorem 3.2 below, as well as the trivial properties of

(monotonicity) $\alpha \leq \alpha'$ and $\beta \leq \beta'$ imply $[\alpha, \beta] \leq [\alpha', \beta']$,

(submultiplicativity) $[\alpha, \beta] \leq \alpha\beta$.

Definition 3.1. Let $\mathbf{A}$ be any algebra, and let $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$.

$M(\alpha, \beta)$ is the set of all *matrices* of the form

$$\begin{vmatrix} t(\overline{a}, \overline{b}) & t(\overline{a}, \overline{b}') \\ t(\overline{a}', \overline{b}) & t(\overline{a}', \overline{b}') \end{vmatrix}$$

where $\overline{a}, \overline{a}' \in A^n$, $\overline{b}, \overline{b}' \in A^m$, for some $m, n \geq 0$, t is an $(m+n)$-ary term operation of $\mathbf{A}$, and $\overline{a} \alpha \overline{a}'$, $\overline{b} \beta \overline{b}'$. Further, we set

$$K(\alpha, \beta; \gamma) = \left\{ (z, w) \middle| \begin{vmatrix} x & y \\ z & w \end{vmatrix} \in M(\alpha, \beta), \text{ for some } x\gamma y \right\},$$

$$[\alpha, \beta | 0] = 0_{\mathbf{A}}, \qquad [\alpha, \beta | n+1] = Cg(K(\alpha, \beta; [\alpha, \beta | n])),$$

where Cg means "the congruence generated by".

In the results that follow, $[\frac{m-1}{2}]$ denotes the *integer part* of $\frac{m-1}{2}$.

Theorem 3.2. *(i) If β and γ m-permute, then for every n, $[\beta + \gamma, \alpha | n] \leq \alpha\beta_{mn}$.*

(ii) If $\mathcal{V}$ is an m-permutable variety, then there is a ternary term d such that

$$d(b, b, a) \equiv a \equiv d(a, b, b) \pmod{[\alpha, \alpha | n]}$$

for every algebra $\mathbf{A} \in \mathcal{V}$, every congruence $\alpha \in \operatorname{Con} \mathbf{A}$ and elements $a \alpha b \in A$, and where $n = [\frac{m-1}{2}]$.

Clause (i) in Theorem 3.2 is from [L1, Lemma 1(i)]. Condition (ii) is an easy corollary of the proof of [T, Theorem 2], as noticed in [L1, Lemma 2, and Remark (c) on p. 162]. Full details are given in the proof of [L2, Theorem 1.2(c)]. Replace n, s, t, m there by, respectively, $m-2$, $[\frac{m-2}{2}]$, $[\frac{m-1}{2}]$, 1.

Proposition 3.3. *Let $m \geq 3$ and $\mathcal{V}$ be an m-permutable variety, and put $k = m[\frac{m-1}{2}]$. Then for all $j \geq k-1$ all algebras in $\mathcal{V}$ satisfy*

$$\beta_{j+1} = \beta_k \circ \alpha\gamma_j \circ \beta_k.$$

Proof. By Theorem 3.2(ii) and by [L4, Lemma 3.1(iii)] with $F(\delta) = [\delta, \delta|n]$, and $n = [\frac{m-1}{2}]$, we get

$$(*) \qquad \delta + \varepsilon = \big([\delta, \delta|n] + [\varepsilon, \varepsilon|n]\big) \circ \delta \circ \varepsilon \circ \big([\delta, \delta|n] + [\varepsilon, \varepsilon|n]\big)$$

for every pair of congruences δ and ε in every algebra in $\mathcal{V}$.

By Theorem 3.2(i), $[\beta+\gamma, \alpha|n] \leq \alpha\beta_{mn}$. Hence, by monotonicity, and since, for all j, $\alpha\gamma_j \leq \alpha(\beta+\gamma)$, we have $[\alpha\gamma_j, \alpha\gamma_j|n] \leq [\alpha(\beta+\gamma), \alpha(\beta+\gamma)|n] \leq [\beta+\gamma, \alpha|n] \leq \alpha\beta_k \leq \beta_k$, for all j.

Thus, by submultiplicativity and $(*)$ above,

$$\begin{aligned}
\beta_{j+1} &= \beta + \alpha\gamma_j \\
&= \big([\beta, \beta|n] + [\alpha\gamma_j, \alpha\gamma_j|n]\big) \circ \beta \circ \alpha\gamma_j \circ \big([\beta, \beta|n] + [\alpha\gamma_j, \alpha\gamma_j|n]\big) \\
&\subseteq \big(\beta + [\alpha\gamma_j, \alpha\gamma_j|n]\big) \circ \alpha\gamma_j \circ \big(\beta + [\alpha\gamma_j, \alpha\gamma_j|n]\big) \\
&\subseteq \beta_k \circ \alpha\gamma_j \circ \beta_k.
\end{aligned}$$

For the reverse inclusion, notice that, trivially, $\beta_{j+1} \geq \alpha\gamma_j$ and $\beta_{j+1} \geq \beta_k$, since $j \geq k-1$; thus $\beta_{j+1} \geq \beta_k + \alpha\gamma_j \supseteq \beta_k \circ \alpha\gamma_j \circ \beta_k$. □

Proposition 3.4. *Let $m \geq 3$ and $\mathcal{V}$ be an m-permutable variety, and put $k = m[\frac{m-1}{2}]$. Then for all $n > 0$ all algebras in $\mathcal{V}$ satisfy*

$$\beta_{k+n} = \beta_k \circ \alpha(\gamma_k \circ \alpha(\beta_k \circ \dots \alpha(\gamma_k^\bullet \circ \alpha(\beta_k^\bullet \circ \alpha\gamma_k^\bullet \circ \beta_k^\bullet) \circ \gamma_k^\bullet) \dots \circ \beta_k) \circ \gamma_k) \circ \beta_k$$

with exactly $n-1$ open brackets and where $\beta_k^\bullet = \beta_k$, $\gamma_k^\bullet = \gamma_k$ if n is odd, and $\beta_k^\bullet = \gamma_k$, $\gamma_k^\bullet = \beta_k$ if n is even.

Proof. By Proposition 3.3 with $j = k$, we get $\alpha\beta_{k+1} = \alpha(\beta_k \circ \alpha\gamma_k \circ \beta_k)$ and, by symmetry, $\alpha\gamma_{k+1} = \alpha(\gamma_k \circ \alpha\beta_k \circ \gamma_k)$.

By the above identity, and by taking $j = k+1$ in Proposition 3.3 we have $\beta_{k+2} = \beta_k \circ \alpha\gamma_{k+1} \circ \beta_k = \beta_k \circ \alpha(\gamma_k \circ \alpha\beta_k \circ \gamma_k) \circ \beta_k$, and $\alpha\beta_{k+2} = \alpha(\beta_k \circ \alpha(\gamma_k \circ \alpha\beta_k \circ \gamma_k) \circ \beta_k)$, as well as the symmetrical identities.

The proposition is obtained by iterating the above arguments. □

Notice that, so far, we have not used the results of Section 2.

Theorem 3.5. *For $m \geq 3$, every m-permutable variety satisfies the congruence identity $\alpha\beta_h = \alpha\gamma_h$, for $h = m[\frac{m+1}{2}] - 1$.*

Proof. First notice that if $k = m[\frac{m-1}{2}]$, then $h = m[\frac{m+1}{2}] - 1 = m[\frac{m-1}{2}] + m - 1 = k + m - 1$. By Proposition 3.4 with $n = m$, and by Theorem 2.2 with β_k, γ_k and

$\gamma_k^\bullet$ in place of, respectively, β, γ and δ, we have

$$\begin{aligned}\alpha\beta_h &= \alpha\beta_{k+m-1} \leq \alpha\beta_{k+m} \\ &= \alpha(\beta_k \circ \alpha(\gamma_k \circ \alpha(\beta_k \circ \dots \alpha(\gamma_k^\bullet \circ \alpha(\beta_k^\bullet \circ \alpha\gamma_k^\bullet \circ \beta_k^\bullet) \circ \gamma_k^\bullet) \dots \circ \beta_k) \circ \gamma_k) \circ \beta_k) \\ &= \alpha(\gamma_k \circ \alpha(\beta_k \circ \alpha(\gamma_k \circ \dots \alpha(\beta_k^\bullet \circ \alpha(\gamma_k^\bullet \circ \alpha\gamma_k^\bullet \circ \gamma_k^\bullet) \circ \beta_k^\bullet) \dots \circ \gamma_k) \circ \beta_k) \circ \gamma_k) \\ &= \alpha(\gamma_k \circ \alpha(\beta_k \circ \alpha(\gamma_k \circ \dots \alpha(\beta_k^\bullet \quad \circ \quad \alpha\gamma_k^\bullet \quad \circ \quad \beta_k^\bullet) \dots \circ \gamma_k) \circ \beta_k) \circ \gamma_k) \\ &= \alpha\gamma_{k+m-1} = \alpha\gamma_h\end{aligned}$$

since the last two lines are equal because of Proposition 3.4 with $n = m - 1$ and γ in place of β.

Thus, we have proved that $\alpha\beta_h \leq \alpha\gamma_h$. By symmetry $\alpha\gamma_h \leq \alpha\beta_h$, from which we reach the conclusion. □

In the particular case of locally finite m-permutable varieties, the value $n = [\frac{m-1}{2}]$ in Theorem 3.2(ii) can be improved to $n = 1$, because of [HMK, Theorems 9.8 and 9.14], and because of the result stated in the last line of [L1, p. 163]. K. Kearnes [K2] has communicated us results which imply that Theorem 3.2(ii) holds with $n = 1$ for every m-permutable variety. Thus, modulo the above results, Theorem 3.5 holds for $h = 2m - 1$.

Can h be improved further?

Considering small values of m suggests that h can actually be improved. Permutable and 3-permutable varieties are congruence modular, hence they satisfy $\alpha\beta_2 = \alpha\gamma_2$. We know that an m-permutable variety $\mathcal{V}$ satisfies $\alpha\beta_m = \alpha\gamma_m$ if at least one of the following conditions is satisfied: (a) $m = 4$ (no use of commutator theory); (b) $\mathcal{V}$ is semidistributive [K2]; (c) $\mathcal{V}$ has a difference term for $[\alpha, \alpha|1]$ (that is, a term satisfying condition (ii) in Theorem 3.2 with $n = 1$ and with one "≡" replaced by "=").

The proof of Theorem 3.5 applies to a more general context. First, notice that, in the proof of Theorem 3.5, in place of (X_m), it is enough to assume the following weaker property $(X_m)^*$:

$$\begin{aligned}&\alpha(\beta \circ \alpha(\gamma \circ \alpha(\beta \circ \dots \alpha(\gamma^\bullet \circ \alpha(\beta^\bullet \circ \alpha\delta \circ \beta^\bullet) \circ \gamma^\bullet) \dots \circ \beta) \circ \gamma) \circ \beta) \\ &\qquad \subseteq \Big(\alpha(\gamma \circ \alpha(\beta \circ \alpha(\gamma \circ \dots \alpha(\beta^\bullet \circ \alpha(\gamma^\bullet \circ \alpha\delta \circ \gamma^\bullet) \circ \beta^\bullet) \dots \circ \gamma) \circ \beta) \circ \gamma)\Big)^*\end{aligned}$$

with m normal-sized open parentheses on each side, where * denotes transitive closure.

We have a long technical proof showing that if a variety satisfies $(X_m)^*$, then for some n and k the commutator $[\alpha, \beta|n]$ satisfies $[\beta + \gamma, \alpha|n] \leq \alpha\beta_k$, and there exists a term d as in condition (ii) in Theorem 3.2. Thus we get: *If a variety $\mathcal{V}$ satisfies $(X_m)^*$ for some m, then there is some h (depending on $\mathcal{V}$) such that $\mathcal{V}$ satisfies $\alpha\beta_h = \alpha\gamma_h$.*

Is the converse true?

Problem 3.6. Is it true that if $\mathcal{V}$ satisfies $\alpha\beta_h = \alpha\gamma_h$ for some h, then $\mathcal{V}$ satisfies $(X_m)^*$ for some m?

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References

[C] G. Czédli, A characterization for congruence semidistributivity, in "Universal Algebra and Lattice Theory", Springer Lecture Notes No. 1004, 1983. MR716177 (85g:08006)

[HM] J. Hagemann, A. Mitschke, On n-permutable congruences, Algebra Universalis, 3 (1973), 8-12. MR0330010 (48:8349)

[HMK] D. Hobby, R. McKenzie, The structure of finite algebras, Contemp. Math. 76 (1988). MR958685 (89m:08001)

[J] B. Jónsson, Congruence varieties, Algebra Universalis, 10 (1980), 355–394. MR564122 (81e:08004)

[JR] B. Jónsson, I. Rival, Lattice varieties covering the smallest non-modular variety, Pacific Journal of Mathematics, 82 (1979), 463–478. MR551703 (81j:06007)

[K1] K. Kearnes, Congruence join semidistributivity is equivalent to a congruence identity, Algebra Universalis, 46 (2001), 373-387. MR1857204 (2002f:08008)

[K2] K. Kearnes, personal communication, December 2004.

[L1] P. Lipparini, n-permutable varieties satisfy non-trivial congruence identities, Algebra Universalis 33 (1995), 159-168. MR1318980 (96c:08010)

[L2] P. Lipparini, Congruence identities satisfied in n-permutable varieties, Bollettino U.M.I. (7) 8-B (1994), 851-868. MR1315822 (95m:08006)

[L3] P. Lipparini, A non-trivial congruence implication between identities weaker than modularity, Acta Scientiarum Mathematicarum, 68 (2002), 593–609. MR1954538 (2003i:08004)

[L4] P. Lipparini, Commutator theory without join-distributivity, Trans. Amer. Math. Soc. 346, 177-202 (1994). MR1257643 (95c:08009)

[L5] P. Lipparini, A congruence identity satisfied by m-permutable varieties, arXiv: math.GM/0508548 (2005).

[T] W. Taylor, Some applications of the term condition, Algebra Universalis 14 (1982), 11-24. MR634412 (83d:08004)

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