A GENERALIZED BANACH CONTRACTION PRINCIPLE THAT CHARACTERIZES METRIC COMPLETENESS

TOMONARI SUZUKI

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Abstract. We prove a fixed point theorem that is a very simple generalization of the Banach contraction principle and characterizes the metric completeness of the underlying space. We also discuss the Meir-Keeler fixed point theorem.

1. Introduction

Throughout this paper we denote by \( \mathbb{N} \) the set of all positive integers and by \( \mathbb{R} \) the set of all real numbers.

The following famous theorem is referred to as the Banach contraction principle.

**Theorem 1** (Banach [1]). Let \( (X, d) \) be a complete metric space and let \( T \) be a contraction on \( X \), i.e., there exists \( r \in [0, 1) \) such that

\[
d(Tx, Ty) \leq rd(x, y)
\]

for all \( x, y \in X \). Then \( T \) has a unique fixed point.

This theorem is very forceful and simple, and it became a classical tool in nonlinear analysis. Moreover, it has many generalizations; see [2, 3, 4, 8, 9, 14, 15, 17, 18, 21, 23, 24, 25] and others. On the other hand, Connell [6] gave an example of a metric space \( X \) such that \( X \) is not complete and every contraction on \( X \) has a fixed point. Thus, Theorem 1 cannot characterize the metric completeness of \( X \) which means the notion of contractions is too strong from this point of view.

A mapping \( T \) on a metric space \( (X, d) \) is called **Kannan** if there exists \( \alpha \in [0, 1/2) \) such that

\[
d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)
\]

for all \( x, y \in X \). Kannan [11] proved that if \( X \) is complete, then every Kannan mapping has a fixed point. We note that Kannan’s theorem is not an extension of Theorem 1. In our opinion, Kannan’s fixed point theorem is also very important because Subrahmanyam [22] proved that Kannan’s theorem characterizes the metric completeness. That is, a metric space \( X \) is complete if and only if every Kannan mapping on \( X \) has a fixed point. Also, several mathematicians have studied the

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metric completeness. For example, Kirk [13] proved that Caristi’s fixed point theorem [2, 3] characterizes the metric completeness. For other results in this setting, see [7, 10, 19, 20, 26] and others.

In this paper, we prove a fixed point theorem which is a generalization of Theorem 1 and characterizes the metric completeness. Though there are many generalizations of Theorem 1, the direction of our extension is new and very simple. We also generalize the Meir-Keeler fixed point theorem [17].

2. Fixed point theorem

In this section, we prove the following theorem, which is a generalization of the Banach contraction principle (Theorem 1).

**Theorem 2.** Let \((X, d)\) be a complete metric space and let \(T\) be a mapping on \(X\). Define a nonincreasing function \(\theta\) from \([0, 1)\) onto \((1/2, 1]\) by

\[
\theta(r) = \begin{cases} 
1 & \text{if } 0 \leq r \leq (\sqrt{5} - 1)/2, \\
(1 - r)r^{-2} & \text{if } (\sqrt{5} - 1)/2 \leq r \leq 2^{-1/2}, \\
(1 + r)^{-1} & \text{if } 2^{-1/2} \leq r < 1.
\end{cases}
\]

Assume that there exists \(r \in [0, 1)\) such that

\[
\theta(r) d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq r d(x, y)
\]

for all \(x, y \in X\). Then there exists a unique fixed point \(z\) of \(T\). Moreover \(\lim_n T^n x = z\) for all \(x \in X\).

**Proof.** Since \(\theta(r) \leq 1\), \(\theta(r) d(x, Tx) \leq d(x, Tx)\) holds for every \(x \in X\). By hypothesis,

\[
d(Tx, T^2 x) \leq r d(x, Tx)
\]

for all \(x \in X\). We now fix \(u \in X\) and define a sequence \(\{u_n\}\) in \(X\) by \(u_n = T^n u\). Then (1) yields \(d(u_n, u_{n+1}) \leq r^n d(u, Tu)\), so \(\sum_{n=1}^{\infty} d(u_n, u_{n+1}) < \infty\), and a standard argument shows \(\{u_n\}\) is a Cauchy sequence. Since \(X\) is complete, \(\{u_n\}\) converges to some point \(z \in X\). We next show

\[
d(Tx, z) \leq r d(x, z) \quad \text{for all } x \in X \setminus \{z\}.
\]

For \(x \in X \setminus \{z\}\), there exists \(\nu \in \mathbb{N}\) such that \(d(u_n, z) \leq d(x, z)/3\) for all \(n \in \mathbb{N}\) with \(n \geq \nu\). Then we have

\[
\theta(r) d(u_n, Tu_n) \leq d(u_n, Tu_n) = d(u_n, u_{n+1}) \\
\leq d(u_n, z) + d(u_{n+1}, z) \\
\leq (2/3) d(x, z) = d(x, z) - d(x, z)/3 \\
\leq d(x, z) - d(u_n, z) \leq d(u_n, x).
\]

Hence and by hypothesis, \(d(u_{n+1}, Tx) \leq r d(u_n, x)\) for \(n \geq \nu\). Letting \(n\) tend to \(\infty\), we get \(d(Tx, z) \leq r d(x, z)\). That is, we have shown (2). Arguing by contradiction, we assume that \(T^j z \neq z\) for all \(j \in \mathbb{N}\). Then (2) yields

\[
d(T^{j+1} z, z) \leq r^j d(Tz, z) \quad \text{for } j \in \mathbb{N}.
\]

We consider the following three cases:

- \(0 \leq r \leq (\sqrt{5} - 1)/2\),
- \((\sqrt{5} - 1)/2 < r < 2^{-1/2}\),
• $2^{-1/2} \leq r < 1$.

In the case where $0 \leq r \leq (\sqrt{5} - 1)/2$, we note $r^2 + r - 1 \leq 0$ and $2r^2 < 1$. If we assume $d(T^2 z, z) < d(T^2 z, T^3 z)$, then we have

$$d(z, Tz) \leq d(z, T^2 z) + d(Tz, T^2 z)$$
$$< d(T^2 z, T^3 z) + d(Tz, T^2 z)$$
$$\leq r^2 d(z, Tz) + r d(z, Tz)$$
$$< d(z, Tz).$$

This is a contradiction. So we have

$$d(T^2 z, z) \geq d(T^2 z, T^3 z) = \theta(r) d(T^2 z, T \circ T^2 z).$$

By hypothesis and (3), we have

$$d(z, Tz) \leq d(z, T^3 z) + d(T^3 z, Tz)$$
$$\leq r^2 d(z, Tz) + r d(T^2 z, z)$$
$$\leq r^2 d(z, Tz) + r^2 d(Tz, z) = 2r^2 d(z, Tz)$$
$$< d(z, Tz).$$

This is a contradiction. In the case where $(\sqrt{5} - 1)/2 < r < 2^{-1/2}$, we note $2r^2 < 1$.

If we assume $d(T^2 z, z) < \theta(r) d(T^2 z, T^3 z)$, then we have in view of (1)

$$d(z, Tz) \leq d(z, T^2 z) + d(Tz, T^2 z)$$
$$< \theta(r) d(T^2 z, T^3 z) + d(Tz, T^2 z)$$
$$\leq \theta(r) r^2 d(z, Tz) + r d(z, Tz) = d(z, Tz).$$

This is a contradiction. Hence $d(T^2 z, z) \geq \theta(r) d(T^2 z, T \circ T^2 z)$. As in the previous case, we can prove

$$d(z, Tz) \leq 2r^2 d(z, Tz) < d(z, Tz).$$

This is a contradiction. In the third case, where $2^{-1/2} \leq r < 1$, we note that for $x, y \in X$, either

$$\theta(r) d(x, Tx) \leq d(x, y) \quad \text{or} \quad \theta(r) d(Tx, T^2 x) \leq d(Tx, y)$$

holds. Indeed, if

$$\theta(r) d(x, Tx) > d(x, y) \quad \text{and} \quad \theta(r) d(Tx, T^2 x) > d(Tx, y),$$

then we have

$$d(x, Tx) \leq d(x, y) + d(Tx, y)$$
$$< \theta(r) (d(x, Tx) + d(Tx, T^2 x))$$
$$\leq \theta(r) d(x, Tx) + r d(x, Tx)$$
$$= d(x, Tx).$$

This is a contradiction. Since either

$$\theta(r) d(u_{2n}, u_{2n+1}) \leq d(u_{2n}, z) \quad \text{or} \quad \theta(r) d(u_{2n+1}, u_{2n+2}) \leq d(u_{2n+1}, z)$$

holds for every $n \in \mathbb{N}$, either

$$d(u_{2n+1}, Tz) \leq r d(u_{2n}, z) \quad \text{or} \quad d(u_{2n+2}, Tz) \leq r d(u_{2n+1}, z)$$
holds for every \( n \in \mathbb{N} \). Since \( \{u_n\} \) converges to \( z \), the above inequalities imply there exists a subsequence of \( \{u_n\} \) which converges to \( Tz \). This implies \( Tz = z \). This is a contradiction. Therefore in all the cases, there exists \( j \in \mathbb{N} \) such that \( T^jz = z \). Since \( \{T^nz\} \) is a Cauchy sequence, we obtain \( Tz = z \). That is, \( z \) is a fixed point of \( T \). The uniqueness of a fixed point follows easily from (2). This completes the proof. 

The following theorem says that \( \theta(r) \) is the best constant for every \( r \in [0, 1) \).

**Theorem 3.** Define a function \( \theta \) as in Theorem 2. Then for each \( r \in [0, 1) \), there exist a complete metric space \( (X, d) \) and a mapping \( T \) on \( X \) such that \( T \) does not have a fixed point and

\[
\theta(r) d(x, Tx) < d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq r d(x, y)
\]

for all \( x, y \in X \).

**Proof.** In the case where \( 0 \leq r \leq (\sqrt{5} - 1)/2 \), define a complete subset \( X \) of the Euclidean space \( \mathbb{R} \) by \( X = \{\pm 1\} \). We also define a mapping \( T \) on \( X \) by \( Tx = -x \) for \( x \in X \). Then \( T \) does not have a fixed point and

\[
\theta(r) d(x, Tx) = 2 \geq d(x, y)
\]

for all \( x, y \in X \). In the case where \( (\sqrt{5} - 1)/2 < r < 2^{-1/2} \), define a complete subset \( X \) of the Euclidean space \( \mathbb{R} \) as follows: \( X = \{x_n : n \in \mathbb{N} \cup \{0\}\} \), where \( x_0 = 0 \), \( x_1 = 1 \), \( x_2 = 1 - r \) and \( x_n = (1 - r - r^2)(-r)^{n-3} \) for \( n \geq 3 \). Define a mapping \( T \) on \( X \) by \( Tx_n = x_{n+1} \) for \( n \in \mathbb{N} \cup \{0\} \). Then \( T \) satisfies the conclusion. In the other case, where \( 2^{-1/2} \leq r < 1 \), define a complete subset \( X \) of the Euclidean space \( \mathbb{R} \) as follows:

\[
X = \{0, 1\} \cup \{x_n : n \in \mathbb{N} \cup \{0\}\},
\]

where \( x_n = (1 - r)(-r)^n \) for \( n \in \mathbb{N} \cup \{0\} \). Define a mapping \( T \) on \( X \) by \( T0 = 1 \), \( T1 = x_0 \) and \( Tx_n = x_{n+1} \) for \( n \in \mathbb{N} \cup \{0\} \). Let us prove that \( T \) satisfies the conclusion. The following are obvious.

- \( d(T0, T1) = r d(0, 1) \).
- \( \theta(r) d(0, T0) \geq \theta(r) d(x_n, Tx_n) = d(0, x_n) \) for \( n \in \mathbb{N} \cup \{0\} \).
- \( d(Tx_m, Tx_n) = r d(x_m, x_n) \) for \( m, n \in \mathbb{N} \cup \{0\} \).

Also, we have

\[
d(T1, Tx_n) - r d(1, x_n) = 1 - 2r - 2(-r)^{n+1} - 2(-r)^{n+2}
\]

\[
\leq 1 - 2r + 2r^{n+1} - 2r^{n+2} = 1 - 2r + 2r^{n+1}(1 - r)
\]

\[
\leq 1 - 2r + 2r^4(1 - r) = 1 - 2r^2 \leq 0
\]

for \( n \in \mathbb{N} \cup \{0\} \). This completes the proof. 

It is obvious that the set of our contractions in Theorem 2 includes that of the usual contractions. However, our contractions and Kannan mappings are independent. We next show it.

**Example 1.** Define a complete metric space \( X \) by \( X = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\} \) and its metric \( d \) by \( d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| \). Define a mapping \( T \) on \( X \) by

\[
T(x_1, x_2) = \begin{cases} 
(x_1, 0) & \text{if } x_1 \leq x_2, \\
(0, x_2) & \text{if } x_1 > x_2.
\end{cases}
\]
Then $T$ satisfies the assumption in Theorem 2, but $T$ is not a Kannan mapping.

Proof. We first note that $d(Tx,Ty) \leq (4/5)d(x,y)$ if $(x,y) \neq ((4,5),(5,4))$ and $(y,x) \neq ((4,5),(5,4))$. Since

$$\theta(r)d((4,5),T(4,5)) > \frac{5}{2} > 2 = d((4,5),(5,4))$$

and $\theta(r)d((5,4),T(5,4)) > d((5,4),(4,5))$ for every $r \in [0,1)$, $T$ satisfies the assumption in Theorem 2. But, since

$$d(T(5,4),T(4,5)) = 8 > 5 = \frac{1}{2}\left(d((4,5),T(4,5)) + d((5,4),T(5,4))\right),$$

$T$ is not a Kannan mapping. \qed

Example 2. Define a complete metric space $X$ by $X = \{-1,0,1,2\}$ and a mapping $T$ on $X$ by

$$Tx = \begin{cases} 0 & \text{if } x \neq 2, \\ -1 & \text{if } x = 2. \end{cases}$$

Then $T$ is a Kannan mapping, but $T$ does not satisfy the assumption in Theorem 2.

Proof. Since

$$d(Tx,T2) \leq 1 = \frac{1}{3}d(2,T2) \leq \frac{1}{3}d(x,Tx) + \frac{1}{3}d(2,T2)$$

for all $x \in X$, $T$ is a Kannan mapping. But, since

$$\theta(r)d(1,T1) \leq 1 = d(1,2) \quad \text{and} \quad d(T1,T2) = 1 = d(1,2)$$

for every $r \in [0,1)$, $T$ does not satisfy the assumption in Theorem 2. \qed

3. Metric completeness

In this section, we discuss the metric completeness.

Theorem 4. Let $(X,d)$ be a metric space and define a function $\theta$ as in Theorem 2. For $r \in [0,1)$ and $\eta \in (0,\theta(r)]$, let $A_{r,\eta}$ be the family of mappings $T$ on $X$ satisfying the following:

(a) For $x, y \in X$,

$$\eta d(x,Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leq r d(x,y).$$

Let $B_{r,\eta}$ be the family of mappings $T$ on $X$ satisfying (a) and the following:

(b) $T(X)$ is countably infinite.

(c) Every subset of $T(X)$ is closed.

Then the following are equivalent:

(i) $X$ is complete.

(ii) Every mapping $T \in A_{r,\theta(r)}$ has a fixed point for all $r \in [0,1)$.

(iii) There exist $r \in (0,1)$ and $\eta \in (0,\theta(r)]$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.
Proof. By Theorem 2, (i) implies (ii). Since $B_{r, \eta} \subset A_{r, \theta(r)}$ for $r \in [0, 1)$ and $\eta \in (0, \theta(r)]$, (ii) implies (iii). Let us prove (iii) implies (i). We assume (iii). Arguing by contradiction, we also assume that $X$ is not complete. That is, there exists a Cauchy sequence $\{u_n\}$ which does not converge. Define a function from $X$ into $[0, \infty)$ by $f(x) = \lim_n d(x, u_n)$ for $x \in X$. We note that $f$ is well defined because $\{d(x, u_n)\}$ is a Cauchy sequence for every $x \in X$. The following are obvious:

- $f(x) - f(y) \leq d(x, y) \leq f(x) + f(y)$ for $x, y \in X$,
- $f(x) > 0$ for all $x \in X$ and
- $\lim_n f(u_n) = 0$.

Define a mapping $T$ on $X$ as follows: For each $x \in X$, since $f(x) > 0$ and $\lim_n f(u_n) = 0$, there exists $\nu \in \mathbb{N}$ satisfying $f(u_\nu) \leq \frac{\eta r}{3 + \eta r} f(x)$. We put $Tx = u_\nu$. Then it is obvious that

$$f(Tx) \leq \frac{\eta r}{3 + \eta r} f(x) \quad \text{and} \quad Tx \in \{u_n : n \in \mathbb{N}\}$$

for all $x \in X$. Then $Tx \neq x$ for all $x \in X$ because $f(Tx) < f(x)$. That is, $T$ does not have a fixed point. Since $T(X) \subset \{u_n : n \in \mathbb{N}\}$, (b) holds. Also, it is not difficult to prove (c). Let us prove (a). Fix $x, y \in X$ with $\eta d(x, Tx) \leq d(x, y)$. In the case where $f(y) > 2 f(x)$, we have

$$d(Tx, Ty) \leq f(Tx) + f(Ty) \leq \frac{\eta r}{3 + \eta r} (f(x) + f(y))$$

$$\leq \frac{r}{3} (f(x) + f(y))$$

$$\leq \frac{r}{3} (f(x) + f(y)) + \frac{2r}{3} (f(y) - 2 f(x))$$

$$= r (f(y) - f(x)) \leq r d(x, y).$$

In the other case, where $f(y) \leq 2 f(x)$, we have

$$d(x, y) \geq \eta d(x, Tx) \geq \eta (f(x) - f(Tx))$$

$$\geq \eta \left( 1 - \frac{\eta r}{3 + \eta r} \right) f(x) = \frac{3 \eta}{3 + \eta r} f(x)$$

and hence

$$d(Tx, Ty) \leq f(Tx) + f(Ty) \leq \frac{\eta r}{3 + \eta r} (f(x) + f(y))$$

$$\leq \frac{3 \eta r}{3 + \eta r} f(x) \leq r d(x, y).$$

Therefore we have shown (a), that is, $T \in B_{r, \eta}$. By (iii), $T$ has a fixed point which yields a contradiction. Hence we obtain that $X$ is complete. This completes the proof. \qed

As a direct consequence of Theorem 4, we obtain the following.

**Corollary 1.** For a metric space $(X, d)$, the following are equivalent:

(i) $X$ is complete.

(ii) There exists $r \in (0, 1)$ such that every mapping $T$ on $X$ satisfying the following has a fixed point:

- $\frac{1}{10000} d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq r d(x, y)$ for all $x, y \in X$. 


4. The Meir-Keeler theorem

In this section, we prove a generalization of the Meir-Keeler fixed point theorem [17]. See also [5], [12], [16, Theorem 1.5.1].

**Theorem 5.** Let \((X,d)\) be a complete metric space and let \(T\) be a mapping on \(X\). Assume that for each \(\varepsilon > 0\), there exists \(\delta > 0\) such that

1. \((1/2) d(x, Tx) < d(x, y)\) and \(d(x, y) < \varepsilon + \delta\) imply \(d(Tx, Ty) \leq \varepsilon\) and
2. \((1/2) d(x, Tx) < d(x, y)\) implies \(d(Tx, Ty) < d(x, y)\)

for all \(x, y \in X\). Then there exists a unique fixed point \(z\) of \(T\). Moreover \(\lim_n T^n x = z\) for all \(x \in X\).

**Remark.**

(i) The Meir-Keeler fixed point theorem [17] is a generalization of the Banach contraction principle (Theorem 1). However, Theorem 5 is not a generalization of Theorem 2.

(ii) We note \(\lim_{r \to 0} \theta(r) = 1/2\). By Theorem 3, we can prove that 1/2 is the best constant.

**Proof.** If \(Tx \neq x\), then it is obvious that \(d(x, Tx) < 2d(x, Tx)\). So, by hypothesis,

\[d(Tx, T^2x) < d(x, Tx)\]

holds for all \(x \in X\) with \(Tx \neq x\). We also note that

\[d(Tx, T^2x) \leq d(x, Tx)\]

holds for all \(x \in X\). Fix \(u \in X\) and define a sequence \(\{u_n\}\) in \(X\) by \(u_n = T^n u\) for \(n \in \mathbb{N}\). Since \(\{d(u_n, u_{n+1})\}\) is a nonincreasing sequence, \(\{d(u_n, u_{n+1})\}\) converges to some \(\alpha \geq 0\). Arguing by contradiction, we assume \(\alpha > 0\). Then \(\{d(u_n, u_{n+1})\}\) is strictly decreasing. Hence \(d(u_n, u_{n+1}) > \alpha\) for every \(n \in \mathbb{N}\). By hypothesis, there exists \(\delta > 0\) such that

- \(d(x, Tx) < 2d(x, y)\) and \(d(x, y) < \alpha + \delta\) imply \(d(Tx, Ty) \leq \alpha\).

From the definition of \(\alpha\), there exists \(j \in \mathbb{N}\) such that \(d(u_j, u_{j+1}) < \alpha + \delta\). So we have \(d(u_{j+1}, u_{j+2}) \leq \alpha\). This is a contradiction. Therefore \(\alpha = 0\). That is, \(\lim_n d(u_n, u_{n+1}) = 0\) holds. Fix \(\varepsilon > 0\). Then there exists \(\delta \in (0, \varepsilon)\) such that

- \(d(x, Tx) < 2d(x, y)\) and \(d(x, y) < \varepsilon + \delta\) imply \(d(Tx, Ty) < \varepsilon\).

Let \(\ell \in \mathbb{N}\) such that \(d(u_n, u_{n+1}) < \delta\) for all \(n \in \mathbb{N}\) with \(n \geq \ell\). We shall show

\[d(u_\ell, u_{\ell+m}) < \varepsilon + \delta\]

for \(m \in \mathbb{N}\) by induction. It is obvious that (4) holds when \(m = 1\). We assume (4) holds for some \(m \in \mathbb{N}\). In the case where \(d(u_\ell, u_{\ell+m}) \leq \varepsilon\), we have

\[d(u_\ell, u_{\ell+m+1}) \leq d(u_\ell, u_{\ell+m}) + d(u_{\ell+m}, u_{\ell+m+1}) < \varepsilon + \delta.\]

In the other case, where \(\varepsilon < d(u_\ell, u_{\ell+m}) < \varepsilon + \delta\), since

\[d(u_\ell, u_{\ell+1}) < \delta < \varepsilon < d(u_\ell, u_{\ell+m}) < 2d(u_\ell, u_{\ell+m}),\]

we have \(d(u_{\ell+1}, u_{\ell+m+1}) \leq \varepsilon\) and hence

\[d(u_\ell, u_{\ell+m+1}) \leq d(u_\ell, u_{\ell+1}) + d(u_{\ell+1}, u_{\ell+m+1}) < \delta + \varepsilon.\]

So, by induction, (4) holds for every \(m \in \mathbb{N}\). Therefore we have shown

\[\lim_{n \to \infty} \sup_{m > n} d(u_n, u_m) = 0.\]
This implies that \( \{u_n\} \) is Cauchy. Since \( X \) is complete, \( \{u_n\} \) converges to some point \( z \in X \). We shall show that such \( z \) is a fixed point of \( T \), dividing the following two cases:

- There exists \( \nu \in \mathbb{N} \) such that \( u_\nu = u_{\nu+1} \).
- \( u_n \neq u_{n+1} \) for all \( n \in \mathbb{N} \).

In the first case, \( u_n = u_\nu \) for all \( n \in \mathbb{N} \) with \( n \geq \nu \). Since \( \{u_n\} \) converges to \( z \), we have \( u_n = z \) for all \( n \in \mathbb{N} \) with \( n \geq \nu \). This implies \( Tz = z \). In the second case, we note \( u_n \neq Tu_n \) for \( n \in \mathbb{N} \), so \( \{d(u_n, u_{n+1})\} \) is strictly decreasing. If we assume that

\[
d(u_n, u_{n+1}) \geq 2d(u_n, z) \quad \text{and} \quad d(u_{n+1}, u_{n+2}) \geq 2d(u_{n+1}, z)
\]

hold for some \( n \in \mathbb{N} \), then we have

\[
d(u_n, u_{n+1}) \leq d(u_n, z) + d(u_{n+1}, z) \\
\leq \left( d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) \right)/2 \\
< d(u_n, u_{n+1}).
\]

This is a contradiction. That is, either

\[
d(u_n, u_{n+1}) < 2d(u_n, z) \quad \text{or} \quad d(u_{n+1}, u_{n+2}) < 2d(u_{n+1}, z)
\]

holds for all \( n \in \mathbb{N} \). By hypothesis, either

\[
d(u_{n+1}, Tz) < d(u_n, z) \quad \text{or} \quad d(u_{n+2}, Tz) < d(u_{n+1}, z)
\]

holds for all \( n \in \mathbb{N} \). Since \( \{u_n\} \) converges to \( z \), the above inequalities imply there exists a subsequence of \( \{u_n\} \) which converges to \( Tz \). This implies \( Tz = z \). We have shown that \( z \) is a fixed point of \( T \). Finally, arguing by contradiction, suppose there exists another fixed point \( y \) of \( T \). Since

\[
d(z, Tz) = 0 < 2d(z, y),
\]

we have

\[
d(z, y) = d(Tz, Ty) < d(z, y).
\]

This is a contradiction. That is, the fixed point is unique. This completes the proof. \( \square \)

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**References**


DEPARTMENT OF MATHEMATICS, KYUSHU INSTITUTE OF TECHNOLOGY, SENSUICHO, TOBATA, KITAKYUSHU 804-8550, JAPAN
E-mail address: suzuki-t@mns.kyutech.ac.jp