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A GENERALIZED BANACH CONTRACTION PRINCIPLE THAT CHARACTERIZES METRIC COMPLETENESS

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ABSTRACT. We prove a fixed point theorem that is a very simple generalization of the Banach contraction principle and characterizes the metric completeness of the underlying space. We also discuss the Meir-Keeler fixed point theorem.

1. INTRODUCTION

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

The following famous theorem is referred to as the *Banach contraction principle*.

Theorem 1 (Banach [1]). *Let (X, d) be a complete metric space and let T be a contraction on X, i.e., there exists $r \in [0, 1)$ such that*

$$d(Tx, Ty) \leq r\, d(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

This theorem is very forceful and simple, and it became a classical tool in nonlinear analysis. Moreover, it has many generalizations; see [2, 3, 4, 8, 9, 14, 15, 17, 18, 21, 23, 24, 25] and others. On the other hand, Connell [6] gave an example of a metric space X such that X is not complete and every contraction on X has a fixed point. Thus, Theorem 1 cannot characterize the metric completeness of X which means the notion of contractions is too strong from this point of view.

A mapping T on a metric space (X, d) is called *Kannan* if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha\, d(x, Tx) + \alpha\, d(y, Ty)$$

for all $x, y \in X$. Kannan [11] proved that if X is complete, then every Kannan mapping has a fixed point. We note that Kannan's theorem is not an extension of Theorem 1. In our opinion, Kannan's fixed point theorem is also very important because Subrahmanyam [22] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space X is complete if and only if every Kannan mapping on X has a fixed point. Also, several mathematicians have studied the

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metric completeness. For example, Kirk [13] proved that Caristi's fixed point theorem [2, 3] characterizes the metric completeness. For other results in this setting, see [7, 10, 19, 20, 26] and others.

In this paper, we prove a fixed point theorem which is a generalization of Theorem 1 and characterizes the metric completeness. Though there are many generalizations of Theorem 1, the direction of our extension is new and very simple. We also generalize the Meir-Keeler fixed point theorem [17].

2. Fixed point theorem

In this section, we prove the following theorem, which is a generalization of the Banach contraction principle (Theorem 1).

Theorem 2. *Let (X, d) be a complete metric space and let T be a mapping on X. Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)\,r^{-2} & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)\, d(x, Tx) \le d(x, y) \quad \text{implies} \quad d(Tx, Ty) \le r\, d(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Proof. Since $\theta(r) \le 1$, $\theta(r)\, d(x, Tx) \le d(x, Tx)$ holds for every $x \in X$. By hypothesis,

$$d(Tx, T^2x) \le r\, d(x, Tx) \tag{1}$$

for all $x \in X$. We now fix $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$. Then (1) yields $d(u_n, u_{n+1}) \le r^n\, d(u, Tu)$, so $\sum_{n=1}^{\infty} d(u_n, u_{n+1}) < \infty$, and a standard argument shows $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $z \in X$. We next show

$$d(Tx, z) \le r\, d(x, z) \quad \text{for all } x \in X \setminus \{z\}. \tag{2}$$

For $x \in X \setminus \{z\}$, there exists $\nu \in \mathbb{N}$ such that $d(u_n, z) \le d(x, z)/3$ for all $n \in \mathbb{N}$ with $n \ge \nu$. Then we have

$$\begin{aligned} \theta(r)\, d(u_n, Tu_n) &\le d(u_n, Tu_n) = d(u_n, u_{n+1}) \\ &\le d(u_n, z) + d(u_{n+1}, z) \\ &\le (2/3)\, d(x, z) = d(x, z) - d(x, z)/3 \\ &\le d(x, z) - d(u_n, z) \le d(u_n, x). \end{aligned}$$

Hence and by hypothesis, $d(u_{n+1}, Tx) \le r\, d(u_n, x)$ for $n \ge \nu$. Letting n tend to ∞, we get $d(Tx, z) \le r\, d(x, z)$. That is, we have shown (2). Arguing by contradiction, we assume that $T^j z \ne z$ for all $j \in \mathbb{N}$. Then (2) yields

$$d(T^{j+1}z, z) \le r^j\, d(Tz, z) \qquad \text{for} \quad j \in \mathbb{N}. \tag{3}$$

We consider the following three cases:

- $0 \le r \le (\sqrt{5}-1)/2$,
- $(\sqrt{5}-1)/2 < r < 2^{-1/2}$,

- $2^{-1/2} \le r < 1$.

In the case where $0 \le r \le (\sqrt{5}-1)/2$, we note $r^2 + r - 1 \le 0$ and $2\,r^2 < 1$. If we assume $d(T^2 z, z) < d(T^2 z, T^3 z)$, then we have

$$\begin{aligned} d(z, Tz) &\le d(z, T^2 z) + d(Tz, T^2 z) \\ &< d(T^2 z, T^3 z) + d(Tz, T^2 z) \\ &\le r^2\, d(z, Tz) + r\, d(z, Tz) \\ &\le d(z, Tz). \end{aligned}$$

This is a contradiction. So we have

$$d(T^2 z, z) \ge d(T^2 z, T^3 z) = \theta(r)\, d(T^2 z, T \circ T^2 z).$$

By hypothesis and (3), we have

$$\begin{aligned} d(z, Tz) &\le d(z, T^3 z) + d(T^3 z, Tz) \\ &\le r^2\, d(z, Tz) + r\, d(T^2 z, z) \\ &\le r^2\, d(z, Tz) + r^2\, d(Tz, z) = 2\,r^2\, d(z, Tz) \\ &< d(z, Tz). \end{aligned}$$

This is a contradiction. In the case where $(\sqrt{5}-1)/2 < r < 2^{-1/2}$, we note $2\,r^2 < 1$. If we assume $d(T^2 z, z) < \theta(r)\, d(T^2 z, T^3 z)$, then we have in view of (1)

$$\begin{aligned} d(z, Tz) &\le d(z, T^2 z) + d(Tz, T^2 z) \\ &< \theta(r)\, d(T^2 z, T^3 z) + d(Tz, T^2 z) \\ &\le \theta(r)\, r^2\, d(z, Tz) + r\, d(z, Tz) = d(z, Tz). \end{aligned}$$

This is a contradiction. Hence $d(T^2 z, z) \ge \theta(r)\, d(T^2 z, T \circ T^2 z)$. As in the previous case, we can prove

$$d(z, Tz) \le 2\,r^2\, d(z, Tz) < d(z, Tz).$$

This is a contradiction. In the third case, where $2^{-1/2} \le r < 1$, we note that for $x, y \in X$, either

$$\theta(r)\, d(x, Tx) \le d(x, y) \quad \text{or} \quad \theta(r)\, d(Tx, T^2 x) \le d(Tx, y)$$

holds. Indeed, if

$$\theta(r)\, d(x, Tx) > d(x, y) \quad \text{and} \quad \theta(r)\, d(Tx, T^2 x) > d(Tx, y),$$

then we have

$$\begin{aligned} d(x, Tx) &\le d(x, y) + d(Tx, y) \\ &< \theta(r)\,\big(d(x, Tx) + d(Tx, T^2 x)\big) \\ &\le \theta(r)\,\big(d(x, Tx) + r\, d(x, Tx)\big) \\ &= d(x, Tx). \end{aligned}$$

This is a contradiction. Since either

$$\theta(r)\, d(u_{2n}, u_{2n+1}) \le d(u_{2n}, z) \quad \text{or} \quad \theta(r)\, d(u_{2n+1}, u_{2n+2}) \le d(u_{2n+1}, z)$$

holds for every $n \in \mathbb{N}$, either

$$d(u_{2n+1}, Tz) \le r\, d(u_{2n}, z) \quad \text{or} \quad d(u_{2n+2}, Tz) \le r\, d(u_{2n+1}, z)$$

holds for every $n \in \mathbb{N}$. Since $\{u_n\}$ converges to z, the above inequalities imply there exists a subsequence of $\{u_n\}$ which converges to Tz. This implies $Tz = z$. This is a contradiction. Therefore in all the cases, there exists $j \in \mathbb{N}$ such that $T^j z = z$. Since $\{T^n z\}$ is a Cauchy sequence, we obtain $Tz = z$. That is, z is a fixed point of T. The uniqueness of a fixed point follows easily from (2). This completes the proof. □

The following theorem says that $\theta(r)$ is the best constant for every $r \in [0, 1)$.

Theorem 3. *Define a function θ as in Theorem 2. Then for each $r \in [0, 1)$, there exist a complete metric space (X, d) and a mapping T on X such that T does not have a fixed point and*

$$\theta(r)\, d(x, Tx) < d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \le r\, d(x, y)$$

for all $x, y \in X$.

Proof. In the case where $0 \le r \le (\sqrt{5} - 1)/2$, define a complete subset X of the Euclidean space $\mathbb{R}$ by $X = \{\pm 1\}$. We also define a mapping T on X by $Tx = -x$ for $x \in X$. Then T does not have a fixed point and

$$\theta(r)\, d(x, Tx) = 2 \ge d(x, y)$$

for all $x, y \in X$. In the case where $(\sqrt{5}-1)/2 < r < 2^{-1/2}$, define a complete subset X of the Euclidean space $\mathbb{R}$ as follows: $X = \big\{x_n : n \in \mathbb{N} \cup \{0\}\big\}$, where $x_0 = 0$, $x_1 = 1$, $x_2 = 1 - r$ and $x_n = (1 - r - r^2)\,(-r)^{n-3}$ for $n \ge 3$. Define a mapping T on X by $Tx_n = x_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Then T satisfies the conclusion. In the other case, where $2^{-1/2} \le r < 1$, define a complete subset X of the Euclidean space $\mathbb{R}$ as follows:

$$X = \{0, 1\} \cup \big\{x_n : n \in \mathbb{N} \cup \{0\}\big\},$$

where $x_n = (1 - r)\,(-r)^n$ for $n \in \mathbb{N} \cup \{0\}$. Define a mapping T on X by $T0 = 1$, $T1 = x_0$ and $Tx_n = x_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Let us prove that T satisfies the conclusion. The following are obvious.

- $d(T0, T1) = r\, d(0, 1)$.
- $\theta(r)\, d(0, T0) \ge \theta(r)\, d(x_n, Tx_n) = d(0, x_n)$ for $n \in \mathbb{N} \cup \{0\}$.
- $d(Tx_m, Tx_n) = r\, d(x_m, x_n)$ for $m, n \in \mathbb{N} \cup \{0\}$.

Also, we have

$$\begin{aligned} d(T1, Tx_n) - r\, d(1, x_n) &= 1 - 2\,r - 2\,(-r)^{n+1} - 2\,(-r)^{n+2} \\ &\le 1 - 2\,r + 2\,r^{n+1} - 2\,r^{n+2} = 1 - 2\,r + 2\,r^{n+1}\,(1 - r) \\ &\le 1 - 2\,r + 2\,r^1\,(1 - r) = 1 - 2\,r^2 \le 0 \end{aligned}$$

for $n \in \mathbb{N} \cup \{0\}$. This completes the proof. □

It is obvious that the set of our contractions in Theorem 2 includes that of the usual contractions. However, our contractions and Kannan mappings are independent. We next show it.

Example 1. Define a complete metric space X by $X = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}$ and its metric d by $d\big((x_1, x_2), (y_1, y_2)\big) = |x_1 - y_1| + |x_2 - y_2|$. Define a mapping T on X by

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \le x_2, \\ (0, x_2) & \text{if } x_1 > x_2. \end{cases}$$

Then T satisfies the assumption in Theorem 2, but T is not a Kannan mapping.

Proof. We first note that $d(Tx,Ty) \leq (4/5)\, d(x,y)$ if $(x,y) \neq \big((4,5),(5,4)\big)$ and $(y,x) \neq \big((4,5),(5,4)\big)$. Since

$$\theta(r)\, d\big((4,5),T(4,5)\big) > \frac{5}{2} > 2 = d\big((4,5),(5,4)\big)$$

and $\theta(r)\, d\big((5,4),T(5,4)\big) > d\big((5,4),(4,5)\big)$ for every $r \in [0,1)$, T satisfies the assumption in Theorem 2. But, since

$$d\big(T(5,4),T(4,5)\big) = 8 > 5 = \frac{1}{2}\Big(d\big((4,5),T(4,5)\big) + d\big((5,4),T(5,4)\big)\Big),$$

T is not a Kannan mapping. □

Example 2. Define a complete metric space X by $X = \{-1,0,1,2\}$ and a mapping T on X by

$$Tx = \begin{cases} 0 & \text{if } x \neq 2, \\ -1 & \text{if } x = 2. \end{cases}$$

Then T is a Kannan mapping, but T does not satisfy the assumption in Theorem 2.

Proof. Since

$$d(Tx,T2) \leq 1 = \frac{1}{3} d(2,T2) \leq \frac{1}{3} d(x,Tx) + \frac{1}{3} d(2,T2)$$

for all $x \in X$, T is a Kannan mapping. But, since

$$\theta(r)\, d(1,T1) \leq 1 = d(1,2) \quad \text{and} \quad d(T1,T2) = 1 = d(1,2)$$

for every $r \in [0,1)$, T does not satisfy the assumption in Theorem 2. □

3. Metric completeness

In this section, we discuss the metric completeness.

Theorem 4. *Let (X,d) be a metric space and define a function θ as in Theorem 2. For $r \in [0,1)$ and $\eta \in \big(0,\theta(r)\big]$, let $A_{r,\eta}$ be the family of mappings T on X satisfying the following:*

(a) *For $x,y \in X$,*

$$\eta\, d(x,Tx) \leq d(x,y) \quad \textit{implies} \quad d(Tx,Ty) \leq r\, d(x,y).$$

Let $B_{r,\eta}$ be the family of mappings T on X satisfying (a) *and the following:*

(b) *$T(X)$ is countably infinite.*
(c) *Every subset of $T(X)$ is closed.*

Then the following are equivalent:

(i) *X is complete.*
(ii) *Every mapping $T \in A_{r,\theta(r)}$ has a fixed point for all $r \in [0,1)$.*
(iii) *There exist $r \in (0,1)$ and $\eta \in \big(0,\theta(r)\big]$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.*

Proof. By Theorem 2, (i) implies (ii). Since $B_{r,\eta} \subset A_{r,\theta(r)}$ for $r \in [0,1)$ and $\eta \in \big(0, \theta(r)\big]$, (ii) implies (iii). Let us prove (iii) implies (i). We assume (iii). Arguing by contradiction, we also assume that X is not complete. That is, there exists a Cauchy sequence $\{u_n\}$ which does not converge. Define a function from X into $[0,\infty)$ by $f(x) = \lim_n d(x, u_n)$ for $x \in X$. We note that f is well defined because $\{d(x,u_n)\}$ is a Cauchy sequence for every $x \in X$. The following are obvious:

- $f(x) - f(y) \le d(x,y) \le f(x) + f(y)$ for $x, y \in X$,
- $f(x) > 0$ for all $x \in X$ and
- $\lim_n f(u_n) = 0$.

Define a mapping T on X as follows: For each $x \in X$, since $f(x) > 0$ and $\lim_n f(u_n) = 0$, there exists $\nu \in \mathbb{N}$ satisfying $f(u_\nu) \le \frac{\eta\, r}{3+\eta\, r}\, f(x)$. We put $Tx = u_\nu$. Then it is obvious that

$$f(Tx) \le \frac{\eta\, r}{3 + \eta\, r}\, f(x) \quad \text{and} \quad Tx \in \{u_n : n \in \mathbb{N}\}$$

for all $x \in X$. Then $Tx \ne x$ for all $x \in X$ because $f(Tx) < f(x)$. That is, T does not have a fixed point. Since $T(X) \subset \{u_n : n \in \mathbb{N}\}$, (b) holds. Also, it is not difficult to prove (c). Let us prove (a). Fix $x, y \in X$ with $\eta\, d(x,Tx) \le d(x,y)$. In the case where $f(y) > 2\, f(x)$, we have

$$\begin{aligned} d(Tx,Ty) &\le f(Tx) + f(Ty) \le \frac{\eta\, r}{3+\eta\, r} \big(f(x) + f(y)\big) \\ &\le \frac{r}{3} \big(f(x) + f(y)\big) \\ &\le \frac{r}{3} \big(f(x) + f(y)\big) + \frac{2\, r}{3} \big(f(y) - 2\, f(x)\big) \\ &= r \big(f(y) - f(x)\big) \le r\, d(x,y). \end{aligned}$$

In the other case, where $f(y) \le 2\, f(x)$, we have

$$\begin{aligned} d(x,y) &\ge \eta\, d(x,Tx) \ge \eta \big(f(x) - f(Tx)\big) \\ &\ge \eta \left(1 - \frac{\eta\, r}{3+\eta\, r}\right) f(x) = \frac{3\, \eta}{3 + \eta\, r}\, f(x) \end{aligned}$$

and hence

$$\begin{aligned} d(Tx,Ty) &\le f(Tx) + f(Ty) \le \frac{\eta\, r}{3+\eta\, r} \big(f(x) + f(y)\big) \\ &\le \frac{3\, \eta\, r}{3 + \eta\, r}\, f(x) \le r\, d(x,y). \end{aligned}$$

Therefore we have shown (a), that is, $T \in B_{r,\eta}$. By (iii), T has a fixed point which yields a contradiction. Hence we obtain that X is complete. This completes the proof. □

As a direct consequence of Theorem 4, we obtain the following.

Corollary 1. *For a metric space (X,d), the following are equivalent:*

(i) *X is complete.*

(ii) *There exists $r \in (0,1)$ such that every mapping T on X satisfying the following has a fixed point:*

- $\frac{1}{10000}\, d(x,Tx) \le d(x,y)$ *implies* $d(Tx,Ty) \le r\, d(x,y)$ *for all* $x, y \in X$.

4. The Meir-Keeler theorem

In this section, we prove a generalization of the Meir-Keeler fixed point theorem [17]. See also [5], [12], [16, Theorem 1.5.1].

Theorem 5. *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

- $(1/2)\, d(x, Tx) < d(x, y)$ *and* $d(x, y) < \varepsilon + \delta$ *imply* $d(Tx, Ty) \leq \varepsilon$ *and*
- $(1/2)\, d(x, Tx) < d(x, y)$ *implies* $d(Tx, Ty) < d(x, y)$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Remark.

(i) The Meir-Keeler fixed point theorem [17] is a generalization of the Banach contraction principle (Theorem 1). However, Theorem 5 is not a generalization of Theorem 2.

(ii) We note $\lim_{r \to 1-0} \theta(r) = 1/2$. By Theorem 3, we can prove that $1/2$ is the best constant.

Proof. If $Tx \neq x$, then it is obvious that $d(x, Tx) < 2\, d(x, Tx)$. So, by hypothesis,

$$d(Tx, T^2 x) < d(x, Tx)$$

holds for all $x \in X$ with $Tx \neq x$. We also note that

$$d(Tx, T^2 x) \leq d(x, Tx)$$

holds for all $x \in X$. Fix $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$ for $n \in \mathbb{N}$. Since $\{d(u_n, u_{n+1})\}$ is a nonincreasing sequence, $\{d(u_n, u_{n+1})\}$ converges to some $\alpha \geq 0$. Arguing by contradiction, we assume $\alpha > 0$. Then $\{d(u_n, u_{n+1})\}$ is strictly decreasing. Hence $d(u_n, u_{n+1}) > \alpha$ for every $n \in \mathbb{N}$. By hypothesis, there exists $\delta > 0$ such that

- $d(x, Tx) < 2\, d(x, y)$ and $d(x, y) < \alpha + \delta$ imply $d(Tx, Ty) \leq \alpha$.

From the definition of α, there exists $j \in \mathbb{N}$ such that $d(u_j, u_{j+1}) < \alpha + \delta$. So we have $d(u_{j+1}, u_{j+2}) \leq \alpha$. This is a contradiction. Therefore $\alpha = 0$. That is, $\lim_n d(u_n, u_{n+1}) = 0$ holds. Fix $\varepsilon > 0$. Then there exists $\delta \in (0, \varepsilon)$ such that

- $d(x, Tx) < 2\, d(x, y)$ and $d(x, y) < \varepsilon + \delta$ imply $d(Tx, Ty) \leq \varepsilon$.

Let $\ell \in \mathbb{N}$ such that $d(u_n, u_{n+1}) < \delta$ for all $n \in \mathbb{N}$ with $n \geq \ell$. We shall show

$$d(u_\ell, u_{\ell+m}) < \varepsilon + \delta \tag{4}$$

for $m \in \mathbb{N}$ by induction. It is obvious that (4) holds when $m = 1$. We assume (4) holds for some $m \in \mathbb{N}$. In the case where $d(u_\ell, u_{\ell+m}) \leq \varepsilon$, we have

$$d(u_\ell, u_{\ell+m+1}) \leq d(u_\ell, u_{\ell+m}) + d(u_{\ell+m}, u_{\ell+m+1}) < \varepsilon + \delta.$$

In the other case, where $\varepsilon < d(u_\ell, u_{\ell+m}) < \varepsilon + \delta$, since

$$d(u_\ell, u_{\ell+1}) < \delta < \varepsilon < d(u_\ell, u_{\ell+m}) < 2\, d(u_\ell, u_{\ell+m}),$$

we have $d(u_{\ell+1}, u_{\ell+m+1}) \leq \varepsilon$ and hence

$$d(u_\ell, u_{\ell+m+1}) \leq d(u_\ell, u_{\ell+1}) + d(u_{\ell+1}, u_{\ell+m+1}) < \delta + \varepsilon.$$

So, by induction, (4) holds for every $m \in \mathbb{N}$. Therefore we have shown

$$\lim_{n \to \infty} \sup_{m > n} d(u_n, u_m) = 0.$$

This implies that $\{u_n\}$ is Cauchy. Since X is complete, $\{u_n\}$ converges to some point $z \in X$. We shall show that such z is a fixed point of T, dividing the following two cases:

- There exists $\nu \in \mathbb{N}$ such that $u_\nu = u_{\nu+1}$.
- $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$.

In the first case, $u_n = u_\nu$ for all $n \in \mathbb{N}$ with $n \geq \nu$. Since $\{u_n\}$ converges to z, we have $u_n = z$ for all $n \in \mathbb{N}$ with $n \geq \nu$. This implies $Tz = z$. In the second case, we note $u_n \neq Tu_n$ for $n \in \mathbb{N}$, so $\{d(u_n, u_{n+1})\}$ is strictly decreasing. If we assume that

$$d(u_n, u_{n+1}) \geq 2\, d(u_n, z) \quad \text{and} \quad d(u_{n+1}, u_{n+2}) \geq 2\, d(u_{n+1}, z)$$

hold for some $n \in \mathbb{N}$, then we have

$$\begin{aligned} d(u_n, u_{n+1}) &\leq d(u_n, z) + d(u_{n+1}, z) \\ &\leq \big(d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2})\big)/2 \\ &< d(u_n, u_{n+1}). \end{aligned}$$

This is a contradiction. That is, either

$$d(u_n, u_{n+1}) < 2\, d(u_n, z) \quad \text{or} \quad d(u_{n+1}, u_{n+2}) < 2\, d(u_{n+1}, z)$$

holds for all $n \in \mathbb{N}$. By hypothesis, either

$$d(u_{n+1}, Tz) < d(u_n, z) \quad \text{or} \quad d(u_{n+2}, Tz) < d(u_{n+1}, z)$$

holds for all $n \in \mathbb{N}$. Since $\{u_n\}$ converges to z, the above inequalities imply there exists a subsequence of $\{u_n\}$ which converges to Tz. This implies $Tz = z$. We have shown that z is a fixed point of T. Finally, arguing by contradiction, suppose there exists another fixed point y of T. Since

$$d(z, Tz) = 0 < 2\, d(z, y),$$

we have

$$d(z, y) = d(Tz, Ty) < d(z, y).$$

This is a contradiction. That is, the fixed point is unique. This completes the proof. □

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References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181.

[2] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc., **215** (1976), 241–251. MR0394329 (52:15132)

[3] J. Caristi and W. A. Kirk, *Geometric fixed point theory and inwardness conditions*, Lecture Notes in Math., Vol. 490, pp. 74–83, Springer, Berlin, 1975. MR0399968 (53:3806)

[4] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273. MR0356011 (50:8484)

[5] ———, *A new fixed-point theorem for contractive mappings*, Publ. Inst. Math. (Beograd), **30** (1981), 25–27. MR672538 (83m:54082a)

[6] E. H. Connell, *Properties of fixed point spaces*, Proc. Amer. Math. Soc., **10** (1959), 974–979. MR0110093 (22:976)

[7] J. Dugundji, *Positive definite functions and coincidences*, Fund. Math., **90** (1976), 131–142. MR0400192 (53:4027)

[8] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., **47** (1974), 324–353. MR0346619 (49:11344)

[9] ———, *Nonconvex minimization problems*, Bull. Amer. Math. Soc., **1** (1979), 443–474. MR526967 (80h:49007)

[10] T. K. Hu, *On a fixed-point theorem for metric spaces*, Amer. Math. Monthly, **74** (1967), 436–437. MR0210107 (35:1002)

[11] R. Kannan, *Some results on fixed points – II*, Amer. Math. Monthly, **76** (1969), 405–408. MR0257838 (41:2487)

[12] J. Jachymski, *Equivalent conditions and the Meir-Keeler type theorems*, J. Math. Anal. Appl., **194** (1995), 293–303. MR1353081 (96h:54033)

[13] W. A. Kirk, *Caristi's fixed point theorem and metric convexity*, Colloq. Math., **36** (1976), 81–86. MR0436111 (55:9061)

[14] ———, *Contraction mappings and extensions* in Handbook of metric fixed point theory (W. A. Kirk and B. Sims, Eds.), 2001, pp. 1–34, Kluwer Academic Publishers, Dordrecht. MR1904272 (2003f:54096)

[15] ———, *Fixed points of asymptotic contractions*, J. Math. Anal. Appl., **277** (2003), 645–650. MR1961251 (2003k:47093)

[16] M. Kuczma, B. Choczewski and R. Ger, *Iterative functional equation*, Encyclopedia of Mathematics and Applications, vol. 32, Cambridge University Press, Cambridge, 1990. MR1067720 (92f:39002)

[17] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28** (1969), 326–329. MR0250291 (40:3530)

[18] S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488. MR0254828 (40:8035)

[19] S. Park, *Characterizations of metric completeness*, Colloq. Math., **49** (1984), 21–26. MR774845 (86d:54042)

[20] S. Reich, *Kannan's fixed point theorem*, Boll. Un. Mat. Ital., **4** (1971), 1–11. MR0305163 (46:4293)

[21] P. V. Subrahmanyam, *Remarks on some fixed point theorems related to Banach's contraction principle*, J. Math. Phys. Sci., **8** (1974), 445–457. MR0358749 (50:11208)

[22] ———, *Completeness and fixed-points*, Monatsh. Math., **80** (1975), 325–330. MR0391065 (52:11887)

[23] T. Suzuki, *Generalized distance and existence theorems in complete metric spaces*, J. Math. Anal. Appl., **253** (2001), 440–458. MR1808147 (2002f:49038)

[24] ———, *Several fixed point theorems concerning τ-distance*, Fixed Point Theory Appl., **2004** (2004), 195–209. MR2096951

[25] ———, *Contractive mappings are Kannan mappings, and Kannan mappings are contractive mappings in some sense*, Comment. Math. Prace Mat., **45** (2005), 45–58. MR2199893 (2006m:54055)

[26] J. D. Weston, *A characterization of metric completeness*, Proc. Amer. Math. Soc., **64** (1977), 186–188. MR0458359 (56:16562)

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