PARTITIONING TRIPLES AND PARTIALLY ORDERED SETS

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Abstract. We prove that if $P$ is a partial order and $P \rightarrow (\omega_1^1, \omega)$, then
(a) $P \rightarrow (\omega + \omega + 1, 4)^3$, and
(b) $P \rightarrow (\omega + m, n)^3$ for each $m, n < \omega$.
Together these results represent the best progress known to us on the following question of P. Erdős and others. If $P \rightarrow (\omega_1^1, \omega)$, then does $P \rightarrow (\alpha, n)^3$ for each $\alpha < \omega_1$ and each $n < \omega$?

1. Introduction

The results presented here represent only the most recent leg of the journey begun in 1956 by P. Erdős and R. Rado when they proved the following.

Theorem 1 (P. Erdős and R. Rado, [2]). If $L$ is a real order, then $L \rightarrow (\omega + m, 4)^3$ for each $m < \omega$.

(A real order is an uncountable linear order all of whose wellordered and anti-wellordered subsets are countable.) This result for triples stood alone for over thirty years, until 1987, when E. C. Milner and K. Prikry extended it from real orders to non-special linear orders.

Theorem 2 (E. C. Milner and K. Prikry, [7]). If $L$ is a non-special linear order, then $L \rightarrow (\omega + m, 4)^3$ for each $m < \omega$.

(A partial order $P$ is non-special if it cannot be decomposed into countably many anti-wellfounded subsets.) In particular, this implies that $\omega_1 \rightarrow (\omega + m, 4)^3$ for all $m < \omega$, a previously unknown fact. Six years later, Milner and Prikry improved on this more specific result.

Theorem 3 (E. C. Milner and K. Prikry, [8]). $\omega_1 \rightarrow (\omega + \omega + 1, 4)^3$.

In 1999, we were able to increase the four to an arbitrary finite integer, but at the cost of dropping back below $\omega + \omega$ in the other color.

Theorem 4 (A. L. Jones, [4]). $\omega_1 \rightarrow (\omega + m, n)^3$ for all $m, n < \omega$. 
The results presented below extend the domain of the results referenced above from that of the real orders, the non-special linear orders, or just \( \omega \) to that of all non-special partial orders. Even so, they fall almost unbelievably short of resolving the following.

**Conjecture** (P. Erdős, E. C. Milner, K. Prikry, A. L. Jones, et al.). If \( P \) is a partial order and \( P \to (\omega)_\omega \), then \( P \to (\alpha, n)^3 \) for all \( \alpha < \omega_1 \) and all \( n < \omega \).

We remark that this conjecture is sharp in the sense that if \( P \to (\omega + 1, 4)^3 \), then necessarily \( P \to (\omega)^3_1 \). The interested reader is kindly referred to [5] for more details.

2. **Notation**

Our set theoretic notation is essentially that of [3].

If \( P \) and \( Q \) are partial orders, then let \( [P]^Q = \{ X \subseteq P \mid X \cong Q \} \) be the set of all subsets of \( P \) which are order-isomorphic to \( Q \). Thus, for any ordinal \( \alpha \), \([P]^\alpha\) is the set of wellordered chains of \( P \) of length \( \alpha \). For any ordinal \( \alpha \), let \([P]^{<\alpha} = \bigcup \{ [P]^\beta \mid \beta < \alpha \} \), etc. Also, let \([P]^n_{\alpha, \kappa} = \{ X \subseteq P \mid X \to (\omega)^1_\kappa \} \) be the set of non-special subsets of \( P \). A partial order \( P \) is non-special if \( P \to (\omega)^1_1 \).

If \( A \) and \( B \) are subsets of \( P \), then by \( A < B \) we mean that each element of \( A \) is less than each element of \( B \). If \( A \) is a subset of \( P \) and \( b \) is an element of \( P \), then by \( A < b \) we mean that each element of \( A \) is less than \( b \). If \( a \) is an element of \( P \) and \( B \) is a subset of \( P \), then by \( a < B \) we mean that \( a \) is less than each element of \( B \).

If \( P_1, \ldots, P_n \) and \( Q_1, \ldots, Q_n \) are partial orders, then

\[
[P_1, \ldots, P_n]^{Q_1, \ldots, Q_n} = \{ X_1 \cup \cdots \cup X_n \mid X_1 \in [P_1]^{Q_1}, \ldots, X_n \in [P_n]^{Q_n} \}.
\]

This notation will be most useful to us when \( n \) is small and each \( Q_i \) is a small ordinal. For example, if \( A \) and \( B \) are subsets of \( P \) with \( A < B \), then \([A, B]^{2, 1}\) is the set of three-element chains of \( P \) with two elements in \( A \) and one element in \( B \). Similarly, \([A, B, C]^{1, 1, 1}\) might represent triples with one element in each of the sets \( A, B, \) and \( C \), etc.

If \( x, y \in [P]^{<\omega} \), then put \( x \sqsubset y \) if \( x \neq y \) and there are \( u, v \in [P]^{<\omega} \) with \( u < v \), \( x = u \), and \( y = u \cup v \). Put \( x < y \) if \( x \neq y \) and there are \( u, v_0, v_1 \in [P]^{<\omega} \) with \( u < v_0 < v_1 \), \( x = u \cup v_0 \), and \( y = u \cup v_1 \).

If \( P, Q_0, \) and \( Q_1 \) are partial orders and \( m < \omega \), then the partition relation \( P \to (Q_0, Q_1)^m \) holds if for every partition \([P]^m = K_0 \cup K_1 \) either there is \( X_0 \in [P]^{Q_0} \) with \([X_0]^m \subseteq K_0 \) or there is \( X_1 \in [P]^{Q_1} \) with \([X_1]^m \subseteq K_1 \).

Similarly, if \( P \) and \( Q \) are partial orders, \( \kappa \) is a cardinal, and \( m < \omega \), then the partition relation \( P \to (Q)^m_\kappa \) holds if for every partition \( \beta : [P]^m \to \kappa \) there is \( X \in [P]^Q \) on which \( \beta \) is constant \( (i.e., \text{ there is } i < \kappa \text{ with } \beta^{\circ}[X]^m \subseteq \{i\}) \).

The study of these relations (and much of the notation defined above) was introduced by P. Erdős and R. Rado in [2].

3. **Results**

This section is devoted to our proof of the following.

**Proposition.** If \( P \) is a partial order and \( P \to (\omega)_\omega \), then

(a) \( P \to (\omega + \omega + 1, 4)^3 \), and
(b) \( P \to (\omega + m, n)^3 \) for all \( m, n < \omega \).

By a result of S. Todorcevic it is enough to prove this proposition for non-special trees of cardinality less than that of the continuum, under the additional assumption that \( p = c \). We do so below, and refer the reader to either [10], [1], or [6] for a detailed explanation of this (now somewhat standard) metamathematical reduction.

Our kind reader should be familiar with several fundamental facts.

**Theorem 5** (F. P. Ramsey, [9]). \( \omega \to (\omega)^m_n \) for all \( m, n < \omega \).

A family \( \mathcal{F} \subseteq [\omega]^\omega \) is a filter base if the intersection of any finitely many of its elements is infinite. In particular, any subfamily of a non-principal ultrafilter over \( \omega \) is a filter base. A set \( X \in [\omega]^\omega \) is a pseudo-intersection of \( \mathcal{F} \subseteq [\omega]^\omega \) if \( X \subseteq^* F \) for every \( F \in \mathcal{F} \). Here, \( X \subseteq^* F \) if there is \( N < \omega \) with \( X \setminus N \subseteq F \).

The cardinal \( p \) is the pseudo-intersection number, the minimal cardinality of a filter base for which there is no pseudo-intersection. Most importantly, every filter base of cardinality less than \( p \) must have a pseudo-intersection. As usual, the cardinal \( \omega_1 \) is the cardinality of the continuum. Note that \( \omega_1 \leq p \leq c \).

An ultrafilter \( \mathcal{U} \) over \( \omega \) is a Ramsey ultrafilter if \( \omega \to (\mathcal{U})_m^n \) for all \( m, n < \omega \). If we assume either CH or Martin’s Axiom (or more generally that \( p = c \)), then such filters are easily constructed.

**Theorem 6** (S. Todorcevic, [10]). Non-special trees are much like \( \omega_1 \) in that

(i) if \( T \) is a non-special tree and \( f : T \to T \) is regressive (that is, \( f(t) < t \) for all \( t \in T \) ), then there must be \( S \in [T]^{n.s.} \) on which \( f \) is constant,

(ii) \( \text{n.s. tree} \rightarrow (\text{n.s. tree}, \omega + 1)^2 \), and

(iii) \( \text{n.s. tree} \rightarrow (\alpha)^n_o \) for all \( \alpha < \omega_1 \) and \( n < \omega \).

Each part of the above theorem is a generalization to non-special trees of an important result about \( \omega_1 \):

(i) Fodor’s (Pressing Down) Lemma,

(ii) The Erdős–Dushnik–Miller Theorem, and

(iii) The Baumgartner–Hajnal Theorem.

We refer the interested reader to [3], [2], [1], and [10] for more information on these and other related results.

If \( T \) is a non-special tree and \( P \) is a partial order, then a function \( f : T \to P \) is said to be almost always non-increasing if for every \( s \in T \) the set

\[ I_f(s) = \{ t \in T \mid s < t \land f(s) < f(t) \} \]

is special.

**Corollary 7.** If \( T \) is a non-special tree, \( W \) is a wellordering, and \( f : T \to W \) is almost always non-increasing, then there is \( S \in [T]^{n.s.} \) on which \( f \) is constant.

**Proof.** Let \( I = \{ t \in T \mid \exists s < t [ t \in I_f(s)] \} \) be the diagonal union of \( \{ I_f(s) \mid s \in T \} \). Let \( \tilde{T} = T \setminus I \). By (i), \( \tilde{T} \) is non-special. Define \( [\tilde{T}]^2 = K_0 \cup K_1 \) as follows. For \( s, t \in \tilde{T} \) with \( s < t \), put \( \{ s, t \} \in K_0 \) if \( f(s) = f(t) \) and put \( \{ s, t \} \in K_1 \) if \( f(s) > f(t) \). Because \( W \) is wellfounded there can be no \( X_1 \in [\tilde{T}]^\omega \) with \( [X_1]^2 \subseteq K_1 \). By (ii), there must be \( X_0 \in [\tilde{T}]^{n.s.} \) with \( [X_0]^2 \subseteq K_0 \). Choose \( S \in [X_0]^{n.s.} \) with a unique minimal element. Evidently \( f \) is constant on \( S \).

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\( ^1 \)Part (b) of the proposition first appeared in [4], though the proof given there was much more difficult to follow and was marred by a few typographical errors.
By (i) we can (and do henceforth) assume that each non-special tree $T$ is well-pruned in that $T(b) = \{t \in T \mid b < t\}$ is non-special for each $t \in T$. In particular, if $A < b$ for $A \subseteq T$ and $b \in T$, then there must be $B \in [T]^{n.s.}$ (namely $B = T(b)$) with $A < B$.

We now proceed to our proof of the proposition. Part (a) of the proposition will follow from Lemma A and Lemma C directly, while part (b) of the proposition will follow from Lemma B and Lemma C via a straightforward inductive argument.

**Lemma A.** Let $T$ be a non-special tree with $[T]^3 = K_0 \cup K_1$. If there are $A \in [T]^{n.s.}$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$, then either

(i) there is $X \in [T]^\omega$ with $[X]^3 \subseteq K_0$, or

(ii) there is $Y \in [T]^4$ with $[Y]^3 \subseteq K_1$.

**Proof.** Let $\mathcal{U}$ be a non-principal ultrafilter over $A$. Thus, for each pair $\{r, s\} \in [B]^2$, there are $i_{r,s} \in \{0, 1\}$ and $A_{r,s} \in \mathcal{U}$ with $\{a, r, s\} \in K_{i_{r,s}}$ for each $a \in A_{r,s}$. For each $t \in B$, call the pair $\langle x, y \rangle$ good for $t$ if

(1) $x \in [A]^{<\omega}$, $y \in [B]^{<\omega}$, and $y < t$,

(2) $[x, y]^{1,2} \cup [y, \{t\}]^{1,1} \cup [y, \{t\}]^{2,1} \subseteq K_0$,

(3) $i_{r,s} = 0$ for each pair $\{r, s\} \in [y \cup \{t\}]^{2,1}$.

If $\langle x_0, y_0 \rangle$ and $\langle x_1, y_1 \rangle$ are both good pairs for $t$, then put $\langle x_0, y_0 \rangle \sqsubset \langle x_1, y_1 \rangle$ if both $x_0 \sqsubset x_1$ and $y_0 \sqsubset y_1$. Note that $(\emptyset, \emptyset)$ is a good pair for each $t \in B$.

**Claim.** For some $t \in B$ there is an infinite increasing sequence $\langle x_0, y_0 \rangle \sqsubset \langle x_1, y_1 \rangle \sqsubset \langle x_2, y_2 \rangle \sqsubset \cdots$ of pairs that are good for $t$, then either (i) or (ii) holds.

**Proof.** Let $X = \bigcup \{x_n \mid n < \omega\}$ and $Y = \bigcup \{y_n \mid n < \omega\}$. Then $X, Y \in [T]^\omega$ because $\langle x_n, y_n \rangle \sqsubseteq \langle x_{n+1}, y_{n+1} \rangle$ for each $n < \omega$. Note that $X < Y$ and

$$[X, Y \cup \{t\}]^{2,1} \subseteq K_0$$

because $X \subseteq A$ and $Y \cup \{t\} \subseteq B$. Also,

$$[X, Y]^{1,2} \cup [X, Y \cup \{t\}]^{1,1,1} \cup [Y \cup \{t\}]^{2,1} \subseteq K_0$$

because each $\langle x_n, y_n \rangle$ is a good pair for $t$. This is almost enough to make $[X \cup Y \cup \{t\}]^{3} \subseteq K_0$; all that is lacking is that $[X]^3 \subseteq K_0$ and $[Y]^3 \subseteq K_0$. But because $X \rightarrow (\omega, 4)^3$, either there are $X_0 \in [X]^\omega$ and $Y_0 \in [Y]^\omega$ with $[X_0]^3 \subseteq K_0$ and $[Y_0]^3 \subseteq K_0$, in which case $X_0 \cup Y_0 \cup \{t\} \in [T]^\omega \cdots$ and $[X_0 \cup Y_0 \cup \{t\}]^{3} \subseteq K_0$ (and hence (i) holds), or there is either $Z \in [X]^4$ or $Z \in [Y]^4$ with $[Z]^3 \subseteq K_1$ (and hence (ii) holds).

Without loss of generality, we may therefore assume that for each $t \in B$ there is a $\sqsupset$-maximal good pair $\langle x_t, y_t \rangle$. Moreover, we may assume (by pressing down in $B$, if necessary) that $\langle x_t, y_t \rangle$ is the same pair $\langle x, y \rangle$ for all $t \in B$. Note that this implies that for each pair $\{s, t\} \in [B]^2$, either

(1) there is $r \in x \cup y$ with $\{r, s, t\} \subseteq K_1$, or

(2) $i_{s,t} = 1$.

Otherwise, any $a$ in $\bigcap \{A_{p,q} \mid \{p, q\} \in [y \cup \{s\} \cup \{t\}]^2\}$ would make $\langle x \cup \{a\}, y \cup \{s\} \rangle$ a good pair for $t$, contradicting the supposed maximality of $\langle x, y \rangle$ for $t$.

Because $B \rightarrow (\omega + 1)^{2,1}_{\omega+\omega+1}$, either there is $r \in x \cup y$ and $X \in [B]^{\omega+\omega+1}$ with $\{r, s, t\} \subseteq K_1$ for each pair $\{s, t\} \subseteq [X]^2$, or there is $X \in [B]^{\omega+\omega+1}$ with $i_{s,t} = 1$ for all $\{s, t\} \subseteq [X]^2$. In either case, either $[X]^3 \subseteq K_0$ (and hence (i) holds) or there
are $Y \in [X]^3$ and $r \in \bigcap\{A_{s,t} \mid \{s,t\} \in [Y]^2\}$ with $\{r\} \cup Y^3 \subseteq K_1$ (and hence (ii) holds).

\[ \square \]

**Lemma B.** Suppose that $m, n < \omega$ and that n.s. tree $\rightarrow (\omega + m, n)^3$. Let $T$ be a non-special tree with $[T]^3 = K_0 \cup K_1$. If there are $A \in [T]^\omega$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$, then either

(i) there is $X \in [T]^{\omega+m}$ with $[X]^3 \subseteq K_0$, or

(ii) there is $Y \in [T]^{n+1}$ with $[Y]^3 \subseteq K_1$.

**Proof.** Let $\mathcal{U}$ be a non-principal ultrafilter over $A$. Thus, for each pair $(r, s) \in [B]_2$, there are $i_{r,s} \in \{0, 1\}$ and $A_{r,s} \in \mathcal{U}$ with $\{a, r, s\} \in K_{i_{r,s}}$ for each $a \in A_{r,s}$. Since $B \rightarrow (n.s.\ tree, \omega)^2$, either

(1) there is $C \in [B]^{n.s.}$ with $i_{r,s} = 1$ for each pair $(r, s) \in [C]^2$, or

(2) there is $D \in [B]^\omega$ with $i_{r,s} = 0$ for each pair $(r, s) \in [D]^2$.

If (1) holds, then because $C \rightarrow (\omega + m, n)^3$, either there is $E \in [C]^{\omega+m}$ with $[E]^3 \subseteq K_0$ (and hence (i) holds), or there is $F \in [C]^n$ with $[F]^3 \subseteq K_1$. In the latter case, let $a$ be any element of $\bigcap\{A_{r,s} \mid \{r, s\} \in [F]^2\}$, and let $F = \{a\} \cup F$. Clearly, $F \in [T]^{n+1}$ and $[F]^3 \subseteq K_1$ (and hence (ii) holds).

But if (2) holds, then since $D \rightarrow (\omega, n+1)^3$, either there is $\bar{D} \in [D]^\omega$ with $[\bar{D}]^3 \subseteq K_1$, or there is $F \in [D]^{n+1}$ with $[F]^3 \subseteq K_1$ (and hence (ii) holds). In the former case, choose any $E \in [\bar{D}]^m$ and let $A = \bigcap\{A_{r,s} \mid \{r, s\} \in [E]^2\}$. Since $A \rightarrow (\omega, n+1)^3$, either there is $\bar{A} \in [A]^\omega$ with $[\bar{A}]^3 \subseteq K_0$, or there is $F \in [A]^{n+1}$ with $[\bar{F}]^3 \subseteq K_1$ (and hence (ii) holds). In the former case, let $G = \bar{A} \cup E$. Clearly, $G \in [T]^{\omega+m}$ and $[G]^3 \subseteq K_0$ (and hence (i) holds).

\[ \square \]

**Lemma C.** Let $T$ be a non-special tree with $|T| < \mathfrak{p} = \mathfrak{c}$ and $[T]^3 = K_0 \cup K_1$. Then either

(i) there are $A \in [T]^\omega$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$, or

(ii) for each $n < \omega$ there is $C \in [T]^n$ with $[C]^3 \subseteq K_1$.

Informally, n.s. tree $\rightarrow ((\omega : n.s.\ tree)^{2,1}, n)^3$ for each $n < \omega$.

**Proof.** Let $\prec$ be a fixed wellordering of $H(\mathfrak{c}^+)$, the collection of all sets of hereditary cardinality no greater than the continuum.

For each $n < \omega$ and $x, y \in \omega^n$, note that $x \ll y$ if and only if there are $m < n$, $u \in \omega^m$, and $v_0, v_1 \in \omega^{n-m}$ with $u < v_0 < v_1$, $x = u \cup v_0$, and $y = u \cup v_1$. For each $n < \omega$ and $A \in [T]^{\omega^n}$ there is a unique order isomorphism between $\langle [\omega]^n, \prec\rangle$ and $\langle A, \prec_T\rangle$. For each $x \in [\omega]^n$, let $A(x)$ be the element of $A$ identified with $x$ via this isomorphism.

Fix a Ramsey ultrafilter $\mathcal{U}$ over $\omega$. For $n < \omega$, $m < n$, $A \in [T]^{\omega^m}$, and $b \in T$ with $A < b$, define the partition $f_{A,m}^b : [\omega]^{2n-m} \rightarrow \{0, 1\}$ as follows. Decompose each $x \in [\omega]^{2n-m}$ as $x = u \cup v_0 \cup v_1$ with $u < v_0 < v_1$, $|u| = m$, and $|v_0| = |v_1| = n - m$. Let

\[ f_{A,m}^b(x) = i \quad \text{only if } \quad \{A(u \cup v_0), A(u \cup v_1), b\} \in K_i, \]

Because $\mathcal{U}$ is a Ramsey ultrafilter, there must be $i_{A,m}^b \in \{0, 1\}$ and $X_{A,m}^b \in \mathcal{U}$ with $f_{A,m}^b[X_{A,m}^b]^{2n-m} = \{i_{A,m}^b\}$. Let $X_A$ be the $\prec$-least $X \in [\omega]^{\omega}$ with $X \subseteq^* X_{A,m}^b$ for all $m < n$ and all $b \in T$ with $A < b$. Let $N_{A,m}^b$ be the least $N < \omega$ with $X_A \setminus X \subseteq X_{A,m}^b$. 

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For $n < \omega$, $m < n$, and $A \in [T]^{\omega^n}$ let

$$B_{A,m} = \{ b \in T \mid A < b \land i^b_{A,m} = 0 \}.$$ 

For each $n < \omega$ let $B_n$ be the collection of $B \in [T]^{n.s.}$ for which there exist $A \in [T]^{\omega^n}$ and $m < n$ with $B \subseteq B_{A,m}$.

Claim. If (i) fails, then $B_n$ is empty for all $n < \omega$. (Or contrapositively, if $B_n$ is non-empty for some $n < \omega$, then (i) holds.)

Proof. Fix $n < \omega$ and suppose $B \in B_n$. Choose $A \in [T]^{\omega^n}$ and $m < n$ with $B \subseteq B_{A,m}$. Because $B \to (n.s. \text{ tree})_1^\omega$ there must be $\bar{B} \in [B]^{n.s.}$ and $\bar{N} < \omega$ with $\bar{N} = N^b_{A,m}$ for each $b \in \bar{B}$. Choose $u \in [X_A]^m$ and $\{ v_i \mid i < \omega \} \subseteq [X_A]^{n-m}$ with

$$\bar{N} < u < v_0 < v_1 < v_2 < \ldots.$$ 

Let $\bar{A} = \{ A(u \cup v_i) \mid i < \omega \}$. Note that $\bar{A} \in [T]^\omega$, $\bar{B} \in [T]^{n.s.}$, $\bar{A} < \bar{B}$, and $[\bar{A}, \bar{B}]^{2.1} \subseteq K_0$. Thus (i) holds. □

Suppose $A, B \subseteq T$. Then $A$ is bounded in $B$ if there is $b \in B$ with $A < b$. For each $n < \omega$ let $A_n(B)$ be the collection of all $A \in [B]^{\omega^n}$ with $A$ bounded in $B$ and $\{ A(x), A(y), A(z) \} \in K_1$ for all $x, y, z \in [\omega]^n$ with $x \ll y \ll z$ and $|x \cap z| < |x \cap y|$. Let $A_n = A_n(T)$.

Claim. If $B_n$ is empty for all $n < \omega$, then $A_n$ is non-empty for all $n < \omega$.

Proof. Suppose that $B_n$ is empty for all $n < \omega$. Then for each $A \in [T]^{\omega^n}$ the set

$$B_A = \bigcap \{ B_{A,m} \mid m < n \} = \{ b \in T \mid A < b \land \exists m < n [i^b_{A,m} = 0] \}$$

is special. Consequently, for every $B \in [T]^{n.s.}$ each $A \in A_n(B)$ is nicely bounded in $B$, in that there is $b \in B$ with $A < b$ and $i^b_{A,m} = 1$ for all $m < n$.

We will prove (by induction on $n$) that $A_n(B)$ is non-empty for every $B \in [T]^{n.s.}$. Note that $A_0(B)$ and $A_1(B)$ are certainly non-empty for every $B \in [T]^{n.s.}$, being vacuously equal to $[B]^1$ and $\{ A \mid [B]^1 \not\subseteq B[\exists b \in B(A < b)] \}$, respectively.

Fix a non-zero $n < \omega$. Suppose that $A_n(B)$ is non-empty for every $B \in [T]^{n.s.}$. Fix $B \in [T]^{n.s.}$. We will prove that $A_1^{n+1}(B)$ is non-empty. For each $b \in B$, we first try to construct a sequence $\langle A^b_k \mid k < \omega \rangle \subseteq A_n(B)$. Fix $k < \omega$. Suppose that $A^b_j$ has been defined for each $j < k$. If there is $A \in A_n(B)$ with

1. $A^b_j < A < b$ for all $j < k$,
2. $i^b_{A^b_j,m} = i^b_{A^b_{j,m}} = i^b_{A,m} = 1$ for all $j < k$, $m < n$, and $a \in A$, and
3. $N^b_{A^b_j,m} = N^b_{A^b_{j,m}}$ for all $j < k$, $m < n$, and $a \in A$,

then let $A^b_k$ be the $\preceq$-least such $A \in A_n(B)$. Otherwise, stop the construction and let $k^b = k$.

We claim that there must be $b \in B$ for which the construction of $\langle A^b_k \mid k < \omega \rangle$ succeeds. If not, then by repeated application of Theorem 6 and its corollary, there must be $\bar{B} \in [B]^{n.s.}$, $\bar{k} < \omega$, $\langle A_j \mid j < \bar{k} \rangle \subseteq A_n(B)$, and $\langle \bar{N}_{j,m} \mid j < k, m < n \rangle \subseteq \omega$ with

1. $\bar{k} = k^b$ for all $b \in \bar{B}$,
2. $A_j = A^b_j$ for all $j < k$ and $b \in \bar{B}$,
3. $\bar{N}_{j,m} = N^b_{A^b_j,m}$ for all $j < k$, $m < n$, and $b \in \bar{B}$.
But $A_n(B)$ is non-empty. Choose $\bar{A} \in A_n(B)$ and $\bar{b} \in B$ with $\bar{A} < \bar{b}$ and $i_{A,m}^b = 1$ for all $m < n$. This $A$ is a candidate for $A_k^b$, contradicting our assumption that the construction stopped at step $\bar{k}$ for $\bar{b}$.

Choose $b \in B$ for which the construction of $<A_k^b | k < \omega>$ succeeds. For each $k < \omega$, let $N_k = \max\{N_k^k | m < n\}$. Let $A = \{A_k(x) | x \in [X_{A_k}^b \setminus N_k]^n\}$. It is now easily verified that $A \in A_{n+1}(B)$. \hfill $\square$

Claim. If $A_n$ is non-empty for all $n < \omega$, then (ii) holds. (More specifically, for each $n < \omega$, if $A_n$ is non-empty, then there is $Y \in [T]^{n+1}$ with $|Y|^3 \subseteq K_1$.)

Proof. Fix $n < \omega$. Suppose $A \in A_n$. For each $k \leq n$ let

$$x_{n,k} = \{c | c < n - k\} \cup \{kn + c | c < k\}. $$

For all $i < j < k \leq n$ it is easily verified that $x_{n,i} \ll x_{n,j} \ll x_{n,k}$ and $|x_{n,i} \cap x_{n,k}| = n - k < n - j = |x_{n,i} \cap x_{n,j}|$. Let $Y = \{A(x_{n,k}) | k \leq n\}$. Then $Y \in [T]^{n+1}$ and $|Y|^3 \subseteq K_1$.

The lemma now follows directly from the three claims above. \hfill $\square$

Finally—with these three lemmas in hand—the proposition is easily demonstrated.

Proof of (a). Let $T$ be a non-special tree with $[T]^3 = K_0 \cup K_1$. Suppose that there is no $Y \in [T]^3$ with $|Y|^3 \subseteq K_1$. By Lemma C there must then be $A \in [T]^\omega$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A,B]^{2,1} \subseteq K_0$. It now follows from Lemma A that there must be $X \in [T]^{\omega + \omega + 1}$ with $|X|^3 \subseteq K_0$. As $T$ was arbitrary, we may conclude that n.s. tree $\rightarrow (\omega + \omega + 1, 4)^3$. \hfill $\square$

Proof of (b). Fix $m < \omega$ and proceed by induction on $n < \omega$. Assume that n.s. tree $\rightarrow (\omega + m, n)^3$. Let $T$ be a non-special tree with $[T]^3 = K_0 \cup K_1$. Suppose that there is no $Y \in [T]^{n+1}$ with $|Y|^3 \subseteq K_1$. By Lemma C there must then be $A \in [T]^\omega$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A,B]^{2,1} \subseteq K_0$. It now follows from Lemma B that there must be $X \in [T]^{\omega + n}$ with $|X|^3 \subseteq K_0$. As $T$ was arbitrary, we may conclude that n.s. tree $\rightarrow (\omega + m, n + 1)^3$. \hfill $\square$

4. Conclusion

Finally, we remark that we have recently proven the following result, to be published elsewhere.

**Proposition.** $\omega_1 \rightarrow (\omega + \omega + 1, n)^3$ for all $n < \omega$.

Unfortunately, we have been unable to extend our proof to arbitrary non-special trees. This asymmetry leaves us with the following unbalanced pair of open problems.

**Problem 1.** Does $\omega_1 \rightarrow (\omega + \omega + 2, 4)^3$?

**Problem 2.** Does n.s. tree $\rightarrow (\omega + \omega, 5)^3$?
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