CONVEXITY AND THE EXTERIOR INVERSE PROBLEM OF POTENTIAL THEORY

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Abstract. Let \( \Omega_1 \) and \( \Omega_2 \) be bounded solid domains such that their associated volume potentials agree outside \( \Omega_1 \cup \Omega_2 \). Under the assumption that one of the domains is convex, it is deduced that \( \Omega_1 = \Omega_2 \).

1. Introduction

For any (positive Radon) measure \( \mu \) with compact support in Euclidean space \( \mathbb{R}^N \) (\( N \geq 2 \)), we define the usual potential

\[
U^\mu(x) = \int U_y(x) d\mu(y) \quad (x \in \mathbb{R}^N),
\]

where \( U_y(x) = |x - y|^{2-N} \) if \( N \geq 3 \), and \( U_y(x) = \log(1/|x - y|) \) if \( N = 2 \). In the case where \( \mu \) is the restriction of volume measure \( \lambda \) to a bounded Borel set \( A \), we will write \( U^A \) in place of \( U^{\lambda|A} \). A domain \( \Omega \) in Euclidean space \( \mathbb{R}^N \) is called solid if it is bounded, \( \overline{\Omega}^c = \Omega \) and the complement, \( \overline{\Omega}^c \), of \( \overline{\Omega} \) is connected.

A long-standing open question, known as the exterior inverse problem of potential theory, asks: if \( U^{\Omega_1} = U^{\Omega_2} \) on \( (\Omega_1 \cup \Omega_2)^c \), where \( \Omega_1 \) and \( \Omega_2 \) are solid domains, does it follow that \( \Omega_1 = \Omega_2 \)? (The answer is “no” if we omit the word “solid”, as is obvious from the example of a ball and a suitably chosen concentric annular domain of equal measure.) An early result on this problem, due to Novikov [6], says that the answer is “yes” if we require both \( \Omega_1 \) and \( \Omega_2 \) to be convex (or, more generally, starlike with respect to a common point). More recently, Shahgholian [7] proved that it is enough here for \( \Omega_1 \cap \Omega_2 \) to be convex. Kondraškov [5] has shown that the answer to the question is also “yes” if one of the domains is a ball or an ellipsoid (cf. [1]; an elegant elementary proof for the case of a ball may be found in [9]). In this paper we show that convexity of one of the domains is sufficient to arrive at a positive answer.

Theorem 1. Let \( \Omega_1 \) be a solid domain and \( \Omega_2 \) be a convex domain, and let \( \nu \) be a measure such that \( \nu \geq \lambda|_{\Omega_2} \) and \( \nu(\Omega_2^c) = 0 \). If \( U^{\Omega_1} = U^\nu \) on \( (\Omega_1 \cup \Omega_2)^c \), then \( \Omega_2 \subseteq \Omega_1 \).

Corollary 2. Let \( \Omega_1 \) be a solid domain and \( \Omega_2 \) be a convex domain. If \( U^{\Omega_1} = U^{\Omega_2} \) on \( (\Omega_1 \cup \Omega_2)^c \), then \( \Omega_1 = \Omega_2 \).
There is no implication in either direction between the corollary and the result of Shahgholian mentioned above. However, it is worth noting that Theorem 1 only imposes an additional hypothesis on one of the domains in question. Zalcman [9] has conjectured a stronger version of Corollary 2 in which $U^{Ω_1}$ and $U^{Ω_2}$ are only assumed to agree near infinity. The proof of Theorem 1 will be given in Section 3, following some preliminary material in Section 2 concerning the notion of partial balayage, on which it is based.

2. Partial balayage

Let $a_N = σ_N \max\{1, N - 2\}$, where $σ_N$ denotes the surface area of the unit sphere in $\mathbb{R}^N$, and let $q(x) = a_N |x|^2 / (2N)$. Thus $U^Ω + q$ is harmonic on $Ω$, for any bounded open set $Ω$. If $µ$ is a measure with compact support, it is easy to see that there is a greatest subharmonic minorant $s_µ$, say, of $U^µ + q$ on $\mathbb{R}^N$ (using Theorem 3.7.5 of [2], for example). We need the following facts (see [3], [8]).

**Theorem A.** Let $µ$ and $s_µ$ be as above. Then:

(i) the function $s_µ - q$ can be expressed as $U^{f_λ}$, where $f : \mathbb{R}^N \to [0, 1]$ is a Borel function with compact support;

(ii) the open set $ω(µ) = \{U^µ > U^{f_λ}\}$ is bounded, and $fλ = λ|ω(µ)| + µ|ω(µ)|^c$.

The measure $fλ$ arising in Theorem A is called the partial balayage of $µ$ onto $λ$, and will be denoted by $\tilde{µ}$. Obviously, $U^{\tilde{µ}} \leq U^µ$. The decomposition formula in (ii) arises from the fact that $s_µ$ must be harmonic on $ω(µ)$, by standard balayage arguments. It is clear from the lemma that if we define $Ω(µ)$ to be the largest open set $Ω$ for which $(λ - \tilde{µ})(Ω) = 0$, then $Ω(µ)$ is bounded and contains $ω(µ)$, and

$$V = λ|Ω(µ)| + µ|Ω(µ)|^c.$$

The next result is a generalization of a fact due to Gustafsson (see pp. 205-206 of [3]).

**Lemma 3.** Let $Ω_1$ and $Ω_2$ be bounded open sets, where $λ(∂Ω_2) = 0$, and let $ν$ be a measure such that $ν ≥ λ|Ω_2|$, $ν|Ω_2^c = 0$, and $U^ν$ is continuous on $Ω_2$. If $U^ν = U^{Ω_1}$ on $(Ω_1 \cup Ω_2)^c$, and we denote by $η$ the measure satisfying $U^η = \min\{U^{Ω_1}, U^ν\}$, then $\tilde{η} = λ|η|$.

**Proof.** Let $Ω = Ω(η)$ and

$$D_1 = \{U^{Ω_1} < U^ν\}, \quad D_2 = \{U^{Ω_1} > U^ν\}, \quad S = \{U^{Ω_1} = U^ν\}, \quad A = (Ω_1 \cap D_1) \cup (Ω_2 \cap D_2) \cup (Ω_1 \cap Ω_2).$$

Then $A$ is an open set. Since $U^η = U^{Ω_1}$ on $D_1$, $U^η = U^ν$ on $D_2$, and

$$U^η = U^{Ω_1 \cap Ω_2} + \min\{U^{Ω_1 \setminus Ω_2}, U^ν - λ|Ω_1 \cap Ω_2|\},$$

we see that $η|A ≥ λ|A$ and so $\tilde{η}|A = λ|A$. It follows that $A ⊆ Ω$. Since $(Ω_1 \cup Ω_2)^c ⊆ S$, we see that

$$Ω^c ⊆ A^c ⊆ A_1 \cup A_2 \cup ∂Ω_2,$$

where

$$A_1 = Ω^c_1 \cap (D_1 \cup S) \quad \text{and} \quad A_2 = Ω^c_2 \cap (D_2 \cup S).$$

On $A_1 \cap Ω^c$ we have $U^\tilde{η} = U^η = U^{Ω_1}$, and since $U^\tilde{η}$ and $U^{Ω_1}$ both belong to the Sobolev space $W^{2,2}_{loc}(\mathbb{R}^N)$, we have $\tilde{η}(A_1 \cap Ω^c) = λ|Ω_1 \cap Ω^c| = 0$. Since
\[ U_\gamma = U_\eta = U'_\nu \] on \( A_2 \cap \Omega^c \) and \( U'_\nu \) is harmonic on \( \overline{\Omega}^c \), we similarly have \( \gamma(A_2 \cap \Omega^c) = 0 \). Hence \( \gamma(\Omega^c) = 0 \), in view of (3) and the fact that \( \lambda(\partial \Omega_2) = 0 \). The result now follows by applying (1) to the measure \( \eta \).

We denote a typical point \( x \) of \( \mathbb{R}^N \) by \( (x', x_N) \), where \( x' \in \mathbb{R}^{N-1} \) and \( x_N \in \mathbb{R} \), and define

\[ W_+ = \{ x : x_N > 0 \}, \quad W_- = \{ x : x_N < 0 \} \quad \text{and} \quad H = \{ x : x_N = 0 \}. \]

The following result is due to Gustafsson and Sakai [4]. We give a short proof here for the sake of completeness.

**Lemma 4.** Let \( \mu \) be a measure with compact support contained in \( W_- \cup H \) and let \( A = \{ x' : (x', 0) \in \Omega(\mu) \cap H \} \). Then there is a continuous function \( g : A \to (0, \infty) \), continuously vanishing at \( \partial A \), such that

\[ \Omega(\mu) \cap W_+ = \{ (x', x_N) : x' \in A \text{ and } 0 < x_N < g(x') \}. \]

**Proof.** Let \( u = U'_\mu - U_\gamma \). Thus \( u \geq 0 \). We may assume, by means of a limiting argument, that \( \text{supp} \mu \subset W_- \), and so \( u \) is continuously differentiable on an open set containing \( W_+ \cup H \). Let \( \pi(x) = u(x', -x_N) \). We note that \( U'_\mu - \pi + q \) is subharmonic on \( W_+ \), and \( U_\gamma + q \) is subharmonic on all of \( \mathbb{R}^N \). Since

\[ U'_\mu - \pi + q = U'_\mu - u + q = U_\gamma + q \quad \text{on } H, \]

the function

\[ v = \begin{cases} 
\max \{ U'_\mu - \pi + q, U_\gamma + q \} & \text{on } W_+, \\
U_\gamma + q & \text{on } W_- \cup H
\end{cases} \]

is a subharmonic minorant of \( U'_\mu + q \). Thus \( v \leq U_\gamma + q \) by the definition of \( U_\gamma \), whence \( U'_\mu - \pi \leq U_\gamma \) on \( W_+ \) and so \( u \leq \pi \) there. It follows that \( \partial u / \partial x_N \leq 0 \) on \( H \).

Let \( \Omega_+ = \Omega(\mu) \cap W_+ \). Since \( u = 0 \) on \( \omega(\mu)^c \), and so on \( \Omega(\mu)^c \), and since every point of \( \partial \Omega(\mu) \) is the limit of some sequence of points of Lebesgue density of \( \Omega(\mu)^c \), we see that \( \{ \nabla u \} = 0 \) on \( \partial \Omega_+ \cap W_+ \). We note from (1) that \( \Delta u \) is constant in \( \Omega_+ \), so the function \( \partial u / \partial x_N \) is harmonic there, and hence \( \partial u / \partial x_N \leq 0 \) on \( \Omega_+ \), by the maximum principle. Further, since \( u \) is nonconstant in each component of \( \Omega_+ \), and \( u = 0 \) on \( W_+ \setminus \Omega_+ \), we actually have \( \partial u / \partial x_N < 0 \) on \( \Omega_+ \). We now define

\[ g(x') = \sup \{ t > 0 : (x', t) \in \Omega_+ \} \quad (x' \in A). \]

Clearly \( \Omega(\mu) \cap W_+ \) lies under the graph of \( g \). Conversely, if \( (x', x_N) \) lies under the graph of \( g \) and \( x_N > 0 \), then \( u(x', x_N) > 0 \) and so \( (x', x_N) \in \omega(\mu) \subseteq \Omega(\mu) \). Thus (4) holds.

It remains to check that \( g \) is continuous and vanishes at \( \partial A \). In fact, since \( \Omega(\mu) \) is open and

\[ \{ x' : g(x') > c \} = \{ x' : (x', c) \in \Omega(\mu) \} \quad (c > 0), \]

it is clear that \( g \) is lower semicontinuous. On the other hand, if we apply the result of the previous paragraph with hyperplanes of varying orientation, we see that each point of \( \partial \Omega_+ \cap W_+ \) is the vertex of a vertical cone lying in \( \Omega(\mu)^c \), and so \( g \) is also upper semicontinuous. In fact, \( g \) continuously vanishes at \( \partial A \), since we can apply the preceding reasoning with \( H \) replaced by a slightly lower hyperplane. \( \square \)
3. Proof of Theorem 1

Let \( \Omega_1, \Omega_2 \) and \( \nu \) be as in the statement of Theorem 1. We begin by observing that we may assume \( U^\nu \) to be continuous on \( \Omega_2 \). To see this, let \( (\omega_n) \) be an increasing sequence of regular open sets with union \( \Omega_2 \) such that \( \omega_n \subset \omega_{n+1} \) for each \( n \), and let \( \nu_n = (\nu - \lambda)|_{\omega_n \setminus \omega_{n-1}} \), where \( \omega_0 = \emptyset \). If we define \( u_n = U^\nu|_{\omega_{n+1}} \) and extend \( u_n \) to \( \mathbb{R}^N \) by solving the Dirichlet problem on \( \omega_{n+1} \), then the function \( U^{\Omega_2} + \sum u_n \) is continuous on \( \Omega_2 \), equals \( U^\nu \) on \( \Omega_2^c \) and can be expressed as \( U' \) with \( \mu' \geq \lambda|_{\Omega_2} \) and \( \nu'(\Omega_2^c) = 0 \).

Now let \( \eta \) be as in Lemma 3, and let \( \Omega = \Omega(\eta) \). As we noted earlier, it follows from (2) that \( \Omega_1 \cap \Omega_2 \subseteq \Omega \). We will suppose that

\[
\lambda(\Omega_1 \setminus \Omega) > 0
\]

with a view to reaching a contradiction.

Let \( D = \Omega \cup \Omega_2 \). Our first step is to show that

\[
U^{\tilde{\eta}} = U^{\Omega} = U^\nu \quad \text{on} \quad D^c.
\]

To see this, we note that \( U^{\tilde{\eta}} = U^\eta \) on \( \Omega^c \), since \( \omega(\eta) \subseteq \Omega \), and \( U^{\tilde{\eta}} = U^{\Omega} \), by Lemma 3. On \( D^c \setminus \Omega_1 \), which coincides with \( (\Omega_1 \cup \Omega_2)^c \cap \Omega^c \), we thus have \( U^\nu = U^{\Omega_1} = U^{\eta} = U^{\tilde{\eta}} \). The nonnegative function \( U^{\Omega_1} - U^{\Omega} \) is superharmonic on \( \Omega_1 \). It cannot be constant on \( \Omega_1 \), in view of (5), so it is strictly positive there. Hence \( U^{\Omega_1} > U^{\Omega} = U^{\tilde{\eta}} = U^{\eta} = U^\nu \) on \( D^c \cap \Omega_1 \). We have now proved (6).

Let

\[
E = \Omega_2 \setminus \Omega \quad \text{and} \quad \mu = \nu + \lambda|_E,
\]

whence \( D = \Omega \cup E \). Clearly \( U^{\Omega} = U^{\tilde{\eta}} \leq U^{\eta} \leq U^\nu \), so

\[
U^D = U^{\Omega} + U^E \leq U^\nu + U^E = U^\mu,
\]

and from (6) we see that

\[
U^D = U^{\Omega} + U^E = U^\nu + U^E = U^\mu \quad \text{on} \quad D^c.
\]

We note from (7) that \( U^D + q \) is a subharmonic minorant of \( U^\mu + q \), so \( U^D \leq U^{\tilde{\eta}} \leq U^\mu \). The nonnegative function \( U^{\tilde{\eta}} - U^D \) vanishes on \( D^c \), by (8), and hence on \( \mathbb{R}^N \), since it is subharmonic on \( D \). Thus

\[
\tilde{\mu} = \lambda|_D \quad \text{and} \quad D \subseteq \Omega(\mu).
\]

Further,

\[
0 = (\lambda - \tilde{\mu})(\Omega(\mu)) = \lambda((\Omega(\mu) \setminus \Omega_2) \setminus \Omega) = (\lambda - \tilde{\eta})(\Omega(\mu) \setminus \Omega_2),
\]

so \( \Omega(\mu) \setminus \Omega_2 \subseteq \Omega \subseteq D \). In view of (9) we thus see that

\[
D \setminus \Omega_2 = \Omega(\mu) \setminus \Omega_2.
\]

We now claim that \( \partial \Omega_1 \subset \partial \Omega_2 \). For, if this were not the case, we could choose an open ball \( B \subset \partial \Omega_2 \) that intersects \( \partial \Omega_1 \). Since \( U^{\Omega_1} \geq U^{\eta} \geq U^{\tilde{\eta}} = U^\nu \) on \( D^c \), by (6), the function \( U^{\Omega_1} - U^\nu \), which is nonnegative and superharmonic on \( B \) and attains the value 0 on \( B \setminus \Omega_1 \), must vanish identically on \( B \). This leads to a contradiction, as \( B \cap \Omega_1 \neq \emptyset \).

Next we claim that \( \Omega_1 \setminus \Omega_2 \subset D \). For, if there were a point \( y \in \Omega_1 \setminus (D \cup \Omega_2) \), then we could assume (by choosing a suitable coordinate system) that the closest point of \( \partial \Omega_2 \) to \( y \) is 0, and \( y = (0', |y|) \). Let \( y_0 = (0', t_0) \), where \( t_0 = \sup \{ t : (0', t) \in \Omega_1 \} \). Then \( t_0 > |y| \) and \( y_0 \in \partial \Omega_1 \subset \partial \Omega \subseteq \partial \Omega(\mu) \), by the preceding paragraph and (9).
Also, $y \in \Omega(\mu)^c$, by (10). Since $\text{supp} \mu = \overline{\Omega}_2 \subset W_- \cup H$, Lemma 4 now yields the desired contradiction.

In view of the previous paragraph we see that
$$\lambda(\Omega_1 \setminus \Omega) = \lambda((\Omega_1 \cup \Omega_2) \setminus D) \leq \lambda(\partial \Omega_2) = 0,$$
which contradicts (5). Thus $\lambda(\Omega_1 \setminus \Omega) = 0$ and so $\Omega_1 \subseteq \Omega$, by the definition of $\Omega$. Since $\lambda(\Omega_1) = \lambda(\Omega)$, and $\Omega_1$ is solid, it follows that $\Omega = \Omega_1$. Hence $U^\nu - U^\Omega$ is a nonnegative superharmonic function on $\overline{\Omega}$ which attains the value 0, so $U^\nu = U^\Omega = U^{\Omega_1}$ there, and therefore $\lambda(\Omega_2 \setminus \Omega_1) = 0$. It follows that $\Omega_2 \subset \overline{\Omega_1}$, and so $\Omega_2 \subseteq (\overline{\Omega_1})^\circ = \Omega_1$, as required.

The corollary is immediate, since $\lambda(\Omega_1) = \lambda(\Omega_2)$ in this case.

REFERENCES