CONVEXITY AND THE EXTERIOR INVERSE PROBLEM OF POTENTIAL THEORY

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Abstract. Let $\Omega_1$ and $\Omega_2$ be bounded solid domains such that their associated volume potentials agree outside $\Omega_1 \cup \Omega_2$. Under the assumption that one of the domains is convex, it is deduced that $\Omega_1 = \Omega_2$.

1. Introduction

For any (positive Radon) measure $\mu$ with compact support in Euclidean space $\mathbb{R}^N$ ($N \geq 2$), we define the usual potential

$$U^\mu(x) = \int U_y(x) d\mu(y) \quad (x \in \mathbb{R}^N),$$

where $U_y(x) = |x - y|^{2-N}$ if $N \geq 3$, and $U_y(x) = \log (1/|x - y|)$ if $N = 2$. In the case where $\mu$ is the restriction of volume measure $\lambda$ to a bounded Borel set $A$, we will write $U^A$ in place of $U^{\lambda|A}$. A domain $\Omega$ in Euclidean space $\mathbb{R}^N$ is called solid if it is bounded, $\overline{\Omega} \setminus \Omega = \emptyset$ and the complement, $\Omega^c$, of $\Omega$ is connected.

A long-standing open question, known as the exterior inverse problem of potential theory, asks: if $U^{\Omega_1} = U^{\Omega_2}$ on $(\Omega_1 \cup \Omega_2)^c$, where $\Omega_1$ and $\Omega_2$ are solid domains, does it follow that $\Omega_1 = \Omega_2$? (The answer is “no” if we omit the word “solid”, as is obvious from the example of a ball and a suitably chosen concentric annular domain of equal measure.) An early result on this problem, due to Novikov [6], says that the answer is “yes” if we require both $\Omega_1$ and $\Omega_2$ to be convex (or, more generally, starlike with respect to a common point). More recently, Shahgholian [7] proved that it is enough here for $\Omega_1 \cap \Omega_2$ to be convex. Kondraškov [5] has shown that the answer to the question is also “yes” if one of the domains is a ball or an ellipsoid (cf. [1]; an elegant elementary proof for the case of a ball may be found in [9]). In this paper we show that convexity of one of the domains is sufficient to arrive at a positive answer.

Theorem 1. Let $\Omega_1$ be a solid domain and $\Omega_2$ be a convex domain, and let $\nu$ be a measure such that $\nu \geq \lambda|_{\Omega_2}$ and $\nu(\Omega_2^c) = 0$. If $U^{\Omega_1} = U^{\nu}$ on $(\Omega_1 \cup \Omega_2)^c$, then $\Omega_2 \subseteq \Omega_1$.

Corollary 2. Let $\Omega_1$ be a solid domain and $\Omega_2$ be a convex domain. If $U^{\Omega_1} = U^{\Omega_2}$ on $(\Omega_1 \cup \Omega_2)^c$, then $\Omega_1 = \Omega_2$.

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There is no implication in either direction between the corollary and the result of Shahgholian mentioned above. However, it is worth noting that Theorem 1 only imposes an additional hypothesis on one of the domains in question. Zalcman [9] has conjectured a stronger version of Corollary 2 in which $U^{\Omega_1}$ and $U^{\Omega_2}$ are only assumed to agree near infinity. The proof of Theorem 1 will be given in Section 3, following some preliminary material in Section 2 concerning the notion of partial balayage, on which it is based.

2. Partial balayage

Let $a_N = \sigma_N \max\{1, N - 2\}$, where $\sigma_N$ denotes the surface area of the unit sphere in $\mathbb{R}^N$, and let $q(x) = a_N |x|^2 / (2N)$. Thus $U^\Omega + q$ is harmonic on $\Omega$, for any bounded open set $\Omega$. If $\mu$ is a measure with compact support, it is easy to see that there is a greatest subharmonic minorant $s_\mu$, say, of $U^\mu + q$ on $\mathbb{R}^N$ (using Theorem 3.7.5 of [2], for example). We need the following facts (see [3], [8]).

**Theorem A.** Let $\mu$ and $s_\mu$ be as above. Then:

(i) the function $s_\mu - q$ can be expressed as $U^{f_\lambda}$, where $f : \mathbb{R}^N \to [0, 1]$ is a Borel function with compact support;

(ii) the open set $\omega(\mu) = \{U^\mu > U^{f_\lambda}\}$ is bounded, and $f_\lambda = \lambda|_{\omega(\mu)} + \mu|_{\omega(\mu)^c}$.

The measure $f_\lambda$ arising in Theorem A is called the *partial balayage* of $\mu$ onto $\lambda$, and will be denoted by $\tilde{\mu}$. Obviously, $U^{\tilde{\mu}} \leq U^\mu$. The decomposition formula in (ii) arises from the fact that $s_\mu$ must be harmonic on $\omega(\mu)$, by standard balayage arguments. It is clear from the lemma that if we define $\Omega(\mu)$ to be the largest open set $\Omega$ for which $(\lambda - \tilde{\mu})(\Omega) = 0$, then $\Omega(\mu)$ is bounded and contains $\omega(\mu)$, and

$$
(1) \quad \tilde{\mu} = \lambda|_{\omega(\mu)} + \mu|_{\omega(\mu)^c}.
$$

The next result is a generalization of a fact due to Gustafsson (see pp. 205-206 of [3]).

**Lemma 3.** Let $\Omega_1$ and $\Omega_2$ be bounded open sets, where $\lambda(\partial \Omega_2) = 0$, and let $\nu$ be a measure such that $\nu(\Omega_1) = 0$, $\nu(\Omega_2^c) = 0$, and $U^\nu$ is continuous on $\Omega_2$. If $U^\nu = U^{\Omega_1}$ on $(\Omega_1 \cup \Omega_2)^c$, and we denote by $\eta$ the measure satisfying $U^\eta = \min\{U^{\Omega_1}, U^\nu\}$, then

$$
\eta = \lambda|_{\Omega(\eta)}.
$$

**Proof.** Let $\Omega = \Omega(\eta)$ and

$$
D_1 = \{U^{\Omega_1} < U^\nu\}, \quad D_2 = \{U^{\Omega_1} > U^\nu\}, \quad S = \{U^{\Omega_1} = U^\nu\},
$$

$$
A = (\Omega_1 \cap D_1) \cup (\Omega_2 \cap D_2) \cup (\Omega_1 \cap \Omega_2).
$$

Then $A$ is an open set. Since $U^\eta = U^{\Omega_1}$ on $D_1$, $U^\eta = U^\nu$ on $D_2$, and

$$
(2) \quad U^\eta = U^{\Omega_1 \cap \Omega_2} + \min\{U^{\Omega_1 \setminus \Omega_2}, U^\nu - \lambda|_{\Omega_1 \cap \Omega_2}\},
$$

we see that $\eta|_A \geq \lambda|_A$ and so $\tilde{\eta}|_A = \lambda|_A$. It follows that $A \subseteq \Omega$. Since $(\Omega_1 \cup \Omega_2)^c \subseteq S$, we see that

$$
(3) \quad \Omega^c \subseteq A^c \subseteq A_1 \cup A_2 \cup \partial \Omega_2,
$$

where

$$
A_1 = \Omega_1^c \cap (D_1 \cup S) \quad \text{and} \quad A_2 = \Omega_2^c \cap (D_2 \cup S).
$$

On $A_1 \cap \Omega^c$ we have $U^{\tilde{\eta}} = U^\eta = U^{\Omega_1}$, and since $U^{\tilde{\eta}}$ and $U^{\Omega_1}$ both belong to the Sobolev space $W^{2,2}_{\text{loc}}(\mathbb{R}^N)$, we have $\tilde{\eta}(A_1 \cap \Omega^c) = \lambda|_{\Omega_1}(A_1 \cap \Omega^c) = 0$. Since
\[ U^\eta = U^\nu = U^\omega \] on \( A_2 \cap \Omega^c \) and \( U^\nu \) is harmonic on \( \overline{\Omega^c} \), we similarly have \( \overline{\eta(A_2 \cap \Omega^c)} = 0 \). Hence \( \overline{\eta(\Omega^c)} = 0 \), in view of (3) and the fact that \( \lambda(\partial\Omega_2) = 0 \). The result now follows by applying (1) to the measure \( \eta \).

We denote a typical point \( x \) of \( \mathbb{R}^N \) by \( (x', x_N) \), where \( x' \in \mathbb{R}^{N-1} \) and \( x_N \in \mathbb{R} \), and define

\[ W_+ = \{ x : x_N > 0 \}, \quad W_- = \{ x : x_N < 0 \} \quad \text{and} \quad H = \{ x : x_N = 0 \}. \]

The following result is due to Gustafsson and Sakai [4]. We give a short proof here for the sake of completeness.

**Lemma 4.** Let \( \mu \) be a measure with compact support contained in \( W_- \cup H \) and let \( A = \{ x' : (x', 0) \in \Omega(\mu) \cap H \} \). Then there is a continuous function \( g : A \to (0, \infty) \), continuously vanishing at \( \partial A \), such that

\[
\Omega(\mu) \cap W_+ = \{ (x', x_N) : x' \in A \text{ and } 0 < x_N < g(x') \}.
\]

**Proof.** Let \( u = U^\mu - U^\overline{\mu} \). Thus \( u \geq 0 \). We may assume, by means of a limiting argument, that \( \text{supp} \mu \subset W_- \), and so \( u \) is continuously differentiable on an open set containing \( W_+ \cup H \). Let \( \overline{\mu}(x) = u(x', -x_N) \). We note that \( U^\mu - \overline{\mu} + q \) is subharmonic on \( W_+ \), and \( \overline{\mu} + q \) is subharmonic on all of \( \mathbb{R}^N \). Since

\[ U^\mu - \overline{\mu} + q = U^\mu - u + q = U^\overline{\mu} + q \quad \text{on} \ H, \]

the function

\[ v = \begin{cases} \max\{U^\mu - \overline{\mu} + q, U^\overline{\mu} + q\} & \text{on } W_+, \\ U^\overline{\mu} + q & \text{on } W_- \cup H \end{cases} \]

is a subharmonic minorant of \( U^\mu + q \). Thus \( v \leq U^\overline{\mu} + q \) by the definition of \( U^\overline{\mu} \), whence \( U^\mu - \overline{\mu} \leq U^\overline{\mu} \) on \( W_+ \) and so \( u \leq \overline{\mu} \) there. It follows that \( \partial u / \partial x_N \leq 0 \) on \( H \).

Let \( \Omega_+ = \Omega(\mu) \cap W_+ \). Since \( u = 0 \) on \( \omega(\mu)^c \), and so on \( \Omega(\mu)^c \), and since every point of \( \partial \Omega(\mu) \) is the limit of some sequence of points of Lebesgue density of \( \Omega(\mu)^c \), we see that \( |\nabla u| = 0 \) on \( \partial \Omega_+ \cap W_+ \). We note from (1) that \( \Delta u \) is constant in \( \Omega_+ \), so the function \( \partial u / \partial x_N \) is harmonic there, and hence \( \partial u / \partial x_N \leq 0 \) on \( \Omega_+ \), by the maximum principle. Further, since \( u \) is nonconstant in each component of \( \Omega_+ \), and \( u = 0 \) on \( W_+ \setminus \Omega_+ \), we actually have \( \partial u / \partial x_N < 0 \) on \( \Omega_+ \). We now define

\[ g(x') = \sup\{ t > 0 : (x', t) \in \Omega_+ \} \quad (x' \in A). \]

Clearly \( \Omega(\mu) \cap W_+ \) lies under the graph of \( g \). Conversely, if \( (x', x_N) \) lies under the graph of \( g \) and \( x_N > 0 \), then \( u(x', x_N) > 0 \) and so \( (x', x_N) \in \omega(\mu) \subseteq \Omega(\mu) \). Thus (4) holds.

It remains to check that \( g \) is continuous and vanishes at \( \partial A \). In fact, since \( \Omega(\mu) \) is open and

\[ \{ x' : g(x') > c \} = \{ x' : (x', c) \in \Omega(\mu) \} \quad (c > 0), \]

it is clear that \( g \) is lower semicontinuous. On the other hand, if we apply the result of the previous paragraph with hyperplanes of varying orientation, we see that each point of \( \partial \Omega_+ \cap W_+ \) is the vertex of a vertical cone lying in \( \Omega(\mu)^c \), and so \( g \) is also upper semicontinuous. In fact, \( g \) continuously vanishes at \( \partial A \), since we can apply the preceding reasoning with \( H \) replaced by a slightly lower hyperplane. □
3. Proof of Theorem 1

Let \( \Omega_1, \Omega_2 \) and \( \nu \) be as in the statement of Theorem 1. We begin by observing that we may assume \( U^\nu \) to be continuous on \( \Omega_2 \). To see this, let \( (\omega_n) \) be an increasing sequence of regular open sets with union \( \Omega_2 \) such that \( \omega_n \subset \omega_{n+1} \) for each \( n \), and let \( \nu_n = (\nu - \lambda)|_{\omega_n \setminus \omega_{n-1}} \), where \( \omega_0 = \emptyset \). If we define \( u_n = U^{\nu_n} \) on \( \omega_{n+1} \), and extend \( u_n \) to \( \mathbb{R}^N \) by solving the Dirichlet problem on \( \omega_{n+1} \), then the function \( U^{\Omega_2} + \sum u_n \) is continuous on \( \Omega_2 \), equals \( U^\nu \) on \( \Omega_2 \) and can be expressed as \( U' \) with \( \nu' \geq \lambda|_{\Omega_2} \) and \( \nu'(\Omega_2) = 0 \).

Now let \( \eta \) be as in Lemma 3, and let \( \Omega = \Omega(\eta) \). As we noted earlier, it follows from (2) that \( \Omega_1 \cap \Omega_2 \subseteq \Omega \). We will suppose that

\[
(5) \quad \lambda(\Omega_1 \setminus \Omega) > 0
\]

with a view to reaching a contradiction.

Let \( D = \Omega \cup \Omega_2 \). Our first step is to show that

\[
(6) \quad U^\eta = U^\Omega = U^\nu \quad \text{on} \quad D^c.
\]

To see this, we note that \( U^\eta = U^\nu \) on \( \Omega^c \), since \( \omega(\eta) \subseteq \Omega \), and \( U^\eta = U^\Omega \), by Lemma 3. On \( D^c \setminus \Omega_1 \), which coincides with \( (\Omega_1 \cup \Omega_2)^c \cap \Omega^c \), we thus have \( U^\nu = U^{\Omega_1} = U^\eta = U^\eta \). The nonnegative function \( U^{\Omega_1} - U^\Omega \) is superharmonic on \( \Omega_1 \). It cannot be constant on \( \Omega_1 \), in view of (5), so it is strictly positive there. Hence \( U^{\Omega_1} > U^\Omega = U^\eta = U^\nu \) on \( D^c \cap \Omega_1 \). We have now proved (6).

Let

\[
E = \Omega_2 \setminus \Omega \quad \text{and} \quad \mu = \nu + \lambda|_E,
\]

whence \( D = \Omega \cup E \). Clearly \( U^\Omega = U^\eta \leq U^\nu \leq U^\mu \), so

\[
(7) \quad U^D = U^\Omega + U^E \leq U^\nu + U^E = U^\mu,
\]

and from (6) we see that

\[
(8) \quad U^D = U^\Omega + U^E = U^\nu + U^E = U^\mu \quad \text{on} \quad D^c.
\]

We note from (7) that \( U^D + q \) is a subharmonic minorant of \( U^\mu + q \), so \( U^D \leq U^\mu \leq U^\nu \). The nonnegative function \( U^\mu - U^D \) vanishes on \( D^c \), by (8), and hence on \( \mathbb{R}^N \), since it is subharmonic on \( D \). Thus

\[
(9) \quad \tilde{\mu} = \lambda|_D \quad \text{and} \quad D \subseteq \Omega(\mu).
\]

Further,

\[
0 = (\lambda - \tilde{\mu})(\Omega(\mu)) = \lambda(\Omega(\mu) \setminus D) \geq \lambda((\Omega(\mu) \setminus \Omega_2) \setminus D) = (\lambda - \tilde{\eta})(\Omega(\mu) \setminus \Omega_2),
\]

so \( \Omega(\mu) \setminus \Omega_2 \subseteq \Omega \subseteq D \). In view of (9) we thus see that

\[
(10) \quad D \setminus \Omega_2 = \Omega(\mu) \setminus \Omega_2.
\]

We now claim that \( \partial \Omega_1 \subset \partial D \). For, if this were not the case, we could choose an open ball \( B \subset \overline{D} \) that intersects \( \partial \Omega_1 \). Since \( U^{\Omega_1} \geq U^\eta \geq U^\nu \) on \( D^c \), by (6), the function \( U^{\Omega_1} - U^\nu \), which is nonnegative and superharmonic on \( B \) and attains the value 0 on \( B \setminus \Omega_1 \), must vanish identically on \( B \). This leads to a contradiction, as \( B \cap \Omega_1 \neq \emptyset \).

Next we claim that \( \Omega_1 \setminus \Omega_2 \subset D \). For, if there were a point \( y \in \Omega_1 \setminus (D \cup \Omega_2) \), then we could assume (by choosing a suitable coordinate system) that the closest point of \( \Omega_2 \) to \( y \) is 0, and \( y = (y', |y|) \). Let \( y_0 = (y', t_0) \), where \( t_0 = \sup \{ t : (y', t) \in \Omega_1 \} \). Then \( t_0 > |y| \) and \( y_0 \in \partial \Omega_1 \subset \overline{D} \subseteq \Omega(\mu) \), by the preceding paragraph and (9).
Also, \( y \in \Omega(\mu)^c \), by (10). Since \( \text{supp} \mu = \overline{\Omega_2} \subset W^- \cup H \), Lemma 4 now yields the desired contradiction.

In view of the previous paragraph we see that

\[
\lambda(\Omega_1 \setminus \Omega) = \lambda((\Omega_1 \cup \Omega_2) \setminus D) \leq \lambda(\partial \Omega_2) = 0,
\]

which contradicts (5). Thus \( \lambda(\Omega_1 \setminus \Omega) = 0 \) and so \( \Omega_1 \subseteq \Omega \), by the definition of \( \Omega \). Since \( \lambda(\Omega_1) = \lambda(\Omega) \), and \( \Omega_1 \) is solid, it follows that \( \Omega = \Omega_1 \). Hence \( U^\nu - U^\Omega \) is a nonnegative superharmonic function on \( \overline{\Omega} \) which attains the value 0, so \( U^\nu = U^\Omega = U^{\Omega_1} \) there, and therefore \( \lambda(\Omega_2 \setminus \Omega_1) = 0 \). It follows that \( \Omega_2 \subseteq \overline{\Omega_1} \), and so \( \Omega_2 \subseteq (\Omega_1)^c = \Omega_1 \), as required.

The corollary is immediate, since \( \lambda(\Omega_1) = \lambda(\Omega_2) \) in this case.

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