ANTIHOLOMORPHIC INVOLUTIONS
OF SPHERICAL COMPLEX SPACES

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Abstract. Let $X$ be a holomorphically separable irreducible reduced complex space, $K$ a connected compact Lie group acting on $X$ by holomorphic transformations, $\theta : K \to K$ a Weyl involution, and $\mu : X \to X$ an antiholomorphic map satisfying $\mu^2 = \text{Id}$ and $\mu(kx) = \theta(k)\mu(x)$ for $x \in X$, $k \in K$. We show that if $\mathcal{O}(X)$ is a multiplicity free $K$-module, then $\mu$ maps every $K$-orbit onto itself. For a spherical affine homogeneous space $X = G/H$ of the reductive group $G = K^C$ we construct an antiholomorphic map $\mu$ with these properties.

1. Introduction

Let $X = (X, \mathcal{O})$ be a complex space on which a compact Lie group $K$ acts continuously by holomorphic transformations. Then the Fréchet space $\mathcal{O}(X)$ has the natural structure of a $K$-module. Recall that a $K$-module $W$ is called multiplicity free if any irreducible $K$-module occurs in $W$ with multiplicity 1 or does not occur at all. A self-map $\mu$ of a complex space $X$ is called an antiholomorphic involution if $\mu$ is antiholomorphic and $\mu^2 = \text{Id}$. For complex manifolds, J. Faraut and E. G. F. Thomas gave an interesting and simple geometric condition which implies that $\mathcal{O}(X)$ is a multiplicity free $K$-module; see [FT]. Namely, for a complex manifold $X$ the $K$-action in $\mathcal{O}(X)$ is multiplicity free if

\[
\text{there exists an antiholomorphic involution } \mu : X \to X \text{ with the property that, for every } x \in X, \text{ there is an element } k \in K \text{ such that } \mu(x) = k \cdot x.
\]

(FT)

The proof of Theorem 3 in [FT] goes without changes for irreducible reduced complex spaces. It should be noted that the setting in [FT] is more general. Namely, the authors consider any, not necessarily compact, group of holomorphic transformations of $X$ and study invariant Hilbert subspaces of $\mathcal{O}(X)$. We will give a simplified proof of their result in our context; see Proposition 3.3. Our main purpose, however, is to prove the converse theorem for a special class of manifolds, namely, for Stein (or, equivalently, affine algebraic) homogeneous spaces of complex reductive groups.

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Let $G$ be a connected reductive complex algebraic group and $K \subset G$ a maximal compact subgroup. We prove that an affine homogeneous space $X = G/H$ is spherical (or, equivalently, $O(X)$ is a multiplicity free $K$-module) if and only if the $K$-action on $X$ satisfies (FT); see Theorem 5.5. Recall that a diffeomorphism $\mu$ of a manifold $X$ with a $K$-action is said to be $\theta$-equivariant if $\theta$ is an automorphism of $K$ and $\mu(kx) = \theta(k)\mu(x)$ for all $x \in X$, $k \in K$. For $X = G/H$ spherical we can say more about $\mu$ in (FT). Namely, again by Theorem 5.5, $\mu$ can be chosen $\theta$-equivariant, where $\theta$ is a Weyl involution of $K$.

In order to prove Theorem 5.5, we consider $\theta$-equivariant antiholomorphic involutions in a more general context. Namely, let $X$ be a holomorphically separable irreducible reduced complex space, $K$ a connected compact Lie group of holomorphic transformations of $X$, and $\mu$ an antiholomorphic $\theta$-equivariant involution of $X$. Then our Theorem 4.1 asserts that $O(X)$ is multiplicity free if and only if $\mu(x) \in Kx$ for all $x \in X$, i.e., if $X$ has property (FT) with respect to $\mu$.

Another important ingredient in the proof of Theorem 5.5 is the construction of two commuting involutions of $G$, a Weyl involution and a Cartan involution, which both preserve a given reductive spherical subgroup $H \subset G$; see Theorem 5.4. The proof is based on the results of [AV] and, therefore, on the classification of spherical subgroups.

At the end of our paper we give an example of an affine homogeneous space without $\theta$-equivariant antiholomorphic involutions; see Proposition 6.3.

2. Fourier series of Harish-Chandra

Harish-Chandra carried over the classical Fourier series to the representation theory of compact Lie groups in Fréchet spaces; see [H-C]. In this paper, we will need only the representations in $O(X)$, where $X$ is a complex space. We recall the result of Harish-Chandra in this setting. The details can be found in [H-C]; see also [A], Ch. 5.

Let $K$ be a compact Lie group, $\hat{K}$ its unitary dual, and $dk$ the normalized Haar measure on $K$. For $\delta \in \hat{K}$ let $\chi_\delta$ denote the character of $\delta$ multiplied by the dimension of $\delta$. Suppose that $K$ acts by holomorphic transformations on a complex space $X$. Then we have a continuous representation of $K$ in $C(X)$ and in $O(X)$. We will assume that $X$ is reduced and irreducible, so the representation is given by $k \cdot f(x) = f(k^{-1}x)$, where $k \in K$, $x \in X$.

Define an operator family $\{E_\delta\}_{\delta \in \hat{K}}$ in $O(X)$ by

$$E_\delta f(x) = \int_K \overline{\chi_\delta(k)} \cdot f(k^{-1}x) \cdot dk.$$ 

From orthogonality relations for characters it follows that all $E_\delta$ commute with the representation of $K$. Furthermore, $\{E_\delta\}_{\delta \in \hat{K}}$ is a family of projection operators, i.e., $E_\delta^2 = E_\delta$ and $E_\delta E_\epsilon = 0$ if $\delta \neq \epsilon$. Let $O_\delta(X) = E_\delta O(X)$. Then $O_\delta(X) = \text{Ker}(E_\delta - \text{Id})$, so $O_\delta(X)$ is a closed subspace. Again from orthogonality relations it follows that $O_\delta(X)$ is the isotypic component of type $\delta$, i.e., $O_\delta(X)$ consists of all those vectors in $O(X)$ whose $K$-orbit is contained in a finite-dimensional $K$-submodule where the representation is some multiple of $\delta$.

Harish-Chandra proved that each $f \in O(X)$ can be uniquely written in the form

$$f = \sum_{\delta \in \hat{K}} f_\delta,$$
where \( f_\delta = E_\delta(f) \in \mathcal{O}_\delta(X) \) and the convergence is absolute and uniform on compact subsets in \( X \).

Assume now that \( L \) is another compact Lie group acting by holomorphic transformations of another complex space \( Y \) subject to our assumptions. We will use similar notation for \( L \), in particular, \( \theta_\epsilon \) will denote the character of \( \epsilon \in \hat{L} \) multiplied by the dimension of \( \epsilon \). For the representation of \( K \times L \) in \( \mathcal{O}(X \times Y) \) defined by
\[
(k, l) \cdot f(x, y) = f(k^{-1}x, l^{-1}y), \quad x \in X, \ y \in Y, \ k \in K, \ l \in L,
\]
the type of an isotypic component is determined by a pair \( \delta \in \hat{K}, \epsilon \in \hat{L} \). The corresponding isotypic component will be denoted \( \mathcal{O}_{\delta, \epsilon}(X \times Y) \). Of course, the tensor product \( \mathcal{O}_\delta(X) \otimes \mathcal{O}_\epsilon(Y) \) is contained in \( \mathcal{O}_{\delta, \epsilon}(X \times Y) \). We will need the following lemma.

**Lemma 2.1.** If \( \dim \mathcal{O}_\delta(X) < \infty \) for some \( \delta \in \hat{K} \), then \( \mathcal{O}_{\delta, \epsilon}(X \times Y) = \mathcal{O}_\delta(X) \otimes \mathcal{O}_\epsilon(Y) \) for all \( \epsilon \in \hat{L} \).

**Proof.** Let \( f \in \mathcal{O}_{\delta, \epsilon}(X \times Y) \). Then
\[
\text{Fubini's theorem. The function } x \mapsto \int_K \chi_\delta(k) \cdot f(k^{-1}x, y) \cdot dk
\]
is in \( \mathcal{O}_\delta(X) \) for all \( y \in Y \). Let \( \{ \varphi_i \}_{i=1,...,N} \) be a basis of \( \mathcal{O}_\delta(X) \). Then
\[
\int_K \chi_\delta(k) \cdot f(k^{-1}x, y) \cdot dk = \sum_{i=1}^{N} c_i(y) \varphi_i(x)
\]
with some \( c_i \in \mathcal{O}(Y) \). Replace \( y \) by \( l^{-1}y \) in this equality, multiply it by \( \theta_\epsilon(l) \), and integrate over \( L \) against the Haar measure \( dl \). Then we get
\[
f(x, y) = \sum_{i=1}^{N} \varphi_i(x) \psi_i(y),
\]
where
\[
\psi_i(y) = \int_L \theta_\epsilon(l) \cdot c_i(l^{-1}y) \cdot dl \in \mathcal{O}_\epsilon(Y). \quad \square
\]

### 3. K-action and Complex Conjugation

As in the previous section, \( X \) is an irreducible reduced complex space and \( K \) is a compact group acting on \( X \) by holomorphic transformations.

**Lemma 3.1.** Let \( W \subset \mathcal{O}(X) \) be a finite-dimensional \( K \)-submodule. Introduce a \( K \)-invariant Hermitian inner product and choose a unitary basis \( \{ f_1, \ldots, f_N \} \) in \( W \). The function \( F := \sum_{j=1}^{N} f_j^* f_j \) is \( K \)-invariant. Furthermore, \( F \) does not depend on the choice of basis.
Proof. Let \( \{g_1, \ldots, g_N\} \) be another unitary basis of \( W \). There is a unitary transformation \( A : W \to W \) such that \( A(f_j) = g_j = \sum_{i=1}^N a_{ij} f_i \). We have

\[
\sum_{j=1}^N g_j \overline{g_j} = \sum_{j=1}^N \sum_{i,j'=1}^N a_{ij} \overline{a_{ij'}} f_i f_{j'} = \sum_{i,j'=1}^N \delta_{ij} f_i f_{j'} = F.
\]

Now, \( k \cdot F = \sum_{j=1}^N (k \cdot f_j)(k \cdot f_j) \) for \( k \in K \). But \( \{k \cdot f_1, \ldots, k \cdot f_N\} \) is another unitary basis of \( W \). Since \( F \) does not depend on the choice of basis, it follows that \( k \cdot F = F \) for any \( k \in K \).

\[\square\]

**Lemma 3.2.** If \( W \subset \mathcal{O}(X) \) is a finite-dimensional \( K \)-submodule, then \( \overline{W} \subset \overline{\mathcal{O}(X)} \) is also a \( K \)-submodule, which is isomorphic to the dual module \( W^* \).

**Proof.** Let \( (f, g) \) be a \( K \)-invariant Hermitian product on \( W \). For \( f \in W \), \( \phi \in \overline{W} \) we have the bilinear pairing

\[\langle f, \phi \rangle = \langle f, \overline{\phi} \rangle,
\]

which is obviously \( K \)-invariant and non-degenerate. This shows that \( \overline{W} \) is isomorphic to \( W^* \).

Let \( \mu : X \to X \) be an antiholomorphic involution. Then, by definition, the function \( \mu f(x) = f(\mu x) \) is antiholomorphic for any \( f \in \mathcal{O}(X) \). We want to give a simple proof of the theorem of J. Faraut and E. G. F. Thomas in our setting.

**Proposition 3.3.** If the \( K \)-action on \( X \) satisfies (FT), then \( \mathcal{O}(X) \) is a multiplicity free \( K \)-module.

**Proof.** Assume the contrary. Let \( W, W' \subset \mathcal{O}(X) \) be two irreducible isomorphic \( K \)-submodules such that \( W \neq W' \). Define \( f_1, \ldots, f_N \in W \) as in Lemma 3.1. Fix a \( K \)-equivariant isomorphism \( \phi : W \to W' \) and let \( f_i' = \phi(f_i) \). By Lemma 3.1 the function \( F = \sum f_i f_i' \) is \( K \)-invariant. The same proof shows that the function \( G = \sum f_i f_i' \) is also \( K \)-invariant. By (FT) we have \( \mu F = F \) and \( \mu G = G \). Since the multiplication map \( \mathcal{O}(X) \otimes \overline{\mathcal{O}(X)} \to \mathcal{O}(X) \cdot \overline{\mathcal{O}(X)} \) is an isomorphism of vector spaces, it follows that

\[
\sum_i f_i \otimes f_i = \sum_i f_i \otimes f_i',
\]

and

\[
\sum_i f_i' \otimes f_i = \sum_i f_i \otimes f_i'.
\]

Therefore the linear span of \( \mu f_1, \ldots, \mu f_N \) coincides with the linear span of \( f_1, \ldots, f_N \) and with the linear span of \( \mu f_1', \ldots, \mu f_N' \). Thus \( \overline{\mu W} = \overline{\mu W'} = \overline{W} \), and \( W = W' \), contrary to our assumption.

\[\square\]
**Lemma 3.4.** Let $\theta$ be a Weyl involution of $K$ and let $\mu : X \to X$ be a $\theta$-equivariant antiholomorphic involution of $X$. If $W \subset \mathcal{O}(X)$ is a finite-dimensional $K$-submodule, then $\mu W$ is also a $K$-submodule. Furthermore, $\overline{W}$ and $\mu W$ are isomorphic $K$-modules.

**Proof.** Introduce a $K$-invariant Hermitian inner product and choose a unitary basis $\{f_1, \ldots, f_N\}$ in $W$. Denote the representation in $W$ by $\rho$. The condition $\mu(kx) = \theta(k)\mu(x)$ implies that
\[ k : \mu f(x) = \mu f(k^{-1}x) = f(\mu(k^{-1}x)) = f(\theta(k)^{-1}x) = \theta(k)f(\mu x) = \mu \theta(k)f(x). \]
Hence $\mu W$ is indeed a $K$-submodule with the representation $\rho \circ \theta$. Since $\theta$ is a Weyl involution, this representation is dual to $\rho$. But the representation in $\overline{W}$ is also dual to $\rho$ by Lemma 3.2, and our assertion follows. \qed

**Lemma 3.5.** Keep the notation of Lemma 3.4 and assume in addition that $W$ is irreducible and $\mu W = \overline{W}$. Then for a $K$-invariant Hermitian inner product on $W$ one has
\[ \langle \mu f_1, \mu f_2 \rangle = \langle f_1, f_2 \rangle, \]
where $f_1, f_2 \in W$.

**Proof.** The new Hermitian inner product $\{f_1, f_2\} := \langle \mu f_1, \mu f_2 \rangle$ on $W$ is also $K$-invariant. Since $W$ is an irreducible $K$-module, it follows that $\{f_1, f_2\} = c\langle f_1, f_2 \rangle$, where $c > 0$. But then
\[ \{\mu f_1, \mu f_2\} = \langle \mu f_1, \mu f_2 \rangle \]
and, on the other hand,
\[ \{\mu f_1, \mu f_2\} = \langle f_1, f_2 \rangle \]
because $\mu$ is an involution. Thus
\[ c\langle \mu f_1, \mu f_2 \rangle = \langle f_1, f_2 \rangle = c^{-1}\langle \mu f_1, \mu f_2 \rangle; \]
hence $c^2 = 1$ and $c = 1$. \qed

### 4. Holomorphically separable spaces

Since we assume that $K$ is connected, the irreducible representations of $K$ are determined by their highest weights. We denote by $W_\lambda$ an irreducible $K$-module with highest weight $\lambda$ and we write $\mathcal{O}_\lambda(X)$ instead of $\mathcal{O}_\delta(X)$, where $\delta \in K$ and $\lambda = \lambda(\delta)$ is the highest weight of $\delta$. Those highest weights $\lambda$, for which $W_\lambda$ occurs in our $K$-module $\mathcal{O}(X)$, form an additive semigroup, to be denoted by $\Lambda(X)$. In other words, $\Lambda(X)$ is the set of highest weights such that $\mathcal{O}_\lambda(X) \neq \{0\}$. The subspace of fixed vectors of a $K$-module $W$ is denoted by $W^K$. We remark that if $A$ is an algebra on which $K$ acts as a group of automorphisms, then $A^K$ is a subalgebra of $A$.

**Theorem 4.1.** Let $X$ be a holomorphically separable irreducible reduced complex space, $K$ a connected compact Lie group acting on $X$ by holomorphic transformations, $\theta : K \to K$ a Weyl involution, and $\mu : X \to X$ a $\theta$-equivariant antiholomorphic involution of $X$. Then $\mathcal{O}(X)$ is multiplicity free if and only if $\mu(x) \in Kx$ for all $x \in X$. 


Proof. If $\mu(x) \in Kx$ for all $x \in X$, then (FT) guarantees that the $K$-action on $O(X)$ is multiplicity free; see the Introduction and Proposition 3.3.

We now prove the converse. Let $A(X) = O(X) \cdot \overline{O(X)}$. Since $X$ is holomorphically separable, the algebra $A(X)$ separates points of $X$. By Stone-Weierstrass theorem $A(X)$ is dense in the algebra $C(X)$ of continuous functions on $X$. The standard averaging argument shows that $A(X)^K$ is dense in $C(X)^K$. Now, if $Kx$ and $Ky$ are two different $K$-orbits in $X$, then there is a $K$-invariant continuous function $f \in C(X)$ which separates these orbits. Since this function can be approximated by $K$-invariant functions from $A(X)$, it follows that $A(K)^K$ separates $K$-orbits. Let $\lambda \in \Lambda(X)$ be a highest weight which occurs in the decomposition of the $K$-algebra $O(X)$. Since $O(X)$ is multiplicity free, the isotypic component $O_\lambda(X)$ is irreducible. We can identify this isotypic component with $W_\lambda$, and so we write $W_\lambda = O_\lambda(X)$. Now apply Lemma 3.1 to construct a $K$-invariant function in $W_\lambda \cdot \overline{W_\lambda}$. Call this function $F_\lambda$. We claim that the family $\{F_\lambda\}_{\lambda \in \Lambda(X)}$ also separates $K$-orbits in $X$.

To prove the claim, it is enough to present each $F \in A(X)^K$ as the sum of a series

$$F = \sum_{\lambda \in \Lambda(X)} c_\lambda F_\lambda,$$

where the convergence is absolute and uniform on compact subsets in $X$. In order to prove this decomposition, consider the complex space $\overline{X}$ with the conjugate complex structure. There is a natural $K$-action on $\overline{X}$, and so we obtain an action of $K \times K$ on $X \times \overline{X}$. Since $O(X) = \overline{O(X)}$, the isotypic components of the $K$-module $O(X)$ are just the submodules $\overline{W}_\lambda$. By Lemma 2.1 the isotypic components of the $(K \times K)$-module $O(X \times \overline{X})$ are the tensor products $W_\lambda \otimes \overline{W}_{\lambda'}$.

For any $F \in O(X \times \overline{X})$ the theorem of Harish-Chandra yields the decomposition

$$F = \sum F_{\lambda \lambda'} \text{ with } F_{\lambda \lambda'} \in W_\lambda \otimes \overline{W}_{\lambda'},$$

where the convergence is absolute and uniform on compact subsets in $X \times \overline{X}$. In particular, if $F \in (O(X) \otimes O(X))^K$, then all summands are $K$-invariant. But $\overline{W}_\lambda$ is dual to $W_\lambda$ by Lemma 3.2; hence $F_{\lambda \lambda'} = 0$ for $\lambda' \neq \lambda$ by Schur’s lemma. The remaining summands $F_{\lambda \lambda}$ are $K$-invariant elements in $W_\lambda \otimes \overline{W}_\lambda$. But the space $(W_\lambda \otimes \overline{W}_\lambda)^K$ is one-dimensional, again by Schur’s lemma. Therefore, restricting $F_{\lambda \lambda}$ to the diagonal in $X \times \overline{X}$, we get the functions proportional to the $F_{\lambda \lambda}$’s defined above.

Now, because $O(X)$ is multiplicity free, it follows from Lemma 3.4 that $\overline{W}_\lambda = \mu W_\lambda$. Furthermore, Lemma 3.5 shows that the composition of $\mu$ with complex conjugation preserves a $K$-invariant Hermitian product on $W_\lambda$. Therefore $\mu F_\lambda = F_\lambda$ by Lemma 3.1. Since the family of functions $F_\lambda$ separates $K$-orbits, $\mu$ must preserve each of them or, equivalently, $\mu x \in Kx$ for all $x \in X$. □

Remark. For the torus $T = (S^1)^m$ the Weyl involution is given by $\theta(t) = t^{-1}$. Suppose that $T$ acts on $\mathbb{P}_n$ by $t \cdot (z_0 : \ldots : z_n) = (\chi_0(t)z_0 : \ldots : \chi_0(t)z_n)$ with some characters $\chi_i : T \rightarrow S^1$, $i = 0, \ldots, n$, and $\mu : \mathbb{P}_n \rightarrow \mathbb{P}_n$ is given by $\mu(z_0 : \ldots : z_n) = (\overline{z_0} : \ldots : \overline{z_n})$. Then $\mu$ is obviously $\theta$-equivariant. However, if $m < n$, then $\mu$ cannot map each $T$-orbit onto itself. This shows that holomorphic separability of $X$ in Theorem 4.1 is essential.
Let $\mathfrak{k}$ be the Lie algebra of $K$ and let $\mathfrak{g} = \mathfrak{k}^\mathbb{C}$ be its complexification. An irreducible reduced complex space $X$ is called spherical under the action of a compact connected Lie group $K$ if $X$ is normal and there exists a point $x \in X$ such that the tangent space $T_x X$ is generated by the elements of a Borel subalgebra of $\mathfrak{g}$; see [AH].

**Theorem 4.2.** Let $X$ be a normal Stein space, $K$ a connected compact Lie group acting on $X$ by holomorphic transformations, $\theta : K \to K$ a Weyl involution, and $\mu : X \to X$ a $\theta$-equivariant antiholomorphic involution of $X$. Then $X$ is spherical if and only if $\mu(x) \in Kx$ for all $x \in X$.

**Proof.** It is known that a normal Stein space $X$ is spherical if and only if $\mathcal{O}(X)$ is a multiplicity free $K$-module [AH]. The result follows from Theorem 4.1. □

5. **Weyl involution and Cartan involution**

Throughout this section, except in Theorem 5.5, the word involution means an involutive automorphism of a group. This notion will be used for complex algebraic groups and for Lie groups. Let $G$ be a connected reductive algebraic group over $\mathbb{C}$ and let $K$ be a connected compact Lie group. So far we considered Weyl involutions of $K$, but they can also be defined for $G$. Namely, an involution $\theta : G \to G$ is called a Weyl involution if there exists a maximal algebraic torus $T \subset G$ such that $\theta(t) = t^{-1}$ for all $t \in T$.

**Lemma 5.1.** Let $G$ be a connected reductive complex algebraic group and let $K \subset G$ be a maximal compact subgroup. Any Weyl involution $\theta$ of $K$ extends uniquely to a Weyl involution of $G$.

**Proof.** By Theorem 5.2.11 in [OV] any differentiable automorphism $K \to K$ extends uniquely to a polynomial automorphism $G \to G$. Let $\theta : K \to K$ be a Weyl involution and let $T \subset K$ be a maximal torus of $K$ on which $\theta(t) = t^{-1}$. Now, the complexification $T^\mathbb{C}$ of $T$ is a maximal torus of $G$. The extension of $\theta$ to $G$, which we again denote by $\theta$, is a Weyl involution of $G$ because $\theta(t) = t^{-1}$ for all $t \in T$. □

An algebraic subgroup $H \subset G$ is called spherical if $G/H$ is a spherical variety, i.e., if a Borel subgroup of $G$ acts on $G/H$ with an open orbit. A reductive algebraic subgroup $H \subset G$ is called adapted if there exists a Weyl involution $\theta : G \to G$ such that $\theta(H) = H$ and $\theta|_{H^0}$ is a Weyl involution of the connected component $H^0$. A similar definition is used for compact subgroups of connected compact Lie groups.

**Proposition 5.2.** Any spherical reductive subgroup $H \subset G$ is adapted.

**Proof.** See [AV], Proposition 5.10. □

**Proposition 5.3.** Let $H \subset G$ be an adapted algebraic subgroup, $K \subset G$ and $L \subset H$ maximal compact subgroups, and $L \subset K$. Then $L$ is adapted in $K$.

**Proof.** See [AV], Proposition 5.14. □

**Theorem 5.4.** Let $G$ be a connected reductive group and let $H \subset G$ be a reductive spherical subgroup. Then there exist a Weyl involution $\theta : G \to G$ and a Cartan involution $\tau : G \to G$ such that $\theta \tau = \tau \theta$, $\theta(H) = H$, and $\tau(H) = H$. 
Proof. Let $L$ be a maximal compact subgroup of $H$ and let $K$ be a maximal compact subgroup of $G$ that contains $L$. Then $K$ is the fixed point subgroup $G^\tau$ of some Cartan involution $\tau$. Since $H$ is adapted in $G$, there is a Weyl involution $\theta : K \to K$ such that $\theta(L) = L$. For any $k \in K$, we have $\theta \tau(k) = \theta(k) = \tau \theta(k)$ by the definition of $\tau$. Denote again by $\theta : G \to G$ the unique extension to $G$ of the given Weyl involution of $K$. Since $G$ is connected and the relation $\theta \tau(g) = \tau \theta(g)$ holds on $K$, it also holds on $G$.

**Theorem 5.5.** Let $X = G/H$ be an affine homogeneous space of a connected reductive algebraic group $G$. Let $K$ be a maximal compact subgroup of $G$. Then $X$ is spherical if and only if $\theta$ is satisfied for the action of $K_\tau$ on $X$. Moreover, if $X$ is spherical, one can choose $\mu$ in (FT) to be $\theta$-equivariant, where $\theta$ is a Weyl involution of $K$.

Proof. (FT) implies that $\mathcal{O}(X)$ is multiplicity free or, equivalently, that $X$ is spherical. Conversely, assume that $X = G/H$ is a spherical variety. Since $X$ is affine, $H$ is a reductive subgroup by the Matsushima-Onishchik theorem. Define $\theta$ and $\tau$ as in Theorem 5.4 and put $\mu(g \cdot H) = \theta \tau(g) \cdot H$. The map $\mu : X \to X$ is well defined because $\theta \tau(H) = H$. The lift of $\mu$ to $G$ is an antiholomorphic involutive automorphism, so it is obvious that $\mu$ is an antiholomorphic involution of $X$. Since $\theta \tau = \tau \theta$, it follows that $\theta(K) = K$. Therefore, for any $x = gH \in X$ one has

$$
\mu(kx) = \theta \tau(kg) \cdot H = \theta(k) \theta \tau(g) \cdot H = \theta(k) \mu(x)
$$

for all $k \in K$. From Theorem 4.2 it follows that $\mu(x) \in Kx$ for all $x \in X$. □

6. Non-spherical spaces: an example

We keep the notation of the previous section. For a spherical affine homogeneous space $X = G/H$, we constructed a $\theta$-equivariant antiholomorphic involution $\mu$. In this section we want to show that the sphericity assumption is essential.

**Lemma 6.1.** Let $X$ be an irreducible reduced complex space with a holomorphic action of $G$. Let $\theta$ be any algebraic automorphism of $G$ preserving a maximal compact subgroup $K \subset G$. Denote by $\tau$ the Cartan involution with fixed point subgroup $K$. If $\mu$ is an antiholomorphic involution of $X$ satisfying $\mu(kx) = \theta(k) \mu(x)$ for all $x \in X$, $k \in K$, then one has $\mu(gx) = \theta \tau(g) \mu(x)$ for all $x \in X$, $g \in G$.

Proof. For every fixed $x \in X$ consider two antiholomorphic maps $\varphi_x : G \to X$ and $\psi_x : G \to X$, defined by $\varphi_x(g) = \mu(gx)$ and $\psi_x(g) = \theta \tau(g) \mu(x)$. Since the required identity holds for $g \in K$, the maps $\varphi_x$ and $\psi_x$ coincide on $K$. But $K$ is a maximal totally real submanifold in $G$, so $\varphi_x$ and $\psi_x$ must coincide on $G$. □

**Lemma 6.2.** Let $X = G/H$, where $H \subset G$ is an algebraic reductive subgroup, $\theta$ a Weyl involution of $G$ preserving $K$, and $\mu : X \to X$ an antiholomorphic involution of $X$ satisfying $\mu(kx) = \theta(k) \mu(x)$. Then $H$ and $\theta(H)$ are conjugate by an inner automorphism of $G$.

Proof. Assume first that $\tau(H) = H$. Let $x_0 = e \cdot H$ and $h \in H$. Then $\theta(h) \mu(x_0) = \theta(\tau(h)) \mu(x_0) = \mu(\tau(h)x_0) = \mu(x_0)$ by Lemma 6.1. It follows that $\theta(H)$ is the stabilizer of $\mu(x_0)$, so $H$ and $\theta(H)$ are conjugate.

To remove the above assumption, take a maximal compact subgroup $L \subset H$ and a maximal compact subgroup $K_1 \subset G$, such that $L \subset K_1$. Then $K_1 = gKg^{-1}$ for some $g \in G$. The fixed point subgroup of the Cartan involution
\[ \tau_1 := \text{Ad}(g)\tau\text{Ad}(g)^{-1} \] is exactly \( K_1 \), so \( \tau_1 \) is the identity on \( L \) and, consequently, \( \tau_1(H) = H \). Let \( H_1 := g^{-1}Hg \), then \( \tau(H_1) = (\text{Ad}g)^{-1}\tau_1(\text{Ad}g)H_1 = (\text{Ad}g)^{-1}\tau_1(H) = (\text{Ad}g)^{-1}(H) = g^{-1}Hg = H_1 \). Replacing \( H \) by \( H_1 \), we can apply the above argument.

\[ \square \]

**Proposition 6.3.** Let \( G = \text{SO}_{10}(\mathbb{C}) \) and let \( H \subset G \) be the adjoint group of \( \text{SO}_5(\mathbb{C}) \). Let \( \theta \) be a Weyl involution of the maximal compact subgroup \( K = \text{SO}_{10}(\mathbb{R}) \). Then an antiholomorphic involution of \( X = G/H \) cannot be \( \theta \)-equivariant.

**Proof.** Extend \( \theta \) holomorphically to \( G \). In view of Lemma 6.2 it suffices to show that \( \theta(H) \) and \( H \) are not conjugate by an inner automorphism of \( G \). Assume that \( \theta(H) = g_0Hg_0^{-1} \). Then there is an automorphism \( \phi : H \to H \), such that \( \theta(h) = g_0\phi(h)g_0^{-1} \) for \( h \in H \). All automorphisms of \( H \) are inner, so \( \phi(h) = h_0h_0^{-1} \) for some \( h_0 \in H \). Therefore \( \theta(h) = g_1h_0^{-1} \) for all \( h \in H \), where \( g_1 = g_0h_0 \). Define an automorphism of \( \alpha : G \to G \) by \( \alpha := (\text{Ad}(g_1))^{-1} \cdot \theta \) and note that \( H \subset G^\alpha \), where \( G^\alpha \) is the fixed point subgroup of \( \alpha \).

Recall that E. B. Dynkin classified maximal subgroups of classical groups in [D]. Since \( B_2 \) does not occur in his Table 1, every irreducible representation of \( B_2 \) defines a maximal subgroup by Theorem 1.5 in [D]. In particular, \( H \) is a maximal connected subgroup in \( G \). Therefore, either (i) \( H \) is the connected component of \( G^\alpha \) or (ii) \( \alpha = \text{Id} \). Now, (ii) implies that \( \text{Ad}(g_1) = \theta \), thus \( \theta \) is an inner automorphism of \( G \), which is not the case. So we are left with (i). Applying the same argument to \( \beta = \alpha^2 \), we see that either (1) \( H \) is the connected component of \( G^\beta \) or (2) \( \beta = \text{Id} \). Since \( \beta \) is certainly an inner automorphism, (1) would imply that \( H \) is the centralizer of an element of \( G \). However, all centralizers have even codimension in \( G \) and \( \text{codim}(H) = 35 \). On the other hand, if (2) were true, then \( H \) would be a symmetric subgroup in \( G \). The list of symmetric spaces shows that this is not the case. The contradiction just obtained completes the proof. \[ \square \]

**References**


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