THE DEPTH OF AN IDEAL
WITH A GIVEN HILBERT FUNCTION

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Abstract. Let \( A = K[x_1, \ldots, x_n] \) denote the polynomial ring in \( n \) variables over a field \( K \) with each \( \deg x_i = 1 \). Let \( I \) be a homogeneous ideal of \( A \) with \( I \neq A \) and \( H_{A/I} \) the Hilbert function of the quotient algebra \( A/I \). Given a numerical function \( H : \mathbb{N} \to \mathbb{N} \) satisfying \( H = H_{A/I} \) for some homogeneous ideal \( I \) of \( A \), we write \( A_H \) for the set of those integers \( 0 \leq r \leq n \) such that there exists a homogeneous ideal \( I \) of \( A \) with \( H_{A/I} = H \) and with depth \( A/I = r \). It will be proved that one has either \( A_H = \{0, 1, \ldots, b\} \) for some \( 0 \leq b \leq n \) or \( |A_H| = 1 \).

Introduction

Let \( A = K[x_1, \ldots, x_n] \) denote the polynomial ring in \( n \) variables over a field \( K \) with each \( \deg x_i = 1 \). Let \( I \) be a homogeneous ideal of \( A \) with \( I \neq A \) and \( H_R \) the Hilbert function of the quotient algebra \( R = A/I \). Thus \( H_R(q), q = 0, 1, 2, \ldots, \) is the dimension of the subspace of \( R \) spanned over \( K \) by the homogeneous elements of \( R \) of degree \( q \). A classical result [3, Theorem 4.2.10] due to Macaulay guarantees that, given a numerical function \( H : \mathbb{N} \to \mathbb{N} \), where \( \mathbb{N} \) is the set of nonnegative integers, there exists a homogeneous ideal \( I \) of \( A \) with \( I \neq A \) such that \( H \) is the Hilbert function of the quotient algebra \( R = A/I \) if and only if \( H(0) = 1 \), \( H(1) \leq n \) and \( H(q+1) \leq H(q)^{(q)} \) for \( q = 1, 2, \ldots \), where \( H(q)^{(q)} \) is defined in [3, p. 161].

Given a numerical function \( H : \mathbb{N} \to \mathbb{N} \) satisfying \( H(0) = 1 \), \( H(1) \leq n \) and \( H(q+1) \leq H(q)^{(q)} \) for \( q = 1, 2, \ldots \), we write \( A_H \) for the set of those integers \( 0 \leq r \leq n \) such that there exists a homogeneous ideal \( I \) of \( A \) with \( H_{A/I} = H \) and with depth \( A/I = r \). We will show that, given a numerical function \( H : \mathbb{N} \to \mathbb{N} \) satisfying \( H(0) = 1 \), \( H(1) \leq n \) and \( H(q+1) \leq H(q)^{(q)} \) for \( q = 1, 2, \ldots \), one has (i) \( A_H = \{n - \delta\} \) if \( H \) is of the form (1) of Proposition 1.5 and (ii) \( A_H = \{0, 1, \ldots, b\} \) for some \( b \geq 0 \) if \( H \) cannot be of the form (1). The statement (i) will be proved in Theorem 1.6, and the statement (ii) will be proved in Theorem 2.1. Also, we will introduce a way to determine the integer \( b = \max A_H \) from \( H \) in Theorem 2.2.
1. Universal lexsegment ideals

Let $A = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n$ variables over a field $K$ with each $\deg x_i = 1$ and $A_{[m]} = K[x_1, \ldots, x_{n+m}]$, where $m$ is a positive integer. Work with the lexicographic order $<_{\text{lex}}$ on $A$ induced by the ordering $x_1 > x_2 > \cdots > x_n$ of the variables. Write, as usual, $G(I)$ for the (unique) minimal system of monomial generators of a monomial ideal $I$ of $A$. Recall that a monomial ideal $I$ of $A$ is a lexsegment ideal if, for a monomial $u$ of $A$ belonging to $I$ and for a monomial $v$ of $A$ with $\deg u = \deg v$ and with $v >_{\text{lex}} u$, one has $v \in I$. A lexsegment ideal $I$ of $A$ is called universal lexsegment ([1]) if, for any integer $m \geq 1$, the monomial ideal $I A_{[m]}$ of the polynomial ring $A_{[m]}$ is lexsegment. In other words, a universal lexsegment ideal of $A$ is a lexsegment ideal $I = (u_1, \ldots, u_t)$ of $A$ which remains being lexsegment if we regard $I = (u_1, \ldots, u_t)$ as an ideal of the polynomial ring $A_{[m]}$ for all $m \geq 1$.

Example 1.1. (a) The lexsegment ideal $(x_1^2, x_1x_2^2)$ of $K[x_1, x_2]$ is universal lexsegment. In fact, the ideal $(x_1^2, x_1x_2^2)$ of $K[x_1, \ldots, x_m]$ is lexsegment for all $m \geq 2$.

(b) The lexsegment ideal $(x_1^3, x_1^2x_2, x_1x_2^2)$ of $K[x_1, x_2]$ cannot be universal lexsegment. Indeed, since $x_1x_2^3 <_{\text{lex}} x_1^2x_3$ in $K[x_1, x_2, x_3]$, the ideal $(x_1^3, x_1^2x_2, x_1x_2^2)$ of $K[x_1, x_2, x_3]$ is not lexsegment.

Proposition 1.2.

(a) Let $I$ be a lexsegment ideal of $A$ with $G(I) = \{u_1, \ldots, u_\delta\}$ where $\deg u_1 \leq \cdots \leq \deg u_\delta$ and where $u_{i+1} <_{\text{lex}} u_i$ if $\deg u_i == \deg u_{i+1}$. Let $s_i = \deg u_i - 1$ and $s_i = \deg u_i - \deg u_{i-1}$ for $i = 2, 3, \ldots, \delta$. Then, for $k \leq n$, one has

$$u_k = x_1^{s_1}x_2^{s_2}\cdots x_k^{s_k}+1.$$

(b) Given an integer $1 \leq \delta \leq n$ together with a sequence of integers $1 \leq e_1 \leq \cdots \leq e_\delta$, there is a lexsegment ideal $I$ of $A$ with $G(I) = \{u_1, \ldots, u_\delta\}$ such that $\deg u_i = e_i$ for $i = 1, \ldots, \delta$.

Proof. (a) Since $u_1 = x_1^{\deg u_1}$, one has $u_1 = x_1^{s_1}+1$. Let $1 \leq k \leq n, \delta$ and suppose that $u_{k-1} = x_1^{s_1}x_2^{s_2}\cdots x_{k-1}^{s_{k-1}}+1$. Since the ordering of $u_1, u_2, \ldots, u_\delta$ implies that the monomial ideal $(u_1, \ldots, u_{k-1})$ is lexsegment, the smallest monomial with respect to $<_{\text{lex}}$ of degree $\deg u_k$ belonging to $(u_1, \ldots, u_{k-1})$ is $x_{k-1}^{s_{k-1}}$. Since $u_k$ is the largest monomial with respect to $<_{\text{lex}}$ which satisfies $\deg u_k = (u_{k-1}x_{k}^{s_k})$ and $u_k <_{\text{lex}} u_{k-1}x_{n}^{s_k}$, we have $u_k = (u_{k-1}/x_{k-1})x_{k}^{s_k}+1$. Thus $u_k = x_1^{s_1}x_2^{s_2}\cdots x_{k-1}^{s_{k-1}}x_k^{s_k}+1$, as desired.

(b) This can be easily done by induction on $\delta$. Let $\delta \leq n$ and suppose that $J$ is a lexsegment ideal of $A$ with $G(J) = \{u_1, \ldots, u_{\delta-1}\}$ such that $\deg u_i = e_i$ for $i = 1, 2, \ldots, \delta-1$. Then by (a) we have $G(J) \subset K[x_1, \ldots, x_{\delta-1}]$. Hence $x_{\delta}^{e_{\delta}} \notin J$. Thus there exists a monomial of degree $e_{\delta}$ which does not belong to $J$. Let $u_\delta$ be the largest monomial of degree $e_{\delta}$ with respect to $<_{\text{lex}}$ which does not belong to $J$. Then $(u_1, \ldots, u_{\delta-1}, u_\delta)$ is a lexsegment ideal of $A$.

Corollary 1.3. A lexsegment ideal $I$ of $A$ is universal lexsegment if and only if $|G(I)| \leq n$, where $|G(I)|$ is the number of monomials belonging to $G(I)$.

Proof. Let $G(I) = \{u_1, \ldots, u_\delta\}$, where $\deg u_1 \leq \cdots \leq \deg u_\delta$. If $\delta \geq n+1$, then $I A_{[1]}$ is not a lexsegment ideal of $A_{[1]}$ since Proposition 1.2 (a) says that, for any lexsegment ideal $J$ of $A_{[1]}$ with $|G(J)| \geq n+1$, there exists a generator $v \in G(J)$ such that $x_{n+1}$ divides $v$. Thus $I$ is not a universal lexsegment if $\delta \geq n+1$.
Assume that $\delta \leq n$. For any positive integer $m$, Proposition 1.2 (b) says that there exists the lexsegment ideal $J$ of $A_{[m]}$ such that $G(J) = \{v_1, \ldots, v_\delta\}$ satisfies $\deg v_i = \deg u_i$ for $i = 1, 2, \ldots, \delta$. Then Proposition 1.2 (a) says that $G(I) = G(J)$. Thus $IA_{[m]}$ is a lexsegment ideal of $A_{[m]}$ for all $m \geq 1$ if $\delta \leq n$. □

For any monomial $u$ of $A$, let $m(u)$ be the biggest integer $1 \leq i \leq n$ for which $x_i$ divides $u$. A monomial ideal $I$ of $A$ is said to be stable if $u \in I$ implies $(x_q/x_{m(u)})u \in I$ for any $1 \leq q < m(u)$. Eliahou–Kervaire [5] says that, for a stable ideal $I$ of $A$, the projective dimension $\text{proj dim} A/I$ of the quotient algebra $A/I$ coincides with $\max\{m(u) : u \in G(I)\}$. Since a lexsegment ideal is stable, it follows from Proposition 1.2 (a) together with the Auslander–Buchsbaum formula [3, Theorem 1.3.3] that

**Corollary 1.4.** Let $I$ be a lexsegment ideal of $A$ and depth $A/I$ the depth of the quotient algebra $A/I$ of $A$. Then depth $A/I = \max\{n - |G(I)|, 0\}$.

It is known that, given a numerical function $H : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{(q)}$ for $q = 1, 2, \ldots$, there exists a unique lexsegment ideal $I$ of $A$ with $H_{A/I} = H$. We say that a numerical function $H : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{(q)}$ for $q = 1, 2, \ldots$ is critical if the lexsegment ideal $I$ of $A$ with $H_{A/I} = H$ is universal lexsegment.

**Proposition 1.5.** A numerical function $H : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{(q)}$ for $q = 1, 2, \ldots$ is critical if and only if there is an integer $1 \leq \delta \leq n$ together with a sequence of integers $(e_1, \ldots, e_\delta)$ with $1 \leq e_1 \leq \cdots \leq e_\delta$ such that

$$\label{equation1} H(q) = \binom{n-1+q}{n-1} - \sum_{i=1}^{\delta} \binom{n-i+q-e_i}{n-i}$$

for $q = 0, 1, \ldots$. Moreover, $\delta$ is equal to the number of minimal monomial generators of the universal lexsegment ideal $I$ of $A$ with $H_{A/I} = H$.

**Proof.** First, to prove the “only if” part, let $I$ be a universal lexsegment ideal of $A$ with $G(I) = \{u_1, \ldots, u_\delta\}$, where $\delta \leq n$. Without loss of generality, we can suppose that $\deg u_1 \leq \cdots \leq \deg u_\delta$ and that $u_{i+1} <_{\text{lex}} u_i$ if $\deg u_i = \deg u_{i+1}$. Proposition 1.2 (a) says that, for $1 \leq i < j \leq \delta$, the monomial $x_{i}u_{j}$ is divided by $u_i$ and no monomial belongs to both $u_{i}K[x_1, \ldots, x_n]$ and $u_{j}K[x_1, \ldots, x_n]$. Hence the direct sum decomposition $I = \bigoplus_{i=1}^{\delta} u_{i}K[x_1, \ldots, x_n]$ arises. Let $e_i = \deg u_i$ for $i = 1, 2, \ldots, \delta$. The fact that the number of monomials of degree $q$ belonging to $I$ is $\sum_{i=1}^{\delta} \binom{n-i+q-e_i}{n-i}$ yields the formula (1), as required.

Next we consider the “if” part. Let $H : \mathbb{N} \rightarrow \mathbb{N}$ be a numerical function of the form (1). Since $1 \leq e_1 \leq \cdots \leq e_\delta$ and $\delta \leq n$, Proposition 1.2 (b) and Corollary 1.3 say that there exists a universal lexsegment ideal $I$ with $G(I) = \{u_1, \ldots, u_\delta\}$ such that $\deg(u_i) = e_i$ for all $i$. Then the computation of Hilbert functions in the proof of the “only if” part implies $H_{A/I}(q) = H(q)$ for all $q \in \mathbb{N}$. □

A critical ideal of $A$ is a homogeneous ideal $I$ of $A$ with $I \neq A$ such that the Hilbert function $H_R$ of the quotient algebra $R = A/I$ is critical. In other words, a critical ideal of $A$ is a homogeneous ideal $I$ of $A$ such that the lexsegment ideal $I^\text{lex}$ is universal lexsegment, where $I^\text{lex}$ is the unique lexsegment ideal of $A$ such that $A/I$ and $A/I^\text{lex}$ have the same Hilbert function. Somewhat surprisingly,
Theorem 1.6. Suppose that a homogeneous ideal \( I \) of \( A \) is critical. Then
\[
\text{depth} A/I = \text{depth} A/I^{\text{lex}}.
\]

Proof. Let \( \beta_{ij} \) (resp. \( \beta'_{ij} \)) denote the graded Betti numbers of \( I \) (resp. \( I^{\text{lex}} \)). Let \( G(I^{\text{lex}}) = \{u_1, \ldots, u_\delta\} \) with \( \delta \leq n \), where deg \( u_1 \leq \cdots \leq \text{deg} u_\delta \) and where \( u_{i+1} <_{\text{lex}} u_i \) if \( \text{deg} u_i = \text{deg} u_{i+1} \). Let \( e_i = \text{deg} u_i \) for \( i = 1, \ldots, \delta \). Eliahou–Kervaire [5] together with Proposition 1.2 (a) guarantees that \( \beta_{i,\delta-1+e_\delta} = 0 \) unless \( i = \delta - 1 \) and \( \beta'_{\delta-1,\delta-1+e_\delta} = 1 \). Since \( A/I \) and \( A/I^{\text{lex}} \) have the same Hilbert function, it follows from [3, Lemma 4.1.13] that
\[
\sum_{i \geq 0} (-1)^i \beta_{i,\delta-1+e_\delta} = \sum_{i \geq 0} (-1)^i \beta'_{i,\delta-1+e_\delta}.
\]
Since \( \beta_{ij} \leq \beta'_{ij} \) for all \( i \) and \( j \) ([2], [7] and [8]), it follows that \( \beta_{\delta-1,\delta-1+e_\delta} = 1 \).

Thus in particular proj dim \( A/I \) \( \geq \delta \). Since proj dim \( A/I^{\text{lex}} = \delta \) and proj dim \( A/I \leq \text{proj dim} A/I^{\text{lex}} \), it follows that proj dim \( A/I = \text{proj dim} A/I^{\text{lex}} = \delta \). Thus we have depth \( A/I = \dim A/I^{\text{lex}} = n - \delta \), as desired.

Moreover, in the case of monomial ideals, the graded Betti numbers of a critical ideal are determined by its Hilbert function.

Corollary 1.7. Suppose that a monomial ideal \( I \) of \( A \) is critical. Then \( I \) and \( I^{\text{lex}} \) have the same graded Betti numbers.

Proof. It follows from Taylor’s resolution of monomial ideals (see [5, p. 18]) that
\[
\text{proj dim}(A/I) \leq |G(I)|.
\]

On the other hand, Corollary 1.4 and Theorem 1.6 say that
\[
\text{proj dim}(A/I) = \text{proj dim}(A/I^{\text{lex}}) = |G(I^{\text{lex}})|.
\]

Since the number of elements in \( G(I^{\text{lex}}) \) is always larger than that of \( G(I) \), we have \( |G(I)| = |G(I^{\text{lex}})| \). This means \( \sum_{j \geq 0} \beta_{0j}(I) = \sum_{j \geq 0} \beta_{0j}(I^{\text{lex}}) \). Then it follows from [4, Theorem 1.3] that \( \beta_{ij}(I) = \beta_{ij}(I^{\text{lex}}) \) for all \( i \) and \( j \).

We are not sure that Corollary 1.7 holds for an arbitrary critical ideal.

Example 1.8. Let \( I \) be the monomial ideal \( (x_1 x_4, x_3 x_4) \) of \( K[x_1, x_2, x_3, x_4] \). Since \( I^{\text{lex}} = (x_1^2, x_1 x_2) \) is universal lexsegment, it follows that depth \( A/I = 2 \).

2. Depth and Hilbert Functions

Let, as before, \( A = K[x_1, \ldots, x_n] \) denote the polynomial ring in \( n \) variables over a field \( K \) with each \( \text{deg} x_i = 1 \). Given a numerical function \( H : \mathbb{N} \to \mathbb{N} \) satisfying \( H(0) = 1, H(1) \leq n \) and \( H(q+1) \leq H(q)^{(q)} \) for \( q = 1, 2, \ldots \), we write \( \mathcal{A}_H \) for the set of those integers \( 0 \leq r \leq n \) such that there exists a homogeneous ideal \( I \) of \( A \) with \( H_{A/I} = H \) and with depth \( A/I = r \). It follows from Corollary 1.4 together with Theorem 1.6 that if \( H \) is critical, that is, \( H \) is of the form (1), then \( \mathcal{A}_H = \{n - \delta\} \).

Theorem 2.1. Suppose that a numerical function \( H : \mathbb{N} \to \mathbb{N} \) satisfying \( H(0) = 1, H(1) \leq n \) and \( H(q+1) \leq H(q)^{(q)} \) for \( q = 1, 2, \ldots \) is noncritical. Then \( \mathcal{A}_H = \{0, 1, 2, \ldots, b\} \), where \( b \) is the largest integer for which \( b \in \mathcal{A}_H \).
Proof. We may assume that $K$ is infinite. Let $I$ be a homogeneous ideal of $A$ with $H_{A/I} = H$ and with depth $A/I = b$. Let $0 \leq r \leq b$. Since $K$ is infinite and since depth $A/I = b$, there exists a regular sequence $(\theta_1, \ldots, \theta_r)$ of $A/I$ with each deg $\theta_i = 1$. It then follows that there exists a homogeneous ideal $J$ of $B = K[x_1, \ldots, x_{n-r}]$ such that the ideal $JA$ of $A$ satisfies $H_{A/(JA)} = H$.

We now claim that the lexsegment ideal $J^{\text{lex}} \subset B$ of $J$ cannot be universal lexsegment. In fact, if $J^{\text{lex}}$ is universal lexsegment, then $J^{\text{lex}}$ remains being lexsegment in the polynomial ring $K[x_1, \ldots, x_m]$ for each $m \geq n - r$. In particular the ideal $J^{\text{lex}}A$ of $A$ is universal lexsegment. Since $H_{A/(JA)} = H_{A/(J^{\text{lex}}A)} = H$, the numerical function $H$ is critical, a contradiction.

Since the lexsegment ideal $J^{\text{lex}}$ of $J$ cannot be universal lexsegment, it follows from Corollaries 1.3 and 1.4 that depth $B/J^{\text{lex}} = 0$. Thus depth $A/(J^{\text{lex}}A) = r$. Hence $r \in A_H$, as desired. □

One may ask a way to compute the integer $b = \max A_H$ from $H$. This integer $b$ can be determined as follows: Let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function. The differential $\Delta^1(H)$ of $H$ is the numerical function defined by $\Delta^1(H)(0) = 1$ and $\Delta^1(H)(q) = H(q) - H(q - 1)$ for $q \geq 1$. We define the $p$-th differential $\Delta^p(H) = \Delta^1(\Delta^{p-1}(H))$ inductively.

**Theorem 2.2.** Let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q + 1) \leq H(q)(q)$ for all $q \geq 1$. Then one has

$$\max A_H = \max \{p : \Delta^p(H) \text{ satisfies } \Delta^p(H)(q + 1) \leq \Delta^p(H)(q)(q) \text{ for } q \geq 1\}. $$

**Proof.** If $p$ is an integer which belongs to the right-hand side of (2), then there exists a homogeneous ideal $J$ of $B = K[x_1, \ldots, x_{n-p}]$ such that $H_{B/J} = \Delta^p(H)$. Recall that if $M$ is a graded $R$-module and $\vartheta_1, \ldots, \vartheta_r$ with each deg $\vartheta_i = 1$ is a regular sequence of $M$, then $H_{M/(\vartheta_1, \ldots, \vartheta_r)M} = \Delta^p(H_M)$. Set $M = A/(JA)$. Then, since $x_n, x_{n-1}, \ldots, x_{n-p+1}$ is a regular sequence of $A/(JA)$ and $M/(x_n, \ldots, x_{n-p+1})M \simeq B/J$, we have $H_{A/(JA)} = H$ and depth$(A/(JA)) \geq p$. This says that the left-hand side of (2) is greater than or equal to the right-hand side of (2).

On the other hand, if there exists a homogeneous ideal $I$ of $A$ such that $H = H_{A/I}$ and depth$(A/I) = p$, then, in the same way as Theorem 2.1, there exists a homogeneous ideal $J$ of $B = K[x_1, \ldots, x_{n-p}]$ such that $H_{A/(JA)} = H$ and $H_{B/J} = \Delta^p(H)$. Thus the left-hand side of (2) is less than or equal to the right-hand side of (2). □

**Example 2.3.** Let $I$ be the monomial ideal $(x_1x_4, x_1x_5, x_2x_5, x_3x_5, x_2x_3x_4)$ of $A = K[x_1, x_2, x_3, x_4, x_5]$. Then

$$J^{\text{lex}} = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5^2, x_2^2, x_2x_3, x_2^2x_4^2, x_2x_3x_4^2, x_2^2x_4x_5, x_2^2x_4^2, x_2x_3^2, x_2x_3x_4^2).$$

Thus depth $A/I^{\text{lex}} = 0$ by Corollary 1.4. Since the Hilbert series $\sum_{q=0}^{\infty} H_{A/I}(q)\lambda^q$ of $A/I$ is $(1 + 2\lambda - \lambda^2 - \lambda^3)/(1 - \lambda)^3$, it follows from [3, Corollary 4.1.10] that the Krull dimension of $A/I$ is 3 and $3 \not\in A_H$. Since depth $A/I = 2$, one has $A_H = \{0, 1, 2\}$.

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