A NEW PROOF OF THE INTEGRABILITY
OF THE SUBDIFFERENTIAL
OF A CONVEX FUNCTION ON A BANACH SPACE

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Abstract. We provide a simple proof of the Moreau-Rockafellar theorem that a proper lower semicontinuous convex function on a Banach space is determined up to a constant by its subdifferential.

1. Introduction

Recall the following famous theorem of Rockafellar; see [8, 9]:

Theorem 1.1 (Rockafellar). Let $X$ be a Banach space. Let $f$ and $g$ be proper lower semicontinuous convex functions from $X$ to $\mathbb{R} \cup \{+\infty\}$. If

\begin{equation}
\partial f \subset \partial g,
\end{equation}

then

\begin{equation}
g = f + \text{const}.
\end{equation}

The reader is referred to Section 3 of Phelps book [7] for a possible context of this theorem and some of its important implications.

The purpose of this note is to present a new and more direct proof of Theorem 1.1. First of all, however, we present a brief historical overview. The integrability of the subdifferential of proper lower semicontinuous convex functions on Hilbert space is stated and proved by Moreau in [5]. His proof uses Moreau-Yosida regularisation and works also in reflexive Banach space as mentioned on p. 87 of [6]. While the first complete proof of Theorem 1.1 – that of Rockafellar in [9] – uses duality arguments, most of the others rely on approximating the directional derivative and subsequent reduction to the trivial one-dimensional case. This second line is followed by the original Rockafellar proof in [8]. Although there are some gaps in the latter proof, the work of Taylor [10] contains both a remedy and a different proof (cf. [2]). The idea of directional derivative approximation/one-dimensional reduction is most clearly outlined in the proof of Thibault [11]. A different proof using the mean-value theorem of Zagrodny is due to Thibault and Zagrodny [12] (see also [13]).
The proof we are about to present follows a different route. It is similar to
the proof of the classical calculus theorem that a monotone function is Riemann
integrable. It uses neither duality nor explicit one-dimensional arguments.

The main step in our proof is to show directly that a proper lower semicontinu-
ous convex function on a Banach space differs by a constant from the Rockafellar
function (see [1]) of its subdifferential. Namely, we prove the following

\textbf{Theorem 1.2} (Rockafellar [8, 9]). Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) be a proper lower semicontinuous convex function on the Banach space \( X \). Let \( x_0 \in \text{dom} \, \partial f \) and \( p_0 \in \partial f(x_0) \) be arbitrary. Then for all \( x \in X \),

\begin{equation}
 f(x) = f(x_0) + R_{\partial f,(x_0,p_0)}(x),
\end{equation}

where

\begin{equation}
 R_{\partial f,(x_0,p_0)}(x) = \sup \left\{ \sum_{i=0}^{n-1} p_i(x_{i+1} - x_i) : x_n = x, \ p_i \in \partial f(x_i), \ n \in \mathbb{N} \right\};
\end{equation}

that is, the supremum is taken over all finite sequences in \( \text{gph} \, \partial f \), starting at \((x_0,p_0)\).

With the aid of the novel technical Lemma 3.3 we derive first the case \( x \in \text{dom} \, \partial f \) of (1.2). The general case is completed by reference to the following well-known result of Brøndsted and Rockafellar [3]:

\textbf{Proposition 1.3.} Let \( f \) be a proper lower semicontinuous convex function from a Banach space \( X \) into \( \mathbb{R} \cup \{+\infty\} \). Then for all \( x \in X \),

\begin{equation}
 f(x) = \sup\{f(\bar{x}) + \bar{p}(x - \bar{x}) : (\bar{x}, \bar{p}) \in \text{gph} \, \partial f\}.
\end{equation}

From Theorem 1.2 it readily follows that (1.1) implies

\[ f(x) - f(x_0) \leq g(x) - g(x_0) \]

for any \( x_0 \in \text{dom} \, \partial f \) and all \( x \in X \). In particular, \( f - g \equiv \text{const} \) on \( \text{dom} \, \partial f \).

The proof of Theorem 1.1 is completed by a reference to lower semicontinuity and graphical density of points of subdifferentiability (see [3] and [2]):

\textbf{Proposition 1.4.} Let \( f \) be a proper lower semicontinuous convex function from a Banach space \( X \) into \( \mathbb{R} \cup \{+\infty\} \). Then for any \( \bar{x} \in \text{dom} \, f \) and any \( \varepsilon > 0 \) there exists \( x \in \text{dom} \, \partial f \) such that \( \|x - \bar{x}\| + |f(x) - f(\bar{x})| < \varepsilon \).

The remaining content is organized as follows. After a short section on notation,
we state and prove Lemma 3.3 in Section 3. In Section 4 the proofs of Theorem 1.2
and Theorem 1.1 are assembled.

\section{2. Notation}

The notation used throughout the paper is fairly standard, so we recall it briefly.

Usually \( X \) denotes a real Banach space, that is, a complete normed space over \( \mathbb{R} \). The norm of \( X \) is denoted by \( \| \cdot \| \). The closed unit ball \( \{x \in X : \|x\| \leq 1\} \) of \( X \) is denoted by \( B_X \). The dual space \( X^* \) of \( X \) is the Banach space of all continuous linear functionals \( p \) from \( X \) to \( \mathbb{R} \). The natural norm of \( X^* \) is again denoted by \( \| \cdot \| \). The value of \( p \in X^* \) at \( x \in X \) is denoted by both \( p(x) \) and \( \langle p, x \rangle \), wherever convenient.
The **effective domain** \( \text{dom} \ f \) of an extended real-valued function \( f : X \to \mathbb{R} \cup \{+\infty\} \) is the set of points \( x \) where \( f(x) \in \mathbb{R} \). The function \( f \) is **proper** if \( \text{dom} \ f \neq \emptyset \). It is **lower semicontinuous** if

\[
f(\bar{x}) \leq \liminf_{x \to \bar{x}} f(x)
\]

for all \( \bar{x} \in X \). The **epigraph** of \( f \) is the set \( \text{epi} \ f \subset X \times \mathbb{R} \) defined by

\[
\text{epi} \ f = \{(x, r) : x \in \text{dom} \ f, \ r \geq f(x)\}.
\]

It is clear that \( \text{epi} \ f \) is nonempty whenever \( f \) is proper, and \( \text{epi} \ f \) is closed iff \( f \) is lower semicontinuous.

The **indicator** function of a set \( C \subset X \) is the function \( I_C : X \to \mathbb{R} \cup \{+\infty\} \) defined by:

\[
I_C(x) = 0, \text{ if } x \in C; \quad I_C(x) = \infty, \text{ otherwise.}
\]

Obviously, \( I_C \) is proper iff \( C \) is nonempty and \( I_C \) is lower semicontinuous iff \( C \) is closed.

If \( f : X \to \mathbb{R} \cup \{+\infty\} \) is a convex function, then \( \partial f(\bar{x}) \) denotes the subdifferential of \( f \) at \( \bar{x} \in \text{dom} \ f \) in the sense of Convex Analysis, that is,

\[
\partial f(\bar{x}) = \{p \in X^* : \langle p, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \ \forall x \in X\},
\]

and \( \partial f = \emptyset \) on \( X \setminus \text{dom} \ f \). It is a well-known and easy consequence of convexity that if \( \langle p, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \) holds on some neighbourhood of \( \bar{x} \), rather than on the whole space, then again \( p \in \partial f(\bar{x}) \).

The **domain** \( \text{dom} \ \partial f \) of the multi-valued map \( \partial f : X \to X^* \) consists of all points \( x \in X \) such that \( \partial f(x) \) is nonempty. The **graph** of \( \partial f \) is

\[
\text{gph} \ \partial f = \{(x, p) : x \in \text{dom} \ \partial f, \ p \in \partial f(x)\}.
\]

3. **Variational arguments**

This section is devoted to Lemma 3.3, which can be viewed as a particular enhancement of Proposition 1.4.

Although the technique of proof is standard, we provide full details for the reader’s convenience. First, however, we recall the famous Ekeland Variational Principle (e.g. [7], p.45):

**Theorem 3.1** (Ekeland Variational Principle). Let \( f \) be a proper lower semicontinuous function from a Banach space \( X \) into \( \mathbb{R} \cup \{+\infty\} \). Let \( f \) be bounded below, \( \varepsilon > 0 \), and \( \bar{y} \in \text{dom} \ f \) be such that \( f(\bar{y}) \leq \inf f + \varepsilon \). Then for each \( \lambda > 0 \) there is \( \bar{x} \in \text{dom} \ f \) such that:

(i) \( \lambda\|\bar{x} - \bar{y}\| \leq f(\bar{y}) - f(\bar{x}) \),

(ii) \( \|\bar{x} - \bar{y}\| \leq \varepsilon / \lambda \), and

(iii) \( \lambda\|x - \bar{x}\| + f(x) > f(\bar{x}) \) for all \( x \neq \bar{x} \).

We also use the following well-known sum theorem (e.g. [7], p.47):

**Theorem 3.2** (Moreau-Rockafellar Formula). Let \( f \) be a proper lower semicontinuous convex function from a Banach space \( X \) into \( \mathbb{R} \cup \{+\infty\} \), and let \( g : X \to \mathbb{R} \) be convex and continuous. Then for any \( x \in X \),

\[
\partial(f + g)(x) = \partial f(x) + \partial g(x).
\]

(The right-hand side is the usual sum of sets.)
Lemma 3.3. Let $f$ be a proper lower semicontinuous convex function from a Banach space $X$ into $\mathbb{R} \cup \{+\infty\}$. Let $(y_i)_{i=1}^k \subset \text{dom} f$. Then for each $\varepsilon > 0$ there are $(x_i, p_i) \in \text{gph} \partial f$ such that

\begin{equation}
\|x_i - y_i\| \leq \varepsilon \quad \text{and} \quad \|p_j\| \|x_i - y_i\| \leq \varepsilon, \quad \forall i, j = 1, \ldots, k.
\end{equation}

**Proof.** Due to the lower semicontinuity of $f$ there is $\delta \in (0, \varepsilon)$ such that for all $i = 1, \ldots, k$,

\[ f(y_i + \delta B_X) \geq f(y_i) - \varepsilon. \]

\[ f_i(x) + \lambda \|x - x_i\| \geq f_i(x_i), \quad \forall x \in X. \]

Since $x_i \in y_i + (\varepsilon/\lambda)B_X$ and $\varepsilon/\lambda = \delta/2 < \varepsilon$, we have that

\begin{equation}
\|x_i - y_i\| \leq \varepsilon/\lambda = \delta/2 < \varepsilon.
\end{equation}

On the other hand, (3.2) means that the minimum of the proper convex and lower semicontinuous function $f_i(\cdot) + \lambda \|\cdot - x_i\|$ is attained at $x_i$. Therefore,

\[ 0 \in \partial (f_i(\cdot) + \lambda \|\cdot - x_i\|) (x_i). \]

Since $\lambda \|\cdot - x_i\|$ is continuous, from Theorem 3.2 it follows that there is $p_i \in \partial f_i(x_i)$, such that $-p_i \in \partial (\lambda \|\cdot - x_i\|) (x_i)$.

By definition $f_i \equiv f$ on $y_i + \delta B_X$, and, due to (3.3), $x_i$ is in the interior of the latter set; hence, $p_i(x - x_i) \leq f(x) - f(x_i)$ for all $x$ in a neighbourhood of $x_i$. That is,

\begin{equation}
p_i \in \partial f(x_i).
\end{equation}

Next, as $-p_i$ is a subdifferential of $\lambda \|\cdot - x_i\|$ at $x_i$, we get

\[ p_i(x_i - x') \leq \lambda \|x' - x_i\|, \quad \forall x' \in X. \]

So, $\|p_i\| \leq \lambda$.

Therefore, for each $i = 1, \ldots, k$ we have that $\|p_i\| \leq \lambda$ and $\|x_i - y_i\| \leq \varepsilon/\lambda$; cf. (3.3). Both of these clearly imply

\[ \|p_j\| \|x_i - y_i\| \leq \lambda \frac{\varepsilon}{\lambda} = \varepsilon, \quad \forall i, j = 1, \ldots, k. \]

This, (3.4) and (3.3) are all we had to prove. \hfill \Box

4. Integrability

We are now ready to present the proof mentioned in the title of this note.

**Proof of Theorem 1.2.** For each finite sequence $(x_i, p_i)_{i=1}^{n-1} \subset \text{gph} \partial f$, and each $x \in X$ we have that

\begin{equation}
f(x) \geq f(x_0) + \sum_{i=0}^{n-1} p_i(x_{i+1} - x_i),
\end{equation}

where $x_n = x$.

Indeed, by the definition of the subdifferential, $f(x_{i+1}) - f(x_i) \geq p_i(x_{i+1} - x_i)$, $i = 0, \ldots, n-1$. Summing these, we get $f(x_n) - f(x_0) \geq \sum_{i=0}^{n-1} p_i(x_{i+1} - x_i)$, which is (4.1). So, the left-hand side of (1.2) is greater than its right-hand side.

Assume now that $x \in \text{dom} \partial f$. Fix $p \in \partial f(x)$ and let $(x_n, p_n) = (x, p)$.
Again using the definition of the subdifferential, \( f(x_i) - f(x_{i+1}) \geq p_{i+1}(x_i - x_{i+1}) \), that is, \( f(x_{i+1}) - f(x_i) \leq p_{i+1}(x_{i+1} - x_i) \). So,

\[
f(x_{i+1}) - f(x_i) - p_i(x_{i+1} - x_i) \leq \langle p_{i+1} - p_i, x_{i+1} - x_i \rangle,
\]

for \( i = 0, \ldots, n - 1 \). Therefore,

\[
0 \leq f(x_n) - f(x_0) - \sum_{i=0}^{n-1} p_i(x_{i+1} - x_i) \leq \sum_{i=0}^{n-1} \langle p_{i+1} - p_i, x_{i+1} - x_i \rangle,
\]

the first inequality being (4.1).

The job will be done when we find such a sequence \( (x_i, p_i)_{i=1}^{n-1} \subset gph \partial f \) that the right-hand side of (4.2) is arbitrarily small.

Fix \( \varepsilon > 0 \).

Fix \( n \in \mathbb{N} \) so large that

\[
n > 2 \frac{\langle p - p_0, x - x_0 \rangle}{\varepsilon}
\]

and let \( (x_n, p_n) = (x, p) \).

Let \( y_i = x_0 + \frac{i}{n}(x - x_0) \), \( i = 0, \ldots, n \). Since \( f \) is convex, \( y_i \in \text{dom } f \).

Using Lemma 3.3, for each \( i = 1, \ldots, n - 1 \) we get \( (x_i, p_i) \in gph \partial f \) such that

\[
\|x_i - y_i\| \leq \frac{\varepsilon}{8n(1 + \|p_0\| + \|p_n\|)} \quad \text{and} \quad \|p_j\|\|x_i - y_i\| \leq \frac{\varepsilon}{8n},
\]

for all \( i, j = 1, \ldots, n - 1 \). Then, using the above and \( x_0 = y_0, x_n = y_n \),

\[
\|p_j\|\|x_i - y_i\| \leq \frac{\varepsilon}{8n} \quad \forall i, j = 0, \ldots, n.
\]

Now, for each \( i = 0, \ldots, n - 1 \) we can write

\[
\langle p_{i+1} - p_i, x_{i+1} - x_i \rangle \leq \langle p_{i+1} - p_i, y_{i+1} - y_i \rangle + (\|p_i\| + \|p_{i+1}\|)(\|x_i - y_i\| + \|x_{i+1} - y_{i+1}\|)
\]

\[
\leq \langle p_{i+1} - p_i, y_{i+1} - y_i \rangle + \frac{\varepsilon}{2n},
\]

the second inequality because of (4.4). Therefore,

\[
\sum_{i=0}^{n-1} \langle p_{i+1} - p_i, x_{i+1} - x_i \rangle \leq \sum_{i=0}^{n-1} \langle p_{i+1} - p_i, y_{i+1} - y_i \rangle + \frac{\varepsilon}{2}.
\]

But, \( y_{i+1} - y_i = \frac{x - x_0}{n} \), so

\[
\sum_{i=0}^{n-1} \langle p_{i+1} - p_i, y_{i+1} - y_i \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \langle p_{i+1} - p_i, x - x_0 \rangle
\]

\[
= \frac{1}{n} \langle p_n - p_0, x - x_0 \rangle = \frac{1}{n} \langle p - p_0, x - x_0 \rangle < \frac{\varepsilon}{2},
\]

because of (4.3). Therefore, \( \sum_{i=0}^{n-1} \langle p_{i+1} - p_i, x_{i+1} - x_i \rangle < \varepsilon \) and, in the case when \( x \in \text{dom } \partial f \), (1.2) is established.

Now fix an arbitrary \( x \in X \) and a real number \( r \) such that \( r < f(x) \). By Proposition 1.3 there is \((\bar{x}, \bar{p}) \in gph \partial f\) such that

\[
r < f(\bar{x}) + \bar{p}(x - \bar{x}).
\]
Fix \( \varepsilon > 0 \). Since \( \bar{x} \in \text{dom } \partial f \), referring to the case just established we find a finite sequence \( (x_i, p_i)_{i=1}^{n-2} \in \text{gph } \partial f \) such that
\[
f(\bar{x}) < f(x_0) + \sum_{i=0}^{n-2} p_i(x_{i+1} - x_i) + \varepsilon,
\]
where \( x_{n-1} = \bar{x} \).

By letting \( p_{n-1} = \bar{p} \) and \( x_n = x \) and combining the two above we obtain
\[
r < f(x_0) + \sum_{i=0}^{n-1} p_i(x_{i+1} - x_i) + \varepsilon.
\]
Since \( r < f(x) \) and \( \varepsilon > 0 \) were arbitrary, the proof is completed. \( \square \)

**Proof of Theorem 1.1.** Fix \( x_0 \in \text{dom } \partial f \) and let \( c = g(x_0) - f(x_0) \). From Theorem 1.2 it follows that, provided (1.1),
\[
(4.5) \quad f(x) - f(x_0) \leq g(x) - g(x_0) \iff g(x) \geq f(x) + c, \; \forall x \in X.
\]
From this it follows that
\[
(4.6) \quad \text{dom } g \subseteq \text{dom } f,
\]
and, since we can swap \( x \) and \( x_0 \) if \( x \) is also in \( \text{dom } \partial f \),
\[
(4.7) \quad g(x) = f(x) + c, \; \forall x \in \text{dom } \partial f.
\]
Now let \( x \in \text{dom } f \) be arbitrary. From Proposition 1.4 we can find a sequence \( (x_n)_{n=1}^{\infty} \in \text{dom } \partial f \) such that \( \lim x_n = x \) and \( \lim f(x_n) = f(x) \). The lower semicontinuity of \( g \) together with (4.7) gives \( g(x) \leq \lim \inf g(x_n) = \lim \inf (f(x_n) + c) = f(x) + c \). That is,
\[
(4.6) \quad \text{dom } g \supseteq \text{dom } f,
\]
in particular \( \text{dom } g \supseteq \text{dom } f \), meaning that \( \text{dom } g = \text{dom } f \) because of (4.6), and \( g = f + c \) on \( \text{dom } f \) because of (4.5). Outside of their common domain both functions are equal to \( \infty \) by definition, and thus the proof is completed. \( \square \)

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**References**


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