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SPECTRAL AVERAGING FOR TRACE COMPATIBLE OPERATORS

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Abstract. In this note the notions of trace compatible operators and infinitesimal spectral flow are introduced. We define the spectral shift function as the integral of infinitesimal spectral flow. It is proved that the spectral shift function thus defined is absolutely continuous and Kreĭn's formula is established. Some examples of trace compatible affine spaces of operators are given.

1. Introduction

Let H_0 be a self-adjoint operator, and let V be a trace class operator on a Hilbert space $\mathcal{H}$. Then M. G. Kreĭn's famous result [13] says that there is a unique L^1-function $\xi_{H_0+V,H_0}(\lambda)$, known as the Kreĭn spectral shift function, such that for any $C_c^\infty(\mathbb{R})$ function f

$$\mathrm{Tr}(f(H_0+V)-f(H_0)) = \int_{-\infty}^{\infty} f'(\lambda)\xi_{H_0+V,H_0}(\lambda)\,d\lambda. \tag{1}$$

The notion of the spectral shift function was discovered by the physicist I. M. Lifshits [15]. An excellent survey on the spectral shift function can be found in [6].

In 1975, Birman and Solomyak [7] proved the following remarkable formula for the spectral shift function:

$$\xi(\lambda) = \frac{d}{d\lambda}\int_0^1 \mathrm{Tr}(VE^{H_r}_{(-\infty,\lambda]})\,dr, \tag{2}$$

where $H_r = H_0 + rV$, $r \in \mathbb{R}$, and $E^{H_r}_{(-\infty,\lambda]}$ is the spectral projection. Birman-Solomyak's proof relies on double operator integrals. An elementary derivation of (2) was obtained in [10] (without using double operator integrals).

Actually, this spectral averaging formula was discovered for the first time by Javrjan [12] in 1971, in case of a Sturm-Liouville operator on a half-line, perturbation being a perturbation of the boundary condition, so that in this case V was one-dimensional. An important contribution to spectral averaging was made by A. B. Alexandrov [1]. In 1998, B. Simon [18, Theorem 1] gave a simple short proof of the Birman-Solomyak formula. He also noticed that this formula holds for the wide class of Schrödinger operators on $\mathbb{R}^n$ [18, Theorems 3,4]. The connection of

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this formula with the integral formula for spectral flow from non-commutative geometry is outlined in [3]. An interesting approach to spectral averaging via Herglotz functions can be found in [11].

In this note we present an alternative viewpoint to the spectral shift function, and generalize the result of Simon so that it becomes applicable to a class of Dirac operators as well.

The new point of view, which the Birman-Solomyak formula suggests, is that there is a more fundamental notion than that of the spectral shift function. We call this notion the *speed of spectral flow* or *infinitesimal spectral flow* of a self-adjoint operator H under perturbation by a bounded self-adjoint operator V. It was introduced in [3] in the case of operators with compact resolvent. It is defined by the formula

$$\Phi_H(V)(\phi) = \mathrm{Tr}(V\phi(H)), \quad \phi \in C_c^\infty(\mathbb{R}), \tag{3}$$

whenever this definition makes sense. This naturally leads to the notion of *trace compatibility* of two operators. We say that operators H and $H+V$ are trace compatible, if for all $\phi \in C_c^\infty(\mathbb{R})$ the operator $V\phi(H)$ is trace class. The spectral shift function between two trace compatible operators is then considered as the integral of infinitesimal spectral flow. It turns out that the spectral shift function does not depend on the path connecting the initial and final operators, a fact which follows from the aforementioned result of B. Simon in the case of trace class perturbations.

The results of this note are summarized in Theorem 2.9. This theorem extends formulae (1) and (2) to the setting of trace compatible pairs $(H, H+V)$ and also strengthens [18, Theorems 3,4] in the sense that it does not require H to be a positive operator and maximally weakens conditions on the path $H + rV$, $r \in [0,1]$. Our results also hold for a more general setting, when $H = H^*$ is affiliated with a semifinite von Neumann algebra $\mathcal{N}$ and $V = V^* \in \mathcal{N}$.

Our investigation here also strengthens the link between the theory of the Kreĭn spectral shift function and that of spectral flow first discovered in [2]. For exposition of the latter theory we refer to [5] and a detailed discussion of the connection between the two theories in the situation where the resolvent of H is τ-compact (here, τ is an arbitrary faithful normal semifinite trace on $\mathcal{N}$) is contained in [3]. It should be pointed out here that the idea of viewing the spectral shift function as the integral of infinitesimal spectral flow is akin to I. M. Singer's ICM-1974 proposal to define the η invariant (and hence spectral flow) as the integral of a one form. Very general formulae of that type have been produced in the framework of non-commutative geometry (see [5] and references therein). We believe that our present approach will have applications to non-commutative geometry, in particular, it may be useful in avoiding "summability constraints" on H customarily used in that theory.

In semifinite von Neumann algebras $\mathcal{N}$ Kreĭn's formula (1) was proved for the first time in [9] in the case of a bounded self-adjoint operator $H \in \mathcal{N}$ and a trace class perturbation $V = V^* \in \mathcal{L}^1(\mathcal{N}, \tau)$ and in [4] for self-adjoint operators H affiliated with $\mathcal{N}$.

An additional reason to call $\Phi_H(V)$ the speed of spectral flow is the following observation. Let H be the operator of multiplication by λ on $L^2(\mathbb{R}, d\rho(\lambda))$ with some measure ρ and let the perturbation V be an integral operator with a sufficiently

regular (for example C^1) kernel $k(\lambda', \lambda)$. Then for any test function $\phi \in C_c^\infty(\mathbb{R})$

$$\Phi_H(V)(\phi) = \mathrm{Tr}(V\phi(H)) = \int_{\sigma_H} k(\lambda, \lambda)\phi(\lambda)\, d\rho(\lambda).$$

Hence, the infinitesimal spectral flow of H under perturbation by V is the measure on the spectrum of H with density $k(\lambda, \lambda)\, d\rho(\lambda)$. We note that this agrees with the classical formula [14, (38.6)]

$$E_n^{(1)} = V_{nn}$$

from formal perturbation theory. Here $E_n^{(0)}$ is the n-th eigenvalue of the unperturbed operator H_0, $E_n^{(j)}$, $j = 1, 2, \ldots$, is the j-th correction term for the n-th eigenvalue E_n of the perturbed operator $H = H_0 + V$ in the formal perturbation series $E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \ldots$, and $V_{mn} = \langle \psi_m^{(0)} | V | \psi_n^{(0)} \rangle$ is the matrix element of the perturbation operator V with respect to the eigenfunctions $\psi_m^{(0)}$ and $\psi_n^{(0)}$ of the unperturbed operator H_0 [14].

2. Results

Let $\mathcal{N}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with faithful normal semifinite trace τ. Let $\mathcal{A} = H_0 + \mathcal{A}_0$ be an affine space of self-adjoint operators affiliated with $\mathcal{N}$, where H_0 is a self-adjoint operator affiliated with $\mathcal{N}$ and $\mathcal{A}_0$ is a vector subspace of the real Banach space of all self-adjoint operators from $\mathcal{N}$. We say that $\mathcal{A}$ is *trace compatible*, if for all $\phi \in C_c^\infty(\mathbb{R})$, $V \in \mathcal{A}_0$ and $H \in \mathcal{A}$

$$V\phi(H) \in \mathcal{L}^1(\mathcal{N}, \tau), \tag{4}$$

where $\mathcal{L}^1(\mathcal{N}, \tau)$ is the ideal of trace class operators from $\mathcal{N}$, and if $\mathcal{A}_0$ is endowed with a locally convex topology which coincides with or is stronger than the uniform topology, such that the map $(V_1, V_2) \in \mathcal{A}_0^2 \mapsto V_1\phi(H_0 + V_2)$ is $\mathcal{L}^1$ continuous for all $H_0 \in \mathcal{A}$ and $\phi \in C_c^\infty(\mathbb{R})$. In particular, $\mathcal{A}$ is a locally convex affine space. The ideal property of $\mathcal{L}^1(\mathcal{N}, \tau)$ and [16, Theorem VIII.20(a)] imply that, in the definition of trace compatibility, the condition $\phi \in C_c^\infty(\mathbb{R})$ may be replaced by $\phi \in C_c(\mathbb{R})$. It follows from the definition of the topology on $\mathcal{A}_0$ that $H \in \mathcal{A} \mapsto e^{itH}$ is norm continuous.

If $\mathcal{A} = H_0 + \mathcal{A}_0$ is a trace compatible affine space, then we define a (generalized) one-form (on $\mathcal{A}$) of *infinitesimal spectral flow* or *speed of spectral flow* by the formula

$$\Phi_H(V) = \tau(V\delta(H)), \quad H \in \mathcal{A},\ V \in \mathcal{A}_0, \tag{5}$$

where δ is Dirac's delta function. The last formula is to be understood in a generalized function sense, i.e. $\Phi_H(V)(\phi) = \tau(V\phi(H))$, $\phi \in C_c^\infty(\mathbb{R})$. Φ is a generalized function, since if $\phi_n \to 0$ in $C_c^\infty(\mathbb{R})$ such that $\mathrm{supp}(\phi_n) \subseteq \Delta$, then $|\tau(V\phi_n(H))| \leq \left\| V E_\Delta^H \right\|_1 \|\phi_n(H)\| \to 0$. Here $\|A\|_1 = \tau(|A|)$.

Since ϕ can be taken from $C_c(\mathbb{R})$, for each $V \in \mathcal{A}_0$ the infinitesimal spectral flow $\Phi_H(V)$ is actually a measure on the spectrum of H.

By a smooth path $\{H_r\}_{r\in\mathbb{R}}$ in $\mathcal{A}$, we mean a differentiable path, such that its derivative $\frac{dH_r}{dr} \in \mathcal{A}_0$ is continuous.

Let $\Pi = \left\{(s_0, s_1) \in \mathbb{R}^2 \colon s_0 s_1 \geq 0, |s_1| \leq |s_0|\right\}$, and let

$$d\nu_f(s_0, s_1) = \mathrm{sgn}(s_0)\frac{i}{\sqrt{2\pi}}\hat{f}(s_0)\, ds_0\, ds_1.$$

If $f \in C_c^2(\mathbb{R})$, then (Π, ν_f) is a finite measure space [2]. For any $H_0, H_1 \in \mathcal{A}$, any $X \in \mathcal{A}_0$ and any non-negative $f \in C_c^\infty(\mathbb{R})$ set by definition,

$$\begin{aligned} \mathrm{T}_{f^{[1]}}^{H_1,H_0}(X) = \int_\Pi \big(& e^{i(s_0-s_1)H_1}\sqrt{f}(H_1)Xe^{is_1H_0} \\ & + e^{i(s_0-s_1)H_1}X\sqrt{f}(H_0)e^{is_1H_0}\big)\, d\nu_{\sqrt{f}}(s_0,s_1), \end{aligned} \tag{6}$$

where the integral is taken in the so^*-topology. For justification of this notation and details see [3].

Lemma 2.1. *If $\{H_r\} \subset \mathcal{A}$ is a path, continuous (smooth) in the topology of $\mathcal{A}_0$, and if $f \in C_c^2(\mathbb{R})$, then*

$$r \mapsto f(H_r) - f(H_0)$$

takes values in $\mathcal{L}^1(\mathcal{N},\tau)$ and it is $\mathcal{L}^1(\mathcal{N},\tau)$ continuous (smooth).

Proof. We can assume that f is non-negative and that $\sqrt{f} \in C_c^2(\mathbb{R})$. It is proved in [3] that

$$f(H_r) - f(H_0) = \mathrm{T}_{f^{[1]}}^{H_r,H_0}(H_r - H_0). \tag{7}$$

Since $e^{i(s_0-s_1)x}\sqrt{f}(x)$, $e^{is_1x}\sqrt{f}(x) \in C_c^2(\mathbb{R})$, trace compatibility implies that the integrand of the right hand side of (6) takes values in $\mathcal{L}^1$ and is $\mathcal{L}^1$-continuous (smooth), so the dominated convergence theorem completes the proof. □

If $\Gamma = \{H_r\}_{r\in[0,1]}$ is a smooth path in $\mathcal{A}$, then we define the spectral shift function ξ along this path as the integral of infinitesimal spectral flow: $\xi = \int_\Gamma \Phi$, or

$$\xi(\phi) = \int_0^1 \tau\left(\frac{dH_r}{dr}\,\phi(H_r)\right) dr, \quad \phi \in C_c^\infty(\mathbb{R}). \tag{8}$$

Now we prove that the spectral shift function is well-defined in the sense that it does not depend on the path of integration.

A one-form $\alpha_H(V)$ on an affine space $\mathcal{A}$ is called *exact* if there exists a zero-form θ_H on $\mathcal{A}$ such that $d\theta = \alpha$, i.e.

$$\alpha_H(V) = \frac{d}{dr}\theta_{H+rV}\Big|_{r=0}.$$

We say that the generalized one-form Φ is exact if $\Phi(\phi)$ is an exact form for any $\phi \in C_c^\infty(\mathbb{R})$.

The proof of the following proposition follows the lines of the proof of [3, Proposition 3.5].

Proposition 2.2. *The infinitesimal spectral flow Φ is exact.*

Proof. Let $V \in \mathcal{A}_0$, $H_r = H_0 + rV$, $r \in [0,1]$, and let $f \in C_c^\infty(\mathbb{R})$. By (7)

(9)
$$\begin{aligned} f(H_r) - f(H_0) &= \mathrm{T}_{f^{[1]}}^{H_r,H_0}(rV) \\ &= \int_\Pi \big(e^{i(s_0-s_1)H_r}\sqrt{f}(H_r) rV e^{is_1H_0} \\ &\qquad + e^{i(s_0-s_1)H_r} rV \sqrt{f}(H_0) e^{is_1H_0}\big)\, d\nu_{\sqrt{f}}(s_0,s_1) \\ &= \int_\Pi \big(e^{i(s_0-s_1)H_0}\sqrt{f}(H_0) rV e^{is_1H_0} \\ &\qquad + e^{i(s_0-s_1)H_0} rV \sqrt{f}(H_0) e^{is_1H_0}\big)\, d\nu_{\sqrt{f}}(s_0,s_1) \\ &\qquad + \int_\Pi \big((e^{i(s_0-s_1)H_r}\sqrt{f}(H_r) - e^{i(s_0-s_1)H_0}\sqrt{f}(H_0)) rV e^{is_1H_0} \\ &\qquad + (e^{i(s_0-s_1)H_r} - e^{i(s_0-s_1)H_0}) rV \sqrt{f}(H_0) e^{is_1H_0}\big)\, d\nu_{\sqrt{f}}(s_0,s_1) \\ &=: \mathrm{T}_{f^{[1]}}^{H_0,H_0}(rV) + R_1 + R_2. \end{aligned}$$

All three summands here are trace class by the trace compatibility assumption. So, for any $S \in \mathcal{N}$

$$\tau(S(f(H_r) - f(H_0))) = r\tau\left(S\mathrm{T}_{f^{[1]}}^{H_0,H_0}(V)\right) + \tau(SR_1) + \tau(SR_2).$$

Now, Duhamel's formula and (7) show that $\tau(SR_1) = o(r)$ and $\tau(SR_2) = o(r)$. Hence,

$$\frac{d}{dr}\tau(S(f(H_r) - f(H_0))) = \tau\left(S\mathrm{T}_{f^{[1]}}^{H_r,H_r}(V)\right). \tag{10}$$

This implies that for any $S \in \mathcal{N}$

$$\tau(S(f(H_1) - f(H_0))) = \tau\left(\int_0^1 S\mathrm{T}_{f^{[1]}}^{H_r,H_r}(V)\, dr\right).$$

Now let $H_0 \in \mathcal{A}$ be a fixed operator and for any $f \in C_c^\infty(\mathbb{R})$ let

$$\theta_H^f := \int_0^1 \tau(Vf(H_r))\, dr,$$

where $H_r = H_0 + rV$, $H = H_1$. We are going to show that $d\theta_H^f(X) = \Phi_H(X)(f)$ for any $X \in \mathcal{A}_0$.

Following the proof of [3, Proposition 3.5] we have

$$\begin{aligned} (A) := d\theta_H^f(X) &= \lim_{s\to 0}\int_0^1 \tau\big(Xf(H_r + srX)\big)\, dr \\ &\qquad + \lim_{s\to 0}\frac{1}{s}\int_0^1 \tau\Big(V\big(f(H_r + srX) - f(H_r)\big)\Big)\, dr. \end{aligned}$$

By definition of $\mathcal{A}_0$ topology the integrand of the first summand is continuous with respect to r and s. So, the first summand is equal to

$$\int_0^1 \tau\big(Xf(H_r)\big)\, dr.$$

By [2, Theorem 5.3] the second summand is equal to

$$\begin{aligned}\lim_{s\to 0}\frac{1}{s}\int_0^1 \tau\left(V\mathrm{T}_{f^{[1]}}^{H_r+srX,H_r}(srX)\right)\,dr &= \lim_{s\to 0}\int_0^1 \tau\left(V\mathrm{T}_{f^{[1]}}^{H_r+srX,H_r}(rX)\right)\,dr\\ &= \int_0^1 \tau\left(V\,\mathrm{T}_{f^{[1]}}^{H_r,H_r}(rX)\right)\,dr\\ &= \int_0^1 \tau\left(X\,\mathrm{T}_{f^{[1]}}^{H_r,H_r}(V)\right)r\,dr,\end{aligned}$$

where the second equality follows from the definition of $\mathcal{A}_0$-topology and the last equality follows from [3, Lemma 3.2]. Now, using (10) and integrating by parts we get

$$\begin{aligned}(A) &= \tau\left(Xf(H_1)-Xf(H_0)\right)\\ &\quad+\int_0^1\left(\tau(Xf(H_r))-\tau(X[f(H_r)-f(H_0)])\right)dr = \tau\left(Xf(H_1)\right).\end{aligned}\tag{11}$$

□

The argument before [8, Proposition 1.5] now implies

Corollary 2.3. *The spectral shift function given by* (8) *is well-defined.*

Proposition 2.4. *If $r\in\mathbb{R}\mapsto H_r\in\mathcal{A}$ is smooth, then the equality*

$$\tau\left(\frac{df(H_r)}{dr}\phi(H_r)\right)=\tau\left(\dot{H}_r f'(H_r)\phi(H_r)\right)\tag{12}$$

holds for any $f\in C_c^2(\mathbb{R})$ and any bounded measurable function ϕ.

Proof. Without loss of generality, we can assume that $f\geq 0$ and $\sqrt{f}\in C_c^2(\mathbb{R})$. We prove the above equality at $r=0$. The formula (7) and the dominated convergence theorem imply that

$$\begin{aligned}\tau\left(\frac{df(H_r)}{dr}\Big|_{r=0}\phi(H_0)\right) &= \tau\Big(\phi(H_0)\int_\Pi \lim_{r\to 0}\big[e^{i(s_0-s_1)H_r}\sqrt{f}(H_r)\frac{H_r-H_0}{r}e^{is_1H_0}\\ &\qquad + e^{i(s_0-s_1)H_r}\frac{H_r-H_0}{r}\sqrt{f}(H_0)e^{is_1H_0}\big]\,d\nu_{\sqrt{f}}(s_0,s_1)\Big),\end{aligned}$$

where the limit is taken in $\mathcal{L}^1(\mathcal{N},\tau)$. By the $\mathcal{A}_0$-smoothness of $\{H_r\}$, we have

$$\begin{aligned}\tau\left(\frac{df(H_r)}{dr}\Big|_{r=0}\phi(H_0)\right) &= \tau\Big(\phi(H_0)\int_\Pi \big[e^{i(s_0-s_1)H_0}\sqrt{f}(H_0)\dot{H}_r\big|_{r=0}e^{is_1H_0}\\ &\qquad + e^{i(s_0-s_1)H_0}\dot{H}_r\big|_{r=0}\sqrt{f}(H_0)e^{is_1H_0}\big]\,d\nu_{\sqrt{f}}(s_0,s_1)\Big),\end{aligned}$$

so that by [2, Lemmas 3.7, 3.10] and letting $A=\dot{H}_r\big|_{r=0}$

$$\begin{aligned}\tau\left(\frac{df(H_r)}{dr}\Big|_{r=0}\phi(H_0)\right) &= 2\int_\Pi \tau\Big(\phi(H_0)e^{is_0H_0}\sqrt{f}(H_0)A\Big)\,d\nu_{\sqrt{f}}(s_0,s_1)\\ &= 2\tau\Big(A\phi(H_0)\sqrt{f}(H_0)\int_{-\infty}^{\infty}e^{is_0H_0}\frac{i}{\sqrt{2\pi}}s_0\mathcal{F}(\sqrt{f})(s_0)\,ds_0\Big)\\ &= \tau\big(A\phi(H_0)f'(H_0)\big).\end{aligned}$$

□

Proposition 2.5. *The spectral shift function given by* (8) *satisfies Kreĭn's formula, i.e. for any* $f \in C_c^\infty$, $H_0, H_1 \in \mathcal{A}$

$$\tau\left(f(H_1) - f(H_0)\right) = \xi(f').$$

Proof. Taking the integral of (12) with $\phi = 1$ we have

$$\int_0^1 \tau\left(\frac{d(f(H_r) - f(H_0))}{dr}\right) dr = \int_0^1 \tau\left(\dot{H}_r f'(H_r)\right) dr.$$

The right hand side is $\xi(f')$ by definition. It follows from Lemma 2.1 that one can interchange the trace and the derivative in the left hand side. □

Corollary 2.6. *In the case of trace class perturbations, the spectral shift function* ξ *defined by* (8) *coincides with classical definition, given by* [4, Theorem 3.1].

Proof. This follows from Theorem 6.3 and Corollary 6.4 of [2]. □

For trace class perturbations the absolute continuity of the spectral shift function is established in [13] (see also [11]). For the general semifinite case we refer to [4, 2].

Lemma 2.7. *Let* $\mathcal{A}$ *be a trace compatible affine space and let* $f \in C_c^\infty(\mathbb{R})$. *Let* $H_0, H_1 \in \mathcal{A}$, *let* ξ *and* ξ_f *be the spectral shift distributions of the pairs* (H_0, H_1) *and* $f(H_0), f(H_1)$, *respectively. Then for any* $\phi \in C_c^\infty(\mathbb{R})$

$$\xi_f(\phi) = \xi(\phi \circ f \cdot f'). \tag{13}$$

Proof. By Proposition 2.4 for any $f, \phi \in C_c^\infty(\mathbb{R})$,

$$\tau\left(\frac{df(H_r)}{dr}\phi(f(H_r))\right) = \tau\left(\dot{H}_r f'(H_r)\phi(f(H_r))\right) = \tau\left(\dot{H}_r (F \circ f)'(H_r)\right),$$

where $F' = \phi$. Hence, for any smooth path $\Gamma = \{H_r\}_{r\in[0,1]} \subseteq \mathcal{A}$

$$\int_0^1 \tau\left(\frac{df(H_r)}{dr}\phi(f(H_r))\right) dr = \int_0^1 \tau\left(\dot{H}_r (F \circ f)'(H_r)\right) dr, \tag{14}$$

which is (13). □

Proposition 2.8. *Let* $\mathcal{A} = H_0 + \mathcal{A}_0$ *be a trace compatible affine space and let* $\mathcal{A}_0$ *be such that for any* $V \in \mathcal{A}_0$ *there exist positive* $V_1, V_2 \in \mathcal{A}$ *such that* $V = V_1 - V_2$. *Then the spectral shift function* ξ *of any pair* $H, H+V \in \mathcal{A}$ *is absolutely continuous.*

Proof. Since the map $(V_1, V_2) \mapsto \tau\left(V_1\phi(H + V_2)\right)$ is $\mathcal{L}^1(\mathcal{N}, \tau)$-continuous (by definition), it follows that the infinitesimal spectral flow is a uniformly locally finite measure with respect to the path parameter. Hence, the spectral shift function is also a locally finite measure being the integral of locally finite measures, which are uniformly bounded on every segment.

If, for $H_0, H_1, H_2 \in \mathcal{A}$, the spectral shift functions from H_0 to H_1 and from H_1 to H_2 are absolutely continuous, then evidently the spectral shift function from H_0 to H_2 is also absolutely continuous. Hence, if $V = V_1 - V_2$ with $0 \le V_1, V_2 \in \mathcal{A}_0$, then representing the spectral shift function from H to $H + V$ as the sum of the spectral shift function from H to $H + V_1$ and from $H + V_1$ to $H + V$, we see that we can assume that the perturbation V is positive.

By Lemma 2.1 and [4, Theorem 3.1] the spectral shift function ξ_f of the pair $(f(H), f(H + V))$ is absolutely continuous. Let us suppose that the spectral shift function ξ of the pair $(H, H + V)$ has a non-absolutely continuous part μ.

Without loss of generality, we can assume that there exists a set of Lebesgue measure zero $E \subset (\varepsilon, 1-\varepsilon)$ such that $\mu(E) > 0$. For any $a, b \in \mathbb{R}$ with $b - a > 2$ let us consider a "cap"-function $f_{a,b}$, i.e. f is a smooth function which is zero on $(-\infty, a)$ and (b, ∞), it is 1 on $(a+1, b-1)$ and its derivatives on $(a+\varepsilon, a+1-\varepsilon)$ and $(b-1+\varepsilon, b-\varepsilon)$ is 1 and -1, respectively.

Let U be an open set of Lebesgue measure $< \delta$ such that $E \subset U$ and let ϕ be a smoothed indicator of U. Then (13), applied to functions ϕ and $f_{0,b}$ and to functions ϕ and $f_{a,b}$ (with big enough b) implies that $\mu(E) = \mu(a+E)$, i.e. μ is translation invariant. Since it is also locally finite it is some multiple of Lebesgue measure. This yields a contradiction. □

We summarize the results in the following theorem.

Theorem 2.9. *Let $\mathcal{A}$ be a trace compatible affine space of operators in a semifinite von Neumann algebra $\mathcal{N}$ with a normal semifinite faithful trace τ. Let H and $H+V$ be two operators from $\mathcal{A}$. Let the spectral shift (generalized) function $\xi_{H,H+V}$ be defined as the integral of infinitesimal spectral flow by the formula*

$$\xi_{H,H+V}(\phi) = \int_\Gamma \Phi(\phi) = \int_0^1 \Phi_{H_r}(\dot{H}_r)(\phi)\, dr, \quad \phi \in C_c^\infty,$$

where $\Gamma = \{H_r\}_{r\in[0,1]}$ is any piecewise smooth path in $\mathcal{A}$ connecting H and $H+V$. Then the spectral shift function is well-defined in the sense that the integral does not depend on the choice of the piecewise smooth path Γ connecting H and $H+V$, and it satisfies Kreĭn's formula

$$\tau(f(H+V) - f(H)) = \xi(f'), \quad f \in C_c^\infty.$$

Moreover, if for any $V \in \mathcal{A}_0$ there exist $0 = V_1, V_2 \in \mathcal{A}_0$ such that $V = V_1 - V_2$, then $\xi_{H,H+V}$ is an absolutely continuous measure.

Two extreme examples of trace compatible affine spaces are $H_0 + \mathcal{L}^1_{sa}(\mathcal{N}, \tau)$, $H_0 = H_0^* \eta \mathcal{N}$, with the topology induced by $\mathcal{L}^1(\mathcal{N}, \tau)$ [2, 4], and $D_0 + \mathcal{N}_{sa}$, where $(D_0 - i)^{-1}$ is τ-compact, with the topology induced by operator norm [3]. In particular, the space $-\Delta + C(M)$, where (M, g) is a compact Riemannian manifold, and Δ is the Laplacian, is trace compatible.

As an example of an intermediate trace compatible affine space one can consider Schrödinger operators $-\Delta + C_c(\mathbb{R}^n)$ with the inductive topology of uniform convergence. It is proved in [19, Section B9] that for this example condition (4) holds. It also follows from [19, Section B9] that $\|gf(H)\|_1 \le C\|g\|_2$, where C depends only on f, on the support of g and on $\|V_-\|_\infty$, where V_- is the negative part of V, $H = -\Delta + V$. So, the condition on the topology of $\mathcal{A}_0$ is fulfilled by (7).

Another example is given by Dirac operators of the form $D + \mathcal{A}_0$, where $D = \sum_{j=1}^n \alpha_j \frac{\partial}{\partial x_j}$, $\alpha_1, \dots, \alpha_n$ are $m \times m$-matrices such that $\alpha_j\alpha_k + \alpha_k\alpha_j = -2\delta_{jk}$, and

$$\mathcal{A}_0 = \left\{a = a^* \in C_c(\mathbb{R}^n, \mathbf{M}_m(\mathbb{R})) \colon \exists \phi = \phi^* \in C^1(\mathbb{R}^n), \quad iD\phi = a\right\}$$

with the inductive topology of uniform convergence. A proof that the space $D + \mathcal{A}_0$ is trace compatible can be reduced to [17, Theorem 4.5] via the gauge transformation $\psi \mapsto e^{-i\phi(x)}\psi$. We have

$$(D+a)(e^{-i\phi(x)}u) = \sum_{j=1}^n \left(-ie^{-i\phi(x)}\frac{\partial}{\partial x_j}\phi(x)\alpha_j u + e^{-i\phi(x)}\alpha_j \frac{\partial}{\partial x_j} u\right) + ae^{-i\phi(x)}u,$$

where u is an m-column of C_c^∞-functions. So, if $iD\phi = a$, then

$$e^{i\phi(x)}(D+a)(e^{-i\phi(x)}u) = Du.$$

Hence, $(D+a)^2 = e^{-i\phi(x)}D^2e^{i\phi(x)}$. This shows that $gf((D+a)^2)$, $g, a \in \mathcal{A}_0$, $f \in C_c^\infty(\mathbb{R})$, is trace class iff $gf(e^{-i\phi(x)}D^2e^{i\phi(x)}) = e^{-i\phi(x)}ge^{i\phi(x)}f(e^{-i\phi(x)}D^2e^{i\phi(x)})$ is trace class. But the last operator is unitarily equivalent to $gf(D^2)$, which is trace class by [17, Theorem 4.5]. So, if $f \geq 0$, then by the same theorem

$$\|gf(D+a)\|_1 \leq C\,\|g\|_\infty\,\|f\|_\infty\,, \tag{15}$$

where C depends on supports of g and f. Hence, for $g, g_1, a, a_1 \in \mathcal{A}_0$, we have

$$\|gf(D+a) - g_1f(D+a_1)\|_1 \leq \|(g-g_1)f(D+a)\|_1 + \|g_1(f(D+a) - f(D+a_1)\|_1\,.$$

So, the condition on the topology of $\mathcal{A}_0$ is fulfilled by (7) and (15).

In case $n = m = 1$, we have $D + \mathcal{A}_0 = \frac{1}{i}\frac{d}{dx} + C_c(\mathbb{R})$.

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