AN EXTREMAL PROPERTY OF JACOBI POLYNOMIALS
IN TWO-SIDED CHERNOFF-TYPE INEQUALITIES
FOR HIGHER ORDER DERIVATIVES

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Abstract. For a weight function generating the classical Jacobi polynomials, the sharp double estimate of the distance from the subspace of all polynomials of an arbitrary fixed order is established.

1. Introduction

Let $\Delta := (-1, 1)$ and let $\omega(x) > 0$ be a Lebesgue integrable weight function. Denote by $L^2(\omega)$ the real Hilbert space with the inner product $(f, g)_{L^2(\omega)} := \int_{\Delta} fg \omega$ and the norm $\|f\|_{L^2(\omega)} := (f, f)^{1/2}_{L^2(\omega)}$. Let $P$ be the set of all polynomials and let $P_k \subset P$ be the set of all polynomials of degree $\leq k$.

We study the problem on a sharp two-sided estimate of the distance

$$D_{k,\omega}(G) := \left[\text{dist}_{L^2(\omega)}(G, P_{k-1})\right]^2 := \inf_{c_0, \ldots, c_{k-1}} \int_{\Delta} \left| G(x) - \sum_{j=0}^{k-1} c_j x^j \right|^2 \omega(x)dx$$

by an analog with the well-known two-sided inequality from the probability theory

$$(1.1) \quad [E G'(X)]^2 \leq D[G(X)] \leq E[G''(X)]^2,$$

which is valid for any absolutely continuous function $G$ with a finite right-hand side (1.1) and the Gaussian normal random variable $X$. Moreover, both inequalities become the equalities for $G(x) = \text{const} \cdot x$. The right-hand side of (1.1), proved in [1], is often cited as the Chernoff inequality; however, in a form even more general than (1.1), it can be derived from the earlier work [2]. An accurate piece of the history of the problem and the extension of (1.1) for the $L^p$-deviation, $p \geq 1$, can be found in [3]. Unlike [3] and many other works, we extend (1.1) in a different way, increasing the order of derivative, but saving the $L^2$-metric. In this case the role of the extremal functions, when both inequalities become the equalities, is to
play the orthogonal polynomials. A natural extension of (1.1) for the $k$th derivative found in [4] has a form

$$\frac{1}{k!} \left[ \int_{\mathbb{R}} G^{(k)} dF \right]^2 \leq D_{k,F}(G) \leq \frac{1}{k!} \int_{\mathbb{R}} |G^{(k)}|^2 dF,$$

where $dF(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. Moreover, the double equality in (1.2) takes place for the Chebyshev-Hermite polynomials of degree $k$.

In the present paper we consider an analog of (1.1) for the case of a finite interval $\Delta$ and the weight

$$\omega(x) = (1 - x)^\alpha(1 + x)^\beta, \quad \alpha > -1, \beta > -1,$$

when the extremal functions are the classical Jacobi polynomials. Similar to [4], we obtain the result with the help of weighted differential operators for which the Jacobi polynomials are the eigenfunctions (Section 3). Section 2 is devoted to analysis of the case with a general weight $\omega$, and we show that the lower bound is always true with the extremal functions as the orthogonal polynomials with weight $\omega$ and the upper bound with the same extremal functions is valid, roughly speaking, if a polynomial of the second order is the eigenfunction of a related differential operator of the second order, which in turn implies that the weight function $\omega$ has the form (1.3).

We use the symbols $:= \quad \text{and} \quad =: \quad \text{for a definition of new variables}, \ \mathbb{N} \text{ is the set of all natural integers, } \mathbb{N}_0 := \{0\} \cup \mathbb{N}, \ c_i, i = 1, 2, \ldots \text{ are positive constants possibly different from place to place.}$

2. General case

Let $\omega(x) > 0, \ x \in \Delta$, be a weight function such that $\int_{\Delta} \omega(x) dx = 1$. Then the orthonormal system of polynomials $\{\varphi_k\}, \ k \in \mathbb{N}_0$, such that $\varphi_0(x) = 1$,

$$\varphi_k(x) = (-1)^k a_k x^k + \ldots + a_0, \quad k \geq 1,$$

with $a_k > 0$ and

$$\int_{\Delta} \varphi_k^2 \omega = 1$$

is complete in the real Hilbert space $\mathbb{H} := L^2(\omega)$ (see [6], § 1.3.2).

We shall need the Riemann-Liouville transform

$$R_k g(x) := \frac{1}{(k-1)!} \int_{\mathbb{R}} r_k(x, t) g(t) dt,$$

where $\mathbb{R} := (-\infty, \infty)$ and

$$r_k(x, t) := \begin{cases} (x - t)^{k-1}, & 0 \leq t \leq x < \infty, \\ (-1)^k (t - x)^{k-1}, & -\infty < x \leq t < 0, \\ 0, & \text{otherwise}. \end{cases}$$

Let $AC^{(k)}(\Delta)$ denote the class of all functions on $\Delta$ with absolutely continuous $(k - 1)$-th derivative. It easy to see that for any $G \in AC^{(k)}(\Delta)$ the following is
true:

\[(2.1)\quad G(x) = R_k G^{(k)}(x) + \sum_{j=0}^{k-1} \frac{x^j}{j!} G^{(j)}(0),\]

\[(2.2)\quad G - \sum_{j=0}^{k-1} (G, \rho_j)_H \rho_j = R_k G^{(k)} - \sum_{j=0}^{k-1} (R_k G^{(k)}, \rho_j)_H \rho_j.\]

Define

\[\rho_k(x) := \frac{1}{(k-1)!} \int_{-1}^{x} (x-t)^{k-1} \varphi_k(t) \omega(t) dt, \quad k \in \mathbb{N}.\]

Then

\[(2.3)\quad \rho_k^{(n)}(\pm 1) = 0, \quad n = 0, 1, \ldots, k - 1.\]

Moreover, \(\rho_k(x)\) can have alternating signs, but

\[\lambda_k^{1/2} := \int_{\Delta} \rho_k(x) dx = \frac{1}{a_k k!} > 0, \quad k \in \mathbb{N}.\]

If we put

\[\mu_k(x) := \lambda_k^{-1/2} \rho_k(x), \quad k \in \mathbb{N},\]

then for

\[D_{k,\omega}(G) := [\text{dist}_{\mathbb{H}}(G, P_{k-1})]^2, \quad k \in \mathbb{N}\]

the lower bound

\[(2.4)\quad \lambda_k \left[ \int_{\Delta} G^{(k)}(x) \mu_k(x) dx \right]^2 \leq D_{k,\omega}(G)\]

holds (see [4], Lemma 1) provided \(G \in AC^{(k)}(\Delta)\) and \(G^{(k)} \in L^2(\overline{\rho_k})\), where

\[\overline{\rho_k}(t) := \int_{\Delta} |r_k(x, t) \varphi_k(x)| \omega(x) dx.\]

Moreover, (2.4) turns into an equality for \(G = \text{const} \cdot \varphi_k\).

Now, suppose that \(\rho_k(x) \geq 0\) and for all \(2k\) times differentiable functions \(y(x)\) we define differential operators

\[L_k y := (-1)^k D^k [\rho_k D^k y].\]

Then

\[L_k \varphi_k = (-1)^k D^k [\rho_k D^k \varphi_k] = a_k k! \varphi_k^{(k)} = \lambda_k^{-1/2} \varphi_k \omega,\]

so \(\varphi_k\) is an eigenfunction of the weighted differential operator \(L_k\).

Suppose \(\varphi_n, n > k\), are the eigenfunctions of \(L_k\) too, that is,

\[(2.5)\quad L_k \varphi_n = \sigma_{k,n} \varphi_n \omega, \quad n > k \geq 1,\]

and, moreover,

\[(2.6)\quad \lambda_k^{-1/2} = \sigma_{k,k} \leq \sigma_{k,n}, \quad n > k \geq 1.\]

Then for all polynomials \(y \in \mathbb{P}\) such that

\[(y, \varphi_n)_{\mathbb{H}} = 0, \quad n = 0, 1, \ldots, k - 1,\]

the inequality

\[\int_{\Delta} y L_k y \geq \lambda_k^{-1/2} \int_{\Delta} y^2 \omega\]
holds and becomes the equality for \( y = \text{const} \cdot \varphi_k \). By the density \( P \subset L^2(\mu_k) \) it implies the inequality

\[
(2.7) \quad D_k,\omega(G) \leq \lambda_k \int_{\Delta} |G^{(k)}|^2 \mu_k
\]

for all \( G \in AC^{(k)}(\Delta) \) such that \( G^{(k)} \in L^2(\mu_k) \) (see the end of the proof of Theorem 3.1 for details).

Thus, we obtain the following.

**Theorem 2.1.** Let \( k \in \mathbb{N} \) and let a weight function \( \omega(x) > 0, x \in \Delta \), be such that \( \int_{\Delta} \omega = 1 \). Let \( \{\varphi_n\}, n \in \mathbb{N}, \) be the above constructed orthonormal sequence of polynomials in \( \mathbb{H} = L^2(\omega) \). Then for any function \( G \in AC^{(k)}(\Delta) \) such that \( G^{(k)} \in L^2(\mu_k) \), the inequality (2.4) holds. Moreover, if the sequence \( \{\varphi_n\} \) satisfies (2.5), (2.6), and \( \rho_k(x) \geq 0 \), then for any function \( G \in AC^{(k)}(\Delta) \) such that \( G^{(k)} \in L^2(\mu_k) \), the inequality (2.7) holds, and both inequalities (2.4) and (2.7) simultaneously become the equalities, when \( G = \text{const} \cdot \varphi_k \).

Naturally, the sequence of conditions (2.5), (2.6) looks much too restrictive. To this end we consider the first of the equations, (2.5):

\[
(2.8) \quad L_1 \varphi_2 = \sigma_1,2 \varphi_2 \omega =: \sigma \varphi_2 \omega,
\]

where \( \sigma > 0 \) is some number and \( \varphi_2(x) = b_2 x^2 + b_1 x + b_0, b_2 > 0 \). Taking into account that \( \varphi_1 \in P_1, \varphi_2 \in P_2, \) and

\[
\rho_1(x) = \int_{-1}^{x} \varphi_1 \omega,
\]

we obtain

\[
L_1 \varphi_2 = - \rho_1 \varphi_2' = -\varphi_1 \varphi_2' \omega - b_2 \rho_1 = \sigma \varphi_2 \omega.
\]

Consequently,

\[
(2.9) \quad \rho_1 b_2 = [-\varphi_1 \varphi_2' - \sigma \varphi_2] \omega =: q_2 \omega,
\]

where \( q_2 \in P_2 \). It follows from (2.3) and (2.9) that

\[
(2.10) \quad \lim_{x \to \pm 1} q_2(x) \omega(x) = 0.
\]

Moreover, it follows from (2.9) that \( \omega \) is differentiable on the interval \( \Delta \) and satisfies the Pirson equation

\[
(2.11) \quad \frac{\omega'}{\omega} = \frac{b_2 \varphi_1 - q_2}{q_2} =: \frac{q_1}{q_2},
\]

where \( q_1 \in P_1 \). It is known (see [5], Chapter II, § 1) that, under the conditions \( \omega(x) > 0, x \in \Delta, \int_{\Delta} \omega < \infty \) and (2.10), the equation (2.11) has the only solutions of the form

\[
\omega(x) = \text{const}(1 - x)^\alpha (1 + x)^\beta,
\]

where \( \alpha > -1, \beta > -1 \). To close the circle, we show in the next section that polynomials generated by this weight (Jacobi polynomials) satisfy (2.5) and (2.6) as well as \( \rho_k(x) \geq 0 \) for this case, and the condition \( G^{(k)} \in L^2(\mu_k) \) of Theorem 2.1 can be weakened up to \( G^{(k)} \in L^2(\mu_k) \).
3. CHERNOFF-TYPE INEQUALITY

Again let \( H := L^2(\omega) \) be the Hilbert space with the inner product \((f, g)_{L^2(\omega)} := \int_{\Delta} fg \omega \) and the norm \( \|f\|_{L^2(\omega)} := (f,f)_{L^2(\omega)}^{1/2} \), where the weight function \( \omega(x) \) has the form (1.3). Denote by \( P_n(x) \), \( n \in \mathbb{N}_0 \), the orthogonal sequence of Jacobi polynomials. It is known that \( \{ P_n(x) \} \) is complete in \( H \). By Rodrigues formula (see [5], Chapter VII, § 1), we have

\[
(3.1) \quad P_n(x) = \frac{(-1)^n}{2^n n!} (1 - x)^{-a}(1 + x)^{-b} D^n [(1 - x)^{a+n}(1 + x)^{b+n}] ,
\]
where \( D := \frac{d}{dx} \). It is known (see [5], Chapter VII, § 1) that

\[
(3.2) \quad \eta_n^2 := \|P_n\|^2_H = \frac{2^{a+b+1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{n!(\alpha + \beta + 2n + 1) \Gamma(\alpha + \beta + n + 1)} ,
\]
where \( \Gamma(\cdot) \) denotes the gamma function. Let

\[
(3.3) \quad p_n := P_n/\|P_n\|_H
\]
be the orthogonal sequence of Jacobi polynomials. Then

\[
(3.4) \quad D_{k,\omega}(G) = \|G\|^2_H - \sum_{j=0}^{k-1} (G, p_j)^2_H .
\]

For \( n \in \mathbb{N} \) we put

\[
(3.5) \quad \beta_n := \int_{\Delta} (1 - t)^{\alpha+n} (1 + t)^{\beta+n} dt = 2^{\alpha+\beta+2n+1} \frac{\Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 2)} ,
\]

\[
(3.6) \quad \nu_n(x) := \frac{(1 - x)^{\alpha+n} (1 + x)^{\beta+n}}{\beta_n} ,
\]

\[
(3.7) \quad \gamma_n := \frac{\beta_n^2}{\eta_n^2 2^{2n}|n!|^2} .
\]

**Theorem 3.1.** Let \( k \in \mathbb{N} \). Then for any function \( G \in AC^{(k)}(\Delta) \) such that \( G^{(k)} \in L^2(\nu_k) \), the inequalities

\[
(3.8) \quad \gamma_k \left[ \int_{\Delta} G^{(k)} \nu_k \right]^2 \leq D_{k,\omega}(G) \leq \gamma_k \int_{\Delta} |G^{(k)}|^2 \nu_k
\]

hold. Moreover, both inequalities in (3.8) become the equalities for \( G = \text{const} \cdot P_k \).

**Proof.** Observe that if \( G \in AC^{(k)}(\Delta) \) and \( G^{(k)} \in L^2(\nu_k) \), then \( G \in H \). Indeed, it follows from (2.1) that

\[
\int_{\Delta} G^2 \omega \leq c_1 \int_{\Delta} |R_k G^{(k)}|^2 \omega + c_2 \sum_{j=0}^{k-1} |G^{(j)}(0)|^2 \int_{\Delta} x^{2j} \omega(x) dx .
\]

Applying Schwarz’s inequality, we obtain

\[
|R_k G^{(k)}(x)|^2 \leq \int_{\Delta} |r_k(x,t)|^2 dt \int_{\Delta} |r_k(x,t)||G^{(k)}(t)|^2 dt \leq c_3 \int_{\Delta} |r_k(x,t)||G^{(k)}(t)|^2 dt .
\]
Thus,
\[
\int_\Delta G^2 \omega \leq c_4 + c_5 \int_\Delta \omega(x)dx \int_\Delta |r_k(x,t)||G^{(k)}(t)|^2dt
\]
\[= c_4 + c_5 \int_\Delta |G^{(k)}(t)|^2dt \int_\Delta |r_k(x,t)|\omega(x)dx \leq c_4 + c_6 \int_\Delta |G^{(k)}(t)|^2\nu_k(t)dt < \infty.
\]
First we prove the left-hand side of inequality (3.8). It follows from (3.4) and the Bessel inequality that
\[D_{k,\omega}(G) \geq (G,p_k)^2_H
\]
(applying (3.1), (3.2) and (3.3))
\[= \frac{1}{\eta^2_k 2^{2k} [k!]^2} \left[ \int_\Delta G(x)D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}] \right]^2
\]
(using (2.1) and the orthogonality of \(p_k\) to all polynomials of degree \(\leq k-1\))
\[= \frac{1}{\eta^2_k 2^{2k} [k!]^2} \left[ \int_\Delta R_k G^{(k)}(x)D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}] \right]^2.
\]
From the elementary upper bound
\[|D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}]| \leq c_7 \omega(x), \quad x \in \Delta,
\]
applying Schwarz’s inequality twice, we obtain
\[= c_8 \left( \int_\Delta |G^{(k)}(t)|^2dt \int_\Delta |r_k(x,t)|\omega(x)dx \right)^{1/2} \leq c_9 \left( \int_\Delta |G^{(k)}(t)|^2\nu_k(t)dt \right)^{1/2} < \infty.
\]
Therefore, by Fubini’s theorem, we find from (3.9)
\[D_{k,\omega}(G) \geq \frac{1}{\eta^2_k 2^{2k} [k!]^2} \left[ \int_\Delta G^{(k)}(t)dt \int_\Delta r_k(x,t)D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}] \right]^2
\]
\[= \frac{1}{\eta^2_k 2^{2k} [k!]^2} \left[ \int_\Delta G^{(k)}(t)(1-t)^{\alpha+k}(1+t)^{\beta+k}dt \right]^2.
\]
(assuming the definitions (3.5), (3.6) and (3.7))
\[= \gamma_k \left[ \int_\Delta G^{(k)} \nu_k \right]^2.
\]
Consequently, the left-hand side of inequality (3.8) is proved. If \(G = P_k\), then
\[\int_\Delta G^{(k)} \nu_k = \int P_k^{(k)} \nu_k = \frac{1}{\beta_k} \int_\Delta P^{(k)}(x)(1-x)^{\alpha+k}(1+x)^{\beta+k}dx
\]
(integrating by parts)
\[= \frac{(-1)^k}{\beta_k} \int_\Delta P_k(x)D^k[(1-x)^{\alpha+k}(1+x)^{\beta+k}]dx
\]
(assuming (3.1) and then (3.7))
\[= \frac{2^k k!}{\beta_k} \int_\Delta P_k^2 \omega = \frac{2^k k! \eta^2_k}{\beta_k} = \frac{\eta_k}{\gamma_k^{1/2}}.
\]
Thus, we have
\[(3.10)\quad \gamma_k \left[ \int_{\Delta} P_k^{(k)} \nu_k \right]^2 = \eta_k^2 = D_{k,\omega}(P_k),\]
that is, the left-hand side of inequality (3.8) becomes the equality for \( G = \text{const} \cdot P_k \).

Now we show the required upper bound. To this end we observe first that the
Jacobi polynomials \( P_n(x) \) satisfy the differential equations
\[(3.11)\quad (1 - x^2)P_n^{(k+1)}(x) + (\beta - \alpha - (\alpha + \beta + 2k)x)P_n^{(k)}(x) + (\alpha + \beta + n + k)(n - k + 1)P_n^{(k-1)}(x) = 0, \quad k \in \mathbb{N}.\]
It is well known (see [5], Chapter VII, § 3) that every Jacobi polynomial \( P_n(x) \) satisfies the differential equation of the second order
\[(3.12)\quad (1 - x^2)P_n''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)P_n'(x) + n(\alpha + \beta + n + 1)P_n(x) = 0.\]
Also we have
\[
D^{k-1}[(1 - x^2)P_n''(x)] = (1 - x^2)P_n^{(k+1)}(x) - 2(k-1)xP_n^{(k)}(x) - (k-1)(k-2)P_n^{(k-1)}(x),
\]
\[
D^{k-1}[(\beta - \alpha - (\alpha + \beta + 2)x)P_n'(x)] = (\beta - \alpha - (\alpha + \beta + 2)x)P_n^{(k)}(x) - (\alpha + \beta + 2)(k-1)P_n^{(k-1)}(x).
\]
Differentiating (3.12) \( k - 1 \) times and applying these equations, we obtain (3.11). For any natural integer \( k \) we define the differential operation
\[(3.13)\quad L_k y := (-1)^k D^k[(1 - x)^{\alpha+k}(1 + x)^{\beta+k} D^k y].\]
Denote \( \tilde{\nu}_k(x) := (1 - x)^{\alpha+k}(1 + x)^{\beta+k}. \) Let us show that for \( n \geq k \geq 1 \), the equalities
\[(3.14)\quad L_1 P_n(x) = n(\alpha + \beta + n + 1)P_n(x)\omega(x),
L_2 P_n(x) = (\alpha + \beta + n + 1)(\alpha + \beta + n + 1)n(n-1)P_n(x)\omega(x),
\]
\[(3.15)\quad L_n P_n(x) = (\alpha + \beta + 2n) \ldots (\alpha + \beta + n + 1)n!P_n(x)\omega(x)
\]
hold. It is easy to see that (3.14) follows from (3.12). For the proof of the other equalities it is sufficient to establish the recurrent formula
\[(3.16)\quad L_{k+1} P_n = (\alpha + \beta + n + k + 1)(n-k)L_k P_n.\]
Write
\[(3.17)\quad L_{k+1} P_n(x) = (-1)^{k+1} D^k \cdot D \left[ \tilde{\nu}_k(x)(1 - x^2)P_n^{(k+1)}(x) \right].\]
We have
\[(3.18)\quad D\tilde{\nu}_k(x) = \tilde{\nu}_k(x)(1 - x^2)^{-1}(\beta - \alpha - (\alpha + \beta + 2k)x),\]
and also it follows from (3.11) that
\[
D \left[ (1 - x^2)P_n^{(k+1)}(x) \right] = -(\alpha + \beta + n + k + 1)(n-k)P_n^{(k)}(x)
- (\beta - \alpha - (\alpha + \beta + 2k)x)P_n^{(k+1)}(x).
\]
From this, (3.17) and (3.18) follow (3.16) and (3.15). Note that if the equalities
(3.14) and (3.15) are rewritten in the form
\[(3.19)\quad L_k P_n = \sigma_{k,n} P_n\omega, \quad n \geq k \geq 1,\]
then
\[
\sigma_{k,n} \geq \sigma_{k,k} = (\alpha + \beta + 2k) \ldots (\alpha + \beta + k + 1)k!
\]

Now we show the upper estimate (3.8). Note that it trivially holds for a polynomial of degree \( < k \), and for polynomials \( y \in \mathbf{P} \) of degree \( \geq k \) it is equivalent to the inequality
\[
\gamma_k \int_{\Delta} |y^{(k)}|^2 \nu_k \geq \int_{\Delta} y^2 \omega,
\]
provided
\[
(y, p_n)_H = 0, \quad n = 0, 1, \ldots, k - 1.
\]
It follows from (3.22) that there exists \( n_y \in \mathbb{N} \) such that
\[
y(x) = \sum_{n=k}^{n_y} (y, p_n)_H p_n(x).
\]

Hence,
\[
\gamma_k \int_{\Delta} |y^{(k)}|^2 \nu_k = \frac{\gamma_k}{\beta_k} \int_{\Delta} yL_k y = \frac{\gamma_k}{\beta_k} \int_{\Delta} \left[ \sum_{n=k}^{n_y} (y, p_n)_H p_n \right] \left[ \sum_{n=k}^{n_y} (y, p_n)_H L_k p_n \right]
\]
(because of (3.19))
\[
= \frac{\gamma_k}{\beta_k} \sum_{n=k}^{n_y} (y, p_n)_H^2 \sigma_{k,n} \int_{\Delta} \tilde{p}_n^2 \omega
\]
(applying (3.20) and Parseval’s equality)
\[
\geq \frac{\gamma_k}{\beta_k} \sigma_{k,k} \sum_{n=k}^{n_y} (y, p_n)^2 = \frac{\gamma_k}{\beta_k} (\alpha + \beta + 2k) \ldots (\alpha + \beta + k + 1)k! \int_{\Delta} y^2 \omega
\]
(inserting the values of \( \gamma_k \) and \( \beta_k \))
\[
= \int_{\Delta} y^2 \omega,
\]
and (3.21) is proved.

It is well known that Jacobi’s polynomials \( \{p_n\}, \quad n \in \mathbb{N}_0 \), form the complete orthonormal system in \( H \). Therefore, \( \mathbf{P} \) is dense in \( H \) and, since \( \tilde{\nu}_k(x) \) has the form (1.3), then \( \mathbf{P} \) is dense in \( L^2(\nu_k) \). Let \( G \in AC^{(k)}(\Delta) \) be such that \( G^{(k)} \in L^2(\nu_k) \). For any \( \epsilon > 0 \) there exists such a polynomial \( h_\epsilon \in \mathbf{P} \) that
\[
\|G^{(k)} - h_\epsilon\|_{L^2(\nu_k)} \leq \epsilon.
\]
Let \( g_\epsilon := G^{(k)} - h_\epsilon, \quad g := G^{(k)} = h_\epsilon + g_\epsilon \). By Schwarz’s inequality we find
\[
\int_{\Delta} |R_k g_\epsilon|^2 \omega \leq \int_{\Delta} \omega(x) dx \int_{\Delta} |r_k(x, s)| ds \int_{\Delta} |r_k(x, t)||g_\epsilon(t)|^2 dt
\]
\[
\leq c_{10} \int_{\Delta} |g_\epsilon(t)|^2 dt \int_{\Delta} |r_k(x, t)||\omega(x)| dx \leq c_{11} \int_{\Delta} |g_\epsilon|^2 \nu_k \leq c_{11} \epsilon^2.
\]
Put
\[
y_\epsilon := R_k h_\epsilon - \sum_{j=0}^{k-1} (R_k h_\epsilon, p_j)_H p_j.
\]
Then \( y_\varepsilon \in P \) and satisfies the condition (3.22). Using (2.2) we find
\[
G - \sum_{j=0}^{k-1} (G, p_j) H p_j = R_k g - \sum_{j=0}^{k-1} (R_k g, p_j) H p_j = y_\varepsilon + R_k g_\varepsilon - \sum_{j=0}^{k-1} (R_k g_\varepsilon, p_j) H p_j.
\]
Hence,
\[
\left[ D_{k,\omega}(G) \right]^{1/2} \leq \| y_\varepsilon \|_H + \| R_k g_\varepsilon \|_H
\]
then (3.21) and (3.23) imply
\[
\leq \gamma_k^{1/2} \| y_\varepsilon \|^{(k)}_{L^2(\nu_k)} + c_{12} \cdot \varepsilon = \gamma_k^{1/2} \| G^{(k)} - g_\varepsilon \|_{L^2(\nu_k)} + c_{12} \cdot \varepsilon
\]
and the right-hand side of (3.8) follows by \( \varepsilon \to 0 \). If \( G = P_k \), then, integrating by parts, we find
\[
\gamma_k \int_\Delta |P^{(k)}_k|^2 \nu_k = \frac{\gamma_k}{\beta_k} \int_\Delta P_k L_k P_k
\]
\[
= \frac{\gamma_k}{\beta_k} (\alpha + \beta + 2k) \ldots (\alpha + \beta + k + 1)! \int_\Delta P^2_k \omega = D_{k,\omega}(P_k). \]

REFERENCES


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