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ON THE SUM OF THE INDEX OF A PARABOLIC SUBALGEBRA AND OF ITS NILPOTENT RADICAL

RUPERT W. T. YU

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Abstract. In this short note, we investigate the following question of Panyushev stated in 2003: "Is the sum of the index of a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}$ and the index of its nilpotent radical always greater than or equal to the rank of $\mathfrak{g}$?" Using the formula for the index of parabolic subalgebras conjectured by Tauvel and the author and proved by Fauquant-Millet and Joseph in 2005 and Joseph in 2006, we give a positive answer to this question. Moreover, we also obtain a necessary and sufficient condition for this sum to be equal to the rank of $\mathfrak{g}$. This provides new examples of direct sum decomposition of a semisimple Lie algebra verifying the "index additivity condition" as stated by Raïs.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\mathbb{k}$ of characteristic zero. For $f \in \mathfrak{g}^*$, we denote by $\mathfrak{g}^f = \{X \in \mathfrak{g}; f([X,Y]) = 0 \text{ for all } Y \in \mathfrak{g}\}$, the annihilator of f for the coadjoint representation of $\mathfrak{g}$. The *index* of $\mathfrak{g}$, denoted by $\chi(\mathfrak{g})$, is defined to be

$$\chi(\mathfrak{g}) = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}^f.$$

It is well known that if $\mathfrak{g}$ is an algebraic Lie algebra and G denotes its algebraic adjoint group, then $\chi(\mathfrak{g})$ is the transcendence degree of the field of G-invariant rational functions on $\mathfrak{g}^*$.

The index of a semisimple Lie algebra $\mathfrak{g}$ is equal to the rank of $\mathfrak{g}$. This can be obtained easily from the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form. There has been quite a lot of recent work on the determination of the index of certain subalgebras of a semisimple Lie algebra: parabolic subalgebras and related subalgebras ([2], [9], [13], [8]), centralizers of elements and related subalgebras ([10], [1], [15], [7]).

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{u}$ (resp. $\mathfrak{l}$) the nilpotent radical (resp. a Levi factor) of $\mathfrak{p}$. In [10, Corollary 1.5 (i)], Panyushev showed that

$$\chi(\mathfrak{p}) + \chi(\mathfrak{u}) \leq \dim \mathfrak{l}. \tag{1}$$

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He then suggested [10, Remark (ii) of Section 6] that

$$\chi(\mathfrak{p}) + \chi(\mathfrak{u}) \geq \operatorname{rk} \mathfrak{g}. \tag{2}$$

For example, it is well known that if $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ is its nilpotent radical, then $\chi(\mathfrak{b}) + \chi(\mathfrak{n}) = \operatorname{rk} \mathfrak{g}$ (see for example [12], [14, Chapter 40]). It is therefore also interesting to characterise parabolic subalgebras where equality holds in (2). Indeed, in [11], Raïs looked for examples of direct sum decompositions $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}$ verifying the "index additivity condition", namely $\mathfrak{m}$ and $\mathfrak{n}$ are Lie subalgebras of $\mathfrak{g}$ and

$$\chi(\mathfrak{g}) = \chi(\mathfrak{m}) + \chi(\mathfrak{n}).$$

If $\mathfrak{u}_-$ denotes the nilpotent radical of the opposite parabolic subalgebra $\mathfrak{p}_-$ of $\mathfrak{p}$, then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}_-$ and the Lie algebras $\mathfrak{u}$ and $\mathfrak{u}_-$ are isomorphic. Thus parabolic subalgebras such that equality holds in (2) would provide examples of direct sum decompositions verifying the index additivity condition.

Using the formula conjectured in [13] and proved in [3, 6] for the index of parabolic subalgebras, we obtain a formula for the sum $\chi(\mathfrak{p}) + \chi(\mathfrak{u})$. By a careful analysis of root systems, we prove inequality (2) and give a necessary and sufficient condition for the equality to hold in (2) (see Theorem 2.2).

To describe the index of a parabolic subalgebra and the index of its nilpotent radical, we need to recall Kostant's cascade construction of pairwise strongly orthogonal roots ([4], [5], [14]).

Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$. Denote by R, R^+ and $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ respectively the set of roots, positive roots and simple roots with respect to $\mathfrak{h}$ and $\mathfrak{b}$. For any $\alpha \in R$, let $\mathfrak{g}_\alpha$ be the root subspace associated to α. Choose $X_\alpha \in \mathfrak{g}_\alpha$ such that $\alpha([X_\alpha, X_{-\alpha}]) = 2$. We shall write $\alpha^\vee = [X_\alpha, X_{-\alpha}] \in \mathfrak{h}$, and for $\lambda \in \mathfrak{h}^*$, $\langle \lambda, \alpha^\vee \rangle = \lambda(\alpha^\vee)$. For $S \subset \Pi$, we denote by $R_S = R \cap \mathbb{Z}S$, $R_S^+ = R_S \cap R^+$. If S is connected, then we shall denote by ε_S the highest root of R_S.

Let $S \subset \Pi$. We define inductively a set $\mathcal{K}(S)$ whose elements are subsets of Π as follows:

a) $\mathcal{K}(\emptyset) = \emptyset$.
b) If $S_1, \ldots, S_r$ are the connected components of S, then $\mathcal{K}(S) = \mathcal{K}(S_1) \cup \cdots \cup \mathcal{K}(S_r)$.
c) If S is connected, then $\mathcal{K}(S) = \{S\} \cup \mathcal{K}(\widehat{S})$ where $\widehat{S} = \{\alpha \in S; \langle \alpha, \varepsilon_S^\vee \rangle = 0\}$.

It is well known that (see for example [14, Chapter 40]) elements of $\mathcal{K}(S)$ are connected subsets of S. Moreover, if we denote by $\mathcal{R}(S) = \{\varepsilon_K; K \in \mathcal{K}(S)\}$, then $\mathcal{R}(S)$ is a maximal set of pairwise strongly orthogonal roots in R_S.

Let us also recall the following properties of $\mathcal{K}(S)$ which are easy consequences from the definition (see for example [14, Chapter 40]):

Lemma 1.1. *Let S be a subset of Π, $K, K' \in \mathcal{K}(S)$ and set*

$$\begin{aligned} \Gamma^K &= \{\alpha \in R_K; \langle \alpha, \varepsilon_K^\vee \rangle > 0\} \\ &= \{\alpha = \textstyle\sum_{\beta \in K} n_\beta \beta \in R_K^+; n_\beta > 0 \text{ for some } \beta \in K \setminus \widehat{K}\}. \end{aligned}$$

i) *We have either $K \subset K'$ or $K' \subset K$ or K and K' are connected components of $K \cup K'$.*
ii) $\Gamma^K = R_K^+ \setminus \{\beta \in R_K^+; \langle \beta, \varepsilon_K^\vee \rangle = 0\}$. *In particular, R_S^+ is the disjoint union of the Γ^K's, $K \in \mathcal{K}(S)$.*

Table 1

Type	$A_\ell, \ell \geq 1$	$B_\ell, C_\ell, \ell \geq 2$	$D_\ell, \ell \geq 4$	E_6	E_7	E_8	F_4	G_2
$\sharp\mathcal{K}(\Pi)$	$\left[\frac{\ell+1}{2}\right]$	ℓ	$2\left[\frac{\ell}{2}\right]$	4	7	8	4	2

iii) $\sum_{\alpha\in\Gamma^K}\mathfrak{g}_\alpha$ *is a Heisenberg Lie algebra whose centre is* $\mathfrak{g}_{\varepsilon_K}$. *Thus if* $\alpha,\beta\in\Gamma^K$ *verify* $\alpha+\beta\in R$, *then* $\alpha+\beta=\varepsilon_K$.

iv) *Suppose that* $\alpha\in\Gamma^K$ *and* $\beta\in\Gamma^{K'}$ *verify* $\alpha+\beta\in R$, *then either* $K\subset K'$ *and* $\alpha+\beta\in\Gamma^{K'}$ *or* $K'\subset K$ *and* $\alpha+\beta\in\Gamma^K$.

The cardinality of $\mathcal{K}(\Pi)$ is listed in Table 1 for an irreducible root system R, where for any $x\in\mathbb{Q}$, $[x]$ is a unique integer such that $[x]\leq x<[x]+1$.

Examples 1.2. Let R be an irreducible root system. We shall use the numbering of simple roots in [14, Chapter 18]. Set $k=\sharp\mathcal{K}(\Pi)$.

1) Let R be of type A_ℓ. Then $\mathcal{K}(\Pi)=\bigcup_{i=1}^k\{K_i\}$ where $K_i=\{\alpha_i,\dots,\alpha_{\ell+1-i}\}$. For $1\leq i\leq k$,

$$\Gamma^{K_i}=\{\alpha_i+\cdots+\alpha_{i+r},\ \alpha_{\ell+1-i-r}+\cdots+\alpha_{\ell+1-i};0\leq r\leq\ell-2i\}\cup\{\varepsilon_{K_i}\}.$$

2) Let R be of type D_{2n+1}. Then $k=2n$,

$$\mathcal{K}(\Pi)=\bigcup_{i=1}^n\{K_i\}\cup\{L_i\},$$

where $K_i=\{\alpha_{2i-1},\dots,\alpha_{2n+1}\}$ and $L_i=\{\alpha_{2i-1}\}$. For $1\leq i\leq n$,

$$\Gamma^{K_i}=\left\{\sum_{j=2i-1}^{\ell}m_j\alpha_j;m_{2i}\neq 0\right\},\qquad \Gamma^{L_i}=\{\alpha_{2i-1}\}.$$

3) Let R be of type E_6. Then

$$\mathcal{K}(\Pi)=\{\Pi\}\cup\{\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}\}\cup\{\{\alpha_3,\alpha_4,\alpha_5\}\}\cup\{\{\alpha_4\}\}.$$

2. Main result

Recall that for any subset $S\subset\Pi$,

$$\mathfrak{p}_S=\mathfrak{h}\oplus\bigoplus_{\alpha\in R_S\cup R^+}\mathfrak{g}_\alpha$$

is a (standard) parabolic subalgebra of $\mathfrak{g}$. Any parabolic subalgebra of $\mathfrak{g}$ is conjugated to a standard parabolic subalgebra. The Lie subalgebra

$$\mathfrak{u}_S=\bigoplus_{\alpha\in R^+\setminus R_S}\mathfrak{g}_\alpha$$

is the nilpotent radical of $\mathfrak{p}_S$.

For $S\subset\Pi$, denote by V_S the vector subspace of $\mathfrak{h}^*$ spanned by the elements of $\mathcal{R}(S)$ and $\mathcal{R}(\Pi)$. Set

$$\mathcal{E}_S=\{K\in\mathcal{K}(\Pi);X_{\varepsilon_K}\in\mathfrak{u}_S\}=\{K\in\mathcal{K}(\Pi);\varepsilon_K\notin R_S\}$$
$$\text{and}\quad \mathcal{Q}_S=\left(\bigcup_{K\in\mathcal{E}_S}\Gamma^K\right)\cap R_S^+.$$

The subset $\mathcal{E}_S$ has the following simple characterisation.

Lemma 2.1. *Let T_S be the union of the subsets $K \in \mathcal{K}(\Pi)$ verifying $K \subset S$. Then $\mathcal{E}_S = \mathcal{K}(\Pi) \setminus \mathcal{K}(T_S)$.*

Proof. This is straightforward. □

Our main result is the following theorem.

Theorem 2.2. *Let $S \subset \Pi$. Then*

$$\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(S) - \sharp\mathcal{K}(T_S) + 2(\sharp\mathcal{K}(\Pi) - \dim V_S) + \sharp\mathcal{Q}_S. \tag{3}$$

We have $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g}$, and the equality holds if and only if the following conditions are satisfied:

i) $\sharp(\mathcal{K}(S) \cup \mathcal{K}(\Pi)) = \dim V_S$.
ii) *For any connected component S' of S, we have either $S' \in \mathcal{K}(\Pi)$ or $\sharp(S' \setminus T_S) = 1$.*

Proof. We may clearly assume that $\mathfrak{g}$ is simple.

The formula for the sum $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S)$ is a direct consequence of the formula of the index of parabolic subalgebras conjectured in [13] and proved in [3, 6]:

$$\chi(\mathfrak{p}_S) = \operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(\Pi) + \sharp\mathcal{K}(S) - 2\dim V_S, \tag{4}$$

and the formula for the index of $\mathfrak{u}_S$ (see for example [14, Chapter 40]), which, in view of Lemma 1.1, can be expressed in the following way:

$$\chi(\mathfrak{u}_S) = \sharp\mathcal{E}_S + \sum_{K \in \mathcal{E}_S} \sharp\Gamma^K - \dim \mathfrak{u}_S = \sharp\mathcal{E}_S + \sharp\mathcal{Q}_S. \tag{5}$$

Observe that

$$\sharp\mathcal{K}(S) - \sharp\mathcal{K}(T_S) \geq 0. \tag{6}$$

1) Let $S_1, \ldots, S_r$ be the connected components of S. For each i, there is a unique $K_i \in \mathcal{K}(\Pi)$ (see Lemma 1.1) such that $\varepsilon_{S_i} \in \Gamma^{K_i}$. If $K_i = S_i$, then S_i is a connected component of T_S. Otherwise $K_i \in \mathcal{E}_S$, and we have

$$\varepsilon_{S_i} \in \Gamma^{K_i} \cap \Gamma^{S_i} \subset \mathcal{Q}_S.$$

2) It follows from Point 1) that $\mathcal{Q}_S = \emptyset$ if and only if $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$ (or equivalently $S = T_S$).

3) Note that the connected components of T_S are the connected components of $T_S \cap S_i$. It follows again from Point 1) that

$$\begin{aligned} \sharp\mathcal{K}(S) - \sharp\mathcal{K}(T_S) &= \sum_{i=1}^{r} (\sharp\mathcal{K}(S_i) - \sharp\mathcal{K}(T_S \cap S_i)) \\ &= \sum_{K_i \in \mathcal{E}_S} (\sharp\mathcal{K}(S_i) - \sharp\mathcal{K}(T_S \cap S_i)). \end{aligned}$$

4) From Table 1, we have $\dim V_S = \sharp\mathcal{K}(\Pi) = \operatorname{rk}\mathfrak{g}$ in the cases where $\mathfrak{g}$ is of type B_ℓ, C_ℓ, D_{2n}, E_7, E_8, F_4 and G_2. The inequality follows immediately from (3) and (6), and by Point 2) the equality holds if and only if $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$.

On the other hand, since $\dim V_S = \sharp\mathcal{K}(\Pi)$ in these cases, condition i) is equivalent to $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$. Finally, if condition i) is verified, then $S = T_S$, and condition ii) is automatically verified. So we have the result in these cases.

5) Type A_ℓ.

For any i verifying $S_i \neq K_i$, by Point 1), Lemma 1.1 and Examples 1.2, half of $\Gamma^{S_i} \setminus \{\varepsilon_{S_i}\}$ belongs to $\mathcal{Q}_S$. Since $\sharp(\Gamma^{S_i}) = 2\sharp(S_i) - 1$ (Examples 1.2), such an S_i contributes $\sharp(S_i)$ elements of $\mathcal{Q}_S$.

Again, since we are in type A_ℓ, $\mathcal{K}(\Pi)$ is totally ordered by inclusion. It follows that T_S is connected. Without loss of generality, we may assume that $T_S \subset S_1$.

Suppose that $S_1 = T_S$. Then from the previous discussion, we deduce that

$$\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(S \setminus S_1) + 2(\sharp\mathcal{K}(\Pi) - \dim V_S) + \sharp(S \setminus S_1).$$

But our hypothesis implies that

$$\dim V_S \leq \sharp\mathcal{K}(\Pi) + \sharp\mathcal{K}(S \setminus S_1) = \sharp(\mathcal{K}(\Pi) \cup \mathcal{K}(S)), \tag{7}$$

so

$$\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g} + \sharp(S \setminus S_1) - \sharp\mathcal{K}(S \setminus S_1).$$

Hence $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g}$. For the equality to hold, we must have the equality in (7) and

$$\sharp(S \setminus S_1) = \sharp\mathcal{K}(S \setminus S_1).$$

This latter is only possible if $\sharp(S_i) = 1$ for $i \geq 2$, so we have conditions (i) and (ii). Conversely, suppose that conditions (i) and (ii) are verified; then $\sharp(S_i) = 1$ for $i \geq 2$. Consequently $\mathcal{Q}_S = S \setminus S_i$ by Point 1) and the definition of $\mathcal{Q}_S$.

Suppose that $S_1 \supsetneq T_S$ (this includes the case $T_S = \emptyset$). Then $\mathcal{K}(S) \cap \mathcal{K}(\Pi) = \emptyset$. Thus

$$\dim V_S \leq \sharp\mathcal{K}(\Pi) + \sharp\mathcal{K}(S) = \sharp(\mathcal{K}(\Pi) \cup \mathcal{K}(S)). \tag{8}$$

We deduce from Point 1) and the remark in the first paragraph of Point 5) that

$$\begin{aligned} \chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) &\geq \operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(S) - \sharp\mathcal{K}(T_S) + 2(\sharp\mathcal{K}(\Pi) - \dim V_S) + \sharp(S) \\ &\geq \operatorname{rk}\mathfrak{g} + \sharp(S) - \sharp\mathcal{K}(S) - \sharp\mathcal{K}(T_S). \end{aligned}$$

Hence

$$\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g} + \sum_{i=1}^{r} \left(\sharp(S_i) - \left[\frac{\sharp(S_i) + 1}{2} \right] \right) - \left[\frac{\sharp(T_S) + 1}{2} \right].$$

Since $T_S \subsetneq S_1$, we deduce from Table 1 that

$$\sharp(S_1) - \left[\frac{\sharp(S_1) + 1}{2} \right] - \left[\frac{\sharp(T_S) + 1}{2} \right] \geq 0.$$

So we have our inequality $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) \geq \operatorname{rk}\mathfrak{g}$.

Now for the equality $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$ to hold, we must have the equality in (8) and $\sharp\mathcal{Q}_S = \sharp(S)$,

$$\sharp(S_1) - \left[\frac{\sharp(S_1) + 1}{2} \right] - \left[\frac{\sharp(T_S) + 1}{2} \right] = 0 \text{ and } \sharp(S_i) - \left[\frac{\sharp(S_i) + 1}{2} \right] = 0$$

for $i \geq 2$. This implies that $\sharp(S_1) = \sharp(T_S) + 1$, and $\sharp(S_i) = 1$ for $i \geq 2$. So we have conditions (i) and (ii). Conversely, if conditions (i) and (ii) are verified, then $\sharp(S_i) = 1$ for $i \geq 2$, and $\sharp(S_1) = \sharp(T_S) + 1$. In view of the above, to show that $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$, it suffices to check that $\sharp(\mathcal{Q}_S \cap R^+_{S_1}) = \sharp(S_1)$, which is a straightforward verification.

6) Type D_{2n+1}.

In this case, $\sharp\mathcal{K}(\Pi) = \operatorname{rk}\mathfrak{g} - 1$. Let us use the numbering of simple roots in [14, Chapter 18]. We check easily that $\alpha_1, \ldots, \alpha_{\ell-2} \in V_\Pi$.

TABLE 2

S_1	$\dim V_{S_1}$	$T_S \cap S_1$	$\sharp\mathcal{K}(S_1)$	$\sharp\mathcal{K}(T_S \cap S_1)$	K_1	$\sharp(\Gamma^{K_1} \cap \Gamma^{S_1})$
A_1	5	$\emptyset$	1	0	A_3 or A_5	1
A_2	5	$\emptyset$	1	0	A_5	2
A_2	5	A_1	1	1	A_3	2
A_3	6	A_1	2	1	A_5	3
A_3	5	A_1	2	1	E_6	3
A_4	6	A_1	2	1	E_6	4
A_4	6	A_3	2	2	A_5	4
D_4	5	A_3	4	2	E_6	4
D_5	6	A_3	4	2	E_6	9

If $\dim V_S = \sharp\mathcal{K}(\Pi)$, then the inequality follows from (3) and (6), and the condition for equality follows from Point 2).

Suppose now that $\dim V_S = \operatorname{rk}\mathfrak{g}$ and $\alpha_{\ell-1} \in S$ (the case $\alpha_\ell \in S$ being analogous). Then

$$\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(S) - \sharp\mathcal{K}(T_S) - 2 + \sharp(\mathcal{Q}_S), \tag{9}$$

and the connected component S_1 of S containing $\alpha_{\ell-1}$ is not in $\mathcal{K}(\Pi)$; otherwise, we would have $\dim V_S = \sharp\mathcal{K}(\Pi)$.

By Point 1), $\varepsilon_{S_1} \in \mathcal{Q}_S$. By examining the possibilities for S_1 and K_1 (Examples 1.2), we verify that

$$\sharp\mathcal{K}(S_1) - \sharp\mathcal{K}(T_S \cap S_1) + \sharp(\Gamma^{K_1} \cap \Gamma^{S_1}) \geq 2 \tag{10}$$

with equality if and only if S_1 is of type A_1 or A_2. Therefore, we have obtained the inequality.

In fact, we showed in the previous paragraph that already we have

$$\operatorname{rk}\mathfrak{g} + \sharp\mathcal{K}(S_1) - \sharp\mathcal{K}(T_S \cap S_1) - 2 + \sharp(\Gamma^{K_1} \cap \Gamma^{S_1}) \geq \operatorname{rk}\mathfrak{g}.$$

So if $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$, then from (9) and the above inequality, we must have $\mathcal{K}(S \setminus S_1) \subset \mathcal{K}(\Pi)$, and also the equality in (10). Hence conditions (i) and (ii). Conversely, suppose that conditions (i) and (ii) are verified; then the fact that $\alpha_1, \ldots, \alpha_{\ell-2} \in V_\Pi$ implies that $\mathcal{K}(S \setminus S_1) \subset \mathcal{K}(\Pi)$ and $\sharp\mathcal{K}(S_1) = 1$. Hence S_1 is of type A_1, A_2. It is then easy to check that $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$.

7) Type E_6.

Here, we have $\sharp\mathcal{K}(\Pi) = 4$ and $\alpha_2, \alpha_4 \in V_\Pi$. Let S_1 be a connected component of S such that $\dim V_{S_1} > 4$. Under these conditions, the possibilities are shown in Table 2. Thus,

$$\sharp\mathcal{K}(S_1) - \sharp\mathcal{K}(T_S \cap S_1) + \sharp(\Gamma^{S_1} \cap \Gamma^{K_1}) \geq 2(\dim V_S - \sharp\mathcal{K}(\Pi)).$$

A direct verification gives the result. Note that as in the case of type A_ℓ, $\mathcal{K}(\Pi)$ is totally ordered by inclusion, so T_S is connected. □

Remark 2.3. Theorem 2.2 says that if $\mathcal{K}(S) \subset \mathcal{K}(\Pi)$ or, equivalently, $S' \in \mathcal{K}(\Pi)$ for any connected component S' of S, then $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$.

Table 3

Type	$\mathcal{K}(S) \not\subset \mathcal{K}(\Pi)$	$\mathcal{K}(S) \subset \mathcal{K}(\Pi)$
A_ℓ	$i \neq \dfrac{\ell+1}{2}$	$i = \dfrac{\ell+1}{2}$
B_ℓ	none	i odd
C_ℓ	none	$i = \ell$
D_{2n+1}	$i = 2n, 2n+1$	$i < 2n$ odd
D_{2n}	none	i odd or $i = 2n$
E_6	$i \neq 2, 4$	$i = 4$
E_7	none	$i = 2, 3, 5, 7$
E_8	none	$i = 2, 3, 5, 7$
F_4	none	$i = 2$
G_2	none	$i = 1$

Example 2.4. Let us consider the case of minimal parabolic subalgebras. So $S = \{\alpha\}$ and it follows that

$$\chi(\mathfrak{p}) + \chi(\mathfrak{u}) = \begin{cases} \operatorname{rk}\mathfrak{g} & \text{if } \{\alpha\} \in \mathcal{K}(\Pi), \\ \operatorname{rk}\mathfrak{g} & \text{if } \{\alpha\} \notin \mathcal{K}(\Pi) \text{ and } \dim V_S = \sharp\mathcal{K}(\Pi) + 1, \\ \operatorname{rk}\mathfrak{g} + 2 & \text{if } \{\alpha\} \notin \mathcal{K}(\Pi) \text{ and } \dim V_S = \sharp\mathcal{K}(\Pi). \end{cases}$$

Thus the minimal parabolic subalgebras $\mathfrak{p}_S$ verifying $\chi(\mathfrak{p}_S) + \chi(\mathfrak{u}_S) = \operatorname{rk}\mathfrak{g}$ are (in the simple roots numbering of [14, Chapter 18]) as shown in Table 3.

Example 2.5. In the other extreme, it is easy to check that maximal parabolic subalgebras of $\mathfrak{g} = \mathrm{sl}_{\ell+1}$ verifying $\chi(\mathfrak{p}) + \chi(\mathfrak{u}) = \operatorname{rk}\mathfrak{g}$ are exactly the ones associated to simple roots at the extremities of the Dynkin diagram.

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UMR 6086 du C.N.R.S., Département de Mathématiques, Université de Poitiers, Téléport 2 – BP 30179, Boulevard Marie et Pierre Curie, 86962 Futuroscope Chasseneuil Cedex, France

E-mail address: yuyu@math.univ-poitiers.fr