

## ON THE NORMAL BUNDLE OF SUBMANIFOLDS OF $\mathbb{P}^n$

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ABSTRACT. Let  $X$  be a submanifold of dimension  $d \geq 2$  of the complex projective space  $\mathbb{P}^n$ . We prove results of the following type. i) If  $X$  is irregular and  $n = 2d$ , then the normal bundle  $N_{X|\mathbb{P}^n}$  is indecomposable. ii) If  $X$  is irregular,  $d \geq 3$  and  $n = 2d + 1$ , then  $N_{X|\mathbb{P}^n}$  is not the direct sum of two vector bundles of rank  $\geq 2$ . iii) If  $d \geq 3$ ,  $n = 2d - 1$  and  $N_{X|\mathbb{P}^n}$  is decomposable, then the natural restriction map  $\text{Pic}(\mathbb{P}^n) \rightarrow \text{Pic}(X)$  is an isomorphism (and, in particular, if  $X = \mathbb{P}^{d-1} \times \mathbb{P}^1$  is embedded Segre in  $\mathbb{P}^{2d-1}$ , then  $N_{X|\mathbb{P}^{2d-1}}$  is indecomposable). iv) Let  $n \leq 2d$  and  $d \geq 3$ , and assume that  $N_{X|\mathbb{P}^n}$  is a direct sum of line bundles; if  $n = 2d$  assume furthermore that  $X$  is simply connected and  $\mathcal{O}_X(1)$  is not divisible in  $\text{Pic}(X)$ . Then  $X$  is a complete intersection. These results follow from Theorem 2.1 below together with Le Potier's vanishing theorem. The last statement also uses a criterion of Faltings for complete intersection. In the case when  $n < 2d$  this fact was proved by M. Schneider in 1990 in a completely different way.

### INTRODUCTION

It is well known that if  $X$  is a submanifold of the complex projective space  $\mathbb{P}^n$  ( $n \geq 3$ ) of dimension  $d > \frac{n}{2}$ , then a topological result of Lefschetz type, due to Barth and Larsen (see [3], [20]), asserts that the canonical restriction maps

$$H^i(\mathbb{P}^n, \mathbb{Z}) \rightarrow H^i(X, \mathbb{Z})$$

are isomorphisms for  $i \leq 2d - n$ , and injective with torsion-free cokernel for  $i = 2d - n + 1$ . As a consequence, the restriction map

$$\text{Pic}(\mathbb{P}^n) \rightarrow \text{Pic}(X)$$

is an isomorphism if  $d \geq \frac{n+2}{2}$ , and injective with torsion-free cokernel if  $n = 2d - 1$ . In particular, if  $d > \frac{n}{2}$ , then the class of  $\mathcal{O}_X(1)$  is not divisible in  $\text{Pic}(X)$ .

In this paper, in the spirit of the Barth-Larsen theorem, we are going to say something about the normal bundle  $N_{X|\mathbb{P}^n}$  of submanifolds  $X$  of dimension  $d \geq 3$  of  $\mathbb{P}^n$ . Specifically, we shall prove that if  $X$  is a submanifold of dimension  $d \geq 3$  of  $\mathbb{P}^{2d-1}$  whose normal bundle  $N_{X|\mathbb{P}^{2d-1}}$  is decomposable, then the restriction map  $\text{Pic}(\mathbb{P}^{2d-1}) \rightarrow \text{Pic}(X)$  is an isomorphism (see Theorem 3.2, (1) below). In particular, the normal bundle of the image of the Segre embedding  $\mathbb{P}^{d-1} \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{2d-1}$  is indecomposable for every  $d \geq 3$ . This result suggests that the decomposability

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of the normal bundle of a given submanifold  $X$  of  $\mathbb{P}^n$  of dimension  $d \geq 3$  should yield strong geometrical constraints on  $X$ . For illustration, see Theorem 3.2 and its corollaries. For example, Theorem 3.2, (3) asserts that every submanifold of  $\mathbb{P}^n$  of dimension  $d \geq 3$ , whose normal bundle is a direct sum of line bundles, is regular and has  $\text{Num}(X)$  isomorphic to  $\mathbb{Z}$  (here  $\text{Num}(X) := \text{Pic}(X)/\text{numerical equivalence}$ ); moreover, if either  $2d > n$ , or if  $n = 2d$ ,  $X$  is simply connected and  $\mathcal{O}_X(1)$  is not divisible in  $\text{Pic}(X)$ , then  $X$  is a complete intersection (if  $d > \frac{n}{2}$  this result was first proved, in a different way, by M. Schneider in [24]). Another result (Theorem 3.1) asserts the following: (1) the normal bundle of any irregular submanifold of dimension  $d \geq 2$  in  $\mathbb{P}^{2d}$  is indecomposable; (2) the normal bundle of any irregular submanifold of dimension  $d \geq 3$  in  $\mathbb{P}^{2d+1}$  is not the direct sum of two vector bundles of rank  $\geq 2$ .

Although these kinds of results seem to be completely new, the idea behind their proofs is surprisingly simple. Our basic technical result (Theorem 2.1) asserts that for every submanifold  $X \subset \mathbb{P}^n$  of dimension  $d \geq 2$ , the irregularity of  $X$  is equal to  $h^1(N_{X|\mathbb{P}^n}^\vee)$ , and the rank of the Néron-Severi group of  $X$  is  $\leq 1 + h^2(N_{X|\mathbb{P}^n}^\vee)$ . This theorem and a systematic use of Le Potier's vanishing theorem yield the proofs of most of the results of this paper. Certain applications of Theorem 2.1 will also make use of a criterion of Faltings [10] for complete intersection.

The general philosophy according to which there is a close relationship between topological Barth-Lefschetz theorems (see [3], [17]) and vanishing results involving the conormal bundle of the variety in question is not new. For instance, Faltings showed in [11] that for any  $d$ -fold  $X$  in  $\mathbb{P}^n$  the following implication holds:

$$\begin{aligned} H^q(\mathbb{P}^n, X; \mathbb{C}) = 0 \text{ for } q \leq 2d - n + 1 \\ \implies H^q(S^k(N_{X|\mathbb{P}^n}^\vee)) = 0 \text{ for } q \leq 2d - n \text{ and } k \geq 1. \end{aligned}$$

Conversely, Schneider and Zintl proved the following vanishing result (see [25]):

$$(0.1) \quad H^q(S^k(N_{X|\mathbb{P}^n}^\vee)(-i)) = 0 \text{ for } q \leq 2d - n, k \geq 1 \text{ and } i \geq 0,$$

without using the Barth-Lefschetz theorem (here  $E^\vee$  denotes the dual of a vector bundle  $E$ ). Moreover they showed that (0.1) implies the Barth-Lefschetz theorem, i.e.  $H^q(\mathbb{P}^n, X; \mathbb{C}) = 0, \forall q \leq 2d - n + 1$ . Finally, we mention the papers [1], [6] and [22] to illustrate how certain vanishings of the cohomology involving the normal bundle may have interesting geometric consequences concerning small codimensional submanifolds of  $\mathbb{P}^n$ .

## 1. SOME KNOWN RESULTS AND BACKGROUND MATERIAL

All varieties considered here are defined over the field  $\mathbb{C}$  of complex numbers. By a submanifold of  $\mathbb{P}^n$  we mean a smooth closed irreducible subvariety of  $\mathbb{P}^n$ . The rest of the terminology and notation used throughout this paper are standard. In particular, for every projective variety  $X$  one defines:

- $\text{Pic}^0(X)$  (resp.  $\text{Pic}^\tau(X)$ ) as the subgroup of  $\text{Pic}(X)$  of all isomorphism classes of line bundles on  $X$  that are algebraically (resp. numerically) equivalent to zero. One has  $\text{Pic}^0(X) \subseteq \text{Pic}^\tau(X)$ , and a result of Matsusaka asserts that  $\text{Pic}^\tau(X)/\text{Pic}^0(X)$  is a finite group (see e.g. [19]).

- $\text{NS}(X) := \text{Pic}(X)/\text{Pic}^0(X)$  (the Néron-Severi group of  $X$ ) and  $\text{Num}(X) := \text{Pic}(X)/\text{Pic}^\tau(X)$ .

The main tool used in this paper is the following generalization of the Kodaira vanishing theorem due to Le Potier:

**Theorem 1.1** (Le Potier’s vanishing theorem). [23] *Let  $E$  be an ample vector bundle of rank  $r$  on a complex projective manifold  $X$  of dimension  $d \geq 2$ . Then  $H^i(E^\vee) = 0$  for every  $i \leq d - r$ .*

We shall also need the following criterion of Faltings for complete intersection:

**Theorem 1.2** (Faltings [10]). *Let  $X$  be a submanifold of  $\mathbb{P}^n$  such that there is a surjection  $\bigoplus_{i=1}^p \mathcal{O}_X(a_i) \rightarrow N_{X|\mathbb{P}^n}$  for some positive integers  $a_i$ ,  $i = 1, \dots, p$ . If  $p \leq \frac{n}{2}$ , then  $X$  is a complete intersection.*

Now we shall need a definition and some preliminary general results that shall be needed in the sequel. Let

$$(1.1) \quad 0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

be an exact sequence of vector bundles on a projective manifold  $X$ . If  $E$  is ample, it is well known that  $E_2$  is also ample, but this is no longer true in general for  $E_1$ .

**Definition 1.3.** Let  $E$  be an ample vector bundle of rank  $r \geq 2$  on a projective manifold  $X$ . Let  $p$  be a natural number such that  $1 \leq p \leq \frac{r}{2}$ . We say that  $E$  satisfies condition  $\mathcal{A}_p$  if there exists no exact sequence of the form (1.1) with  $E_1$  and  $E_2$  ample vector bundles on  $X$  of rank  $\geq p$ .

Clearly,  $\mathcal{A}_1 \Rightarrow \mathcal{A}_2 \Rightarrow \dots$ . On the other hand, if an ample vector bundle  $E$  satisfies condition  $\mathcal{A}_1$ , then  $E$  is indecomposable; i.e.,  $E$  cannot be written as  $E = E_1 \oplus E_2$ , with  $E_1$  and  $E_2$  vector bundles of rank  $\geq 1$ . We are going to apply Definition 1.3 to the normal bundle  $N_{X|\mathbb{P}^n}$  of a submanifold  $X$  of  $\mathbb{P}^n$ .

First we note the following general essentially well-known fact (see [14] and [6] for some special cases):

**Lemma 1.4.** *Assume that  $X$  is a submanifold of dimension  $d \geq 1$  of the projective space  $\mathbb{P}^n$ , such that the projection  $\pi_P: \mathbb{P}^n \setminus \{P\} \rightarrow \mathbb{P}^{n-1}$  of the center of a general point  $P \notin X$  defines a biregular isomorphism  $X \cong X' := \pi_P(X)$ . Then there exists a canonical exact sequence*

$$(1.2) \quad 0 \rightarrow \mathcal{O}_X(1) \rightarrow N_{X|\mathbb{P}^n} \rightarrow N_{X'|\mathbb{P}^{n-1}} \rightarrow 0.$$

*In particular, under the above hypotheses, the normal bundle  $N_{X|\mathbb{P}^n}$  does not satisfy condition  $\mathcal{A}_1$ .*

The proof is standard and we omit it. We also notice the following well-known and simple fact:

**Lemma 1.5.** *Let  $X$  be a submanifold of  $\mathbb{P}^n$  of dimension  $d \geq 1$ , with  $n \geq 2d + 1$ . Then there exists an exact sequence of vector bundles on  $X$  of the form*

$$0 \rightarrow \mathcal{O}_X(1) \rightarrow N_{X|\mathbb{P}^n} \rightarrow E \rightarrow 0.$$

*In particular, if  $n \geq 2d + 1$ , then  $N_{X|\mathbb{P}^n}$  does not satisfy condition  $\mathcal{A}_1$ .*

*Proof.* From the Euler sequence restricted to  $X$  we get a surjection  $\mathcal{O}_X^{\oplus n+1} \rightarrow N_{X|\mathbb{P}^n}(-1)$ ; i.e.,  $N_{X|\mathbb{P}^n}(-1)$  is generated by its global sections. The hypothesis that  $n \geq 2d + 1$  translates into  $\text{rank}(N_{X|\mathbb{P}^n}(-1)) \geq d + 1$ . Then by a well-known

theorem of Serre, there exists a nowhere-vanishing section  $s \in H^0(N_{X|\mathbb{P}^n}(-1))$ . Thus  $s$  yields an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow N_{X|\mathbb{P}^n}(-1) \rightarrow F \rightarrow 0$$

of vector bundles. Twisting by  $\mathcal{O}_X(1)$  we get the desired exact sequence. □

### 2. A GENERAL RESULT ON SUBMANIFOLDS IN $\mathbb{P}^n$

The aim of this section is to prove the following:

**Theorem 2.1.** *Let  $X$  be a projective submanifold of dimension  $d \geq 2$  of  $\mathbb{P}^n$ . Then:*

- (1)  $h^1(\mathcal{O}_X) = h^1(N_{X|\mathbb{P}^n}^\vee)$ . In particular,  $X$  is regular if and only if  $H^1(N_{X|\mathbb{P}^n}^\vee) = 0$ .
- (2) For every  $i$  such that  $2 \leq i \leq d$  one has  $h^{i-1}(\Omega_X^1) \leq h^{i-2}(\mathcal{O}_X) + h^i(N_{X|\mathbb{P}^n}^\vee)$ . In particular,  $h^1(\Omega_X^1) \leq 1 + h^2(N_{X|\mathbb{P}^n}^\vee)$ . If  $d \geq 3$  and  $H^1(\mathcal{O}_X) = 0$  the latter inequality becomes equality.
- (3)  $\text{rank Num}(X) \leq 1 + h^2(N_{X|\mathbb{P}^n}^\vee)$ . In particular, if  $H^2(N_{X|\mathbb{P}^n}^\vee) = 0$ , then  $\text{Num}(X) \cong \mathbb{Z}$ .

*Proof.* Much of the geometric information about the embedding  $X \subseteq \mathbb{P}^n$  is contained in the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_X & \xrightarrow{\text{id}} & \mathcal{O}_X & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & F & \longrightarrow & \mathcal{O}_X(1)^{\oplus n+1} & \longrightarrow & N_{X|\mathbb{P}^n} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \text{id} \\
 0 & \longrightarrow & T_X & \longrightarrow & T_{\mathbb{P}^n}|_X & \longrightarrow & N_{X|\mathbb{P}^n} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

in which the last row is the normal sequence of  $X$  in  $\mathbb{P}^n$  and the middle column is the Euler sequence of  $\mathbb{P}^n$  restricted to  $X$ . Analogous diagrams have already been used in the literature in a crucial way to prove some results of projective geometry (see e.g. [6], or [2], pages 7 and 25). Notice that the sheaf  $F^\vee$  coincides with  $\mathcal{P}^1(\mathcal{O}_X(1))(-1)$ , where  $\mathcal{P}^1(\mathcal{O}_X(1))$  is the sheaf of first-order principal parts of  $\mathcal{O}_X(1)$ .

Dualizing the middle row and the first column we get the exact sequences

$$(2.1) \quad 0 \rightarrow N_{X|\mathbb{P}^n}^\vee \rightarrow \mathcal{O}_X(-1)^{\oplus n+1} \rightarrow F^\vee \rightarrow 0,$$

$$(2.2) \quad 0 \rightarrow \Omega_X^1 \rightarrow F^\vee \rightarrow \mathcal{O}_X \rightarrow 0.$$

Then the cohomology of (2.1) yields the exact sequence

$$H^{i-1}(\mathcal{O}_X(-1)^{\oplus n+1}) \rightarrow H^{i-1}(F^\vee) \rightarrow H^i(N_{X|\mathbb{P}^n}^\vee) \rightarrow H^i(\mathcal{O}_X(-1)^{\oplus n+1}).$$

By the Kodaira vanishing theorem the first (resp. the last) space is zero for  $1 \leq i \leq d$  (resp. for  $1 \leq i \leq d - 1$ ). Thus

$$(2.3) \quad h^{i-1}(F^\vee) \leq h^i(N_{X|\mathbb{P}^n}^\vee), \text{ for all } 1 \leq i \leq d, \text{ with equality for } 1 \leq i \leq d - 1.$$

On the other hand, the exact sequence (2.2) does not split. Indeed, the dual Euler sequence

$$0 \rightarrow \Omega_{\mathbb{P}^n}^1 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow 0$$

corresponds to a generator of the one-dimensional  $\mathbb{C}$ -vector space  $H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1)$ . Then the exact sequence (2.2) corresponds to the image of this generator under the composite map

$$H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1) \rightarrow H^1(X, \Omega_{\mathbb{P}^n}^1|_X) \rightarrow H^1(X, \Omega_X^1),$$

which is known to be nonzero (otherwise the class of  $\mathcal{O}_X(1)$  would be zero in  $H^1(X, \Omega_X^1)$ ).

Then the cohomology of (2.2) yields the exact sequence

$$0 \rightarrow H^0(\Omega_X^1) \rightarrow H^0(F^\vee) \rightarrow H^0(\mathcal{O}_X) \rightarrow H^1(\Omega_X^1) \rightarrow H^1(F^\vee) \rightarrow H^1(\mathcal{O}_X).$$

Since the exact sequence (2.2) does not split and  $H^0(\mathcal{O}_X) = \mathbb{C}$ , we get the following isomorphism and exact sequence:

$$(2.4) \quad H^0(\Omega_X^1) \cong H^0(F^\vee) \text{ and } 0 \rightarrow H^0(\mathcal{O}_X) \rightarrow H^1(\Omega_X^1) \rightarrow H^1(F^\vee) \rightarrow H^1(\mathcal{O}_X).$$

Moreover, for every  $3 \leq i \leq d$  we have the cohomology sequence

$$(2.5) \quad \dots \rightarrow H^{i-2}(\mathcal{O}_X) \rightarrow H^{i-1}(\Omega_X^1) \rightarrow H^{i-1}(F^\vee) \rightarrow \dots$$

Now we prove (1). From (2.3) we get  $h^1(N_{X|\mathbb{P}^n}^\vee) = h^0(F^\vee)$ , and using the isomorphism of (2.4), this equality becomes  $h^1(N_{X|\mathbb{P}^n}^\vee) = h^0(\Omega_X^1)$ . Then one concludes the proof by Serre’s GAGA and the Hodge symmetry (which yield  $h^0(\Omega_X^1) = h^1(\mathcal{O}_X)$ ).

(2) From the exact sequence (2.5) we get  $h^{i-1}(\Omega_X^1) \leq h^{i-2}(\mathcal{O}_X) + h^{i-1}(F^\vee)$ , and from (2.3),  $h^{i-1}(F^\vee) \leq h^i(N_{X|\mathbb{P}^n}^\vee)$ , whence we get the first part. The second part follows from the first one and from the exact sequence of (2.4).

(3) follows from the last part (2) and from the following standard argument (cf. [16], [8] and [5]). Consider the (logarithmic derivative) map

$$\text{dlog} : \text{Pic}(X) \rightarrow H^1(\Omega_X^1)$$

defined in the following way. If  $Z$  is a scheme let us denote by  $\mathcal{O}_Z^*$  the sheaf of multiplicative groups of all nowhere-vanishing functions on  $Z$ . If  $[L \in \text{Pic}(X)]$  is represented by the 1-cocycle  $\{\xi_{ij}\}_{i,j}$  of  $\mathcal{O}_X^*$  with respect to an affine covering  $\{U_i\}_i$  of  $X$  (with  $\xi_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ ), then  $\text{dlog}(\{\xi_{ij}\})$  is by definition the cohomology class of the 1-cocycle  $\{\frac{d\xi_{ij}}{\xi_{ij}}\}_{i,j}$  of  $\Omega_X^1$ . Since  $\text{dlog}(\text{Pic}^0(X)) = 0$  the map  $\text{dlog}$  yields the map  $\text{dlog} : \text{NS}(X) = \text{Pic}(X)/\text{Pic}^0(X) \rightarrow H^1(\Omega_X^1)$ . Moreover, by a result of Matsusaka,  $\text{Pic}^\tau(X)/\text{Pic}^0(X)$  is a finite subgroup of  $\text{NS}(X)$  (see e.g. [19]). Since the underlying abelian group of the  $\mathbb{C}$ -vector space  $H^1(\Omega_X^1)$  is torsion-free it follows that  $\text{dlog}(\text{Pic}^\tau(X)) = 0$ . In other words, there is a unique map  $\alpha : \text{Num}(X) \rightarrow H^1(\Omega_X^1)$  such that  $\text{dlog} = \alpha \circ \beta$ , where  $\beta : \text{Pic}(X) \rightarrow \text{Num}(X)$  is the canonical surjection. Then (3) follows from the following general well-known fact:

*Claim:*  $\alpha$  induces an injective map  $\alpha_{\mathbb{C}} : \text{Num}(X) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow H^1(\Omega_X^1)$ .

This proves the theorem. □

*Remarks 2.2.* i) Let  $(X, N)$  be a pair consisting of a projective manifold  $X$  of dimension  $d \geq 2$  and a vector bundle  $N$  of rank  $r$  on  $X$ . We may ask the following question: under which conditions there exists a projective embedding of  $X \hookrightarrow \mathbb{P}^{d+r}$  such that  $N_{X|\mathbb{P}^{d+r}} \cong N$ ? Theorem 2.1 provides necessary conditions for  $(X, N)$  for the existence of such a projective embedding. Specifically, the irregularity  $h^1(\mathcal{O}_X)$  should be equal to  $h^1(N^\vee)$  and the inequalities  $h^{i-1}(\Omega_X^1) \leq h^{i-2}(\mathcal{O}_X) + h^i(N^\vee)$  should hold for every  $i$  such that  $2 \leq i \leq d$ .

ii) A simple consequence of Theorem 2.1 is the following weak form of the Barth-Larsen theorem: if  $d \geq \frac{n+2}{2}$ , then  $\text{Pic}(X) \cong \mathbb{Z}$ . Indeed, the inequality  $d \geq \frac{n+2}{2}$  and Le Potier’s vanishing theorem imply that  $H^i(N_{X|\mathbb{P}^n}^\vee) = 0$  for  $i = 1, 2$ . Moreover, the Fulton-Hansen connectedness theorem (see [12], or also [2], Theorem 7.4) implies that  $X$  is also algebraically simply connected and that  $\text{Pic}^\tau(X) = 0$ . Thus by Theorem 2.1,  $\text{Num}(X) = \text{Pic}(X) \cong \mathbb{Z}$ .

### 3. APPLICATIONS

The first application of Theorem 2.1 considers the normal bundle of some irregular  $d$ -folds in  $\mathbb{P}^n$ . By the Barth-Lefschetz theorem, every  $d$ -fold of  $\mathbb{P}^n$ , with  $d > \frac{n}{2}$ , is regular. So the first cases to consider are  $n = 2d$  and  $n = 2d + 1$ . Precisely, we have the following:

**Theorem 3.1.** *Let  $X$  be a submanifold of dimension  $d \geq 2$  of  $\mathbb{P}^n$ . Then:*

- (1) *Assume that  $X$  is irregular and  $n = 2d$ , e.g. an elliptic scroll of dimension  $d \geq 2$  in  $\mathbb{P}^{2d}$  (by [18] such scrolls do exist for every  $d \geq 2$ ). Then  $N_{X|\mathbb{P}^{2d}}$  satisfies condition  $\mathcal{A}_1$  of Definition 1.3, and in particular,  $N_{X|\mathbb{P}^{2d}}$  is indecomposable.*
- (2) *Assume that  $d \geq 3$  and  $n = 2d + 1$ . Then  $N_{X|\mathbb{P}^{2d+1}}$  satisfies condition  $\mathcal{A}_2$ , but never satisfies  $\mathcal{A}_1$ . In particular,  $N_{X|\mathbb{P}^{2d+1}}$  cannot be the direct sum of two vector bundles of rank  $\geq 2$ .*

*Proof.* (1) Assume that there is an exact sequence of the form

$$0 \rightarrow E_1 \rightarrow N_{X|\mathbb{P}^{2d}} \rightarrow E_2 \rightarrow 0,$$

with  $E_1$  and  $E_2$  ample vector bundles on  $X$  of ranks  $\geq 1$ . Dualizing and taking cohomology we get

$$H^1(E_2^\vee) \rightarrow H^1(N_{X|\mathbb{P}^{2d}}^\vee) \rightarrow H^1(E_1^\vee).$$

Since  $E_1$  and  $E_2$  are both ample of rank  $\leq d - 1$  on the projective  $d$ -fold  $X$ , the first and the third space are zero by Le Potier’s vanishing theorem. It follows that  $H^1(N_{X|\mathbb{P}^{2d}}^\vee) = 0$ . Then by Theorem 2.1, (1),  $H^1(\mathcal{O}_X) = 0$ . But this is impossible because  $X$  was an irregular manifold by hypothesis.

(2) We proceed similarly as in case (1). First, the fact that  $N_{X|\mathbb{P}^{2d+1}}$  does not satisfy condition  $\mathcal{A}_1$  follows from Lemma 1.5. To check condition  $\mathcal{A}_2$ , assume that there exists an exact sequence of the form

$$0 \rightarrow E_1 \rightarrow N_{X|\mathbb{P}^{2d+1}} \rightarrow E_2 \rightarrow 0,$$

with  $E_1$  and  $E_2$  ample vector bundles on  $X$  of rank  $\geq 2$ ; in particular,  $E_1$  and  $E_2$  both have rank  $\leq d - 1$ . Thus by Le Potier’s vanishing theorem,  $H^1(E_1^\vee) = H^1(E_2^\vee) = 0$ , whence the cohomology sequence

$$H^1(E_2^\vee) \rightarrow H^1(N_{X|\mathbb{P}^{2d+1}}^\vee) \rightarrow H^1(E_1^\vee)$$

yields  $H^1(N_{X|\mathbb{P}^{2d+1}}^\vee) = 0$ . Then Theorem 2.1, (1), implies  $H^1(\mathcal{O}_X) = 0$ , a contradiction.  $\square$

As a second application of Theorem 2.1 we have the following:

**Theorem 3.2.** *Let  $X$  be a submanifold of dimension  $d \geq 2$  of  $\mathbb{P}^n$ . Then:*

- (1) *Assume  $d \geq 3$  and  $n = 2d - 1$ . If  $N_{X|\mathbb{P}^{2d-1}}$  does not satisfy condition  $\mathcal{A}_1$  (e.g. if  $N_{X|\mathbb{P}^n}$  is decomposable), then  $\text{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)]$ .*
- (2) *Assume  $d \geq 4$  and  $n = 2d$ . If  $N_{X|\mathbb{P}^{2d}}$  does not satisfy condition  $\mathcal{A}_2$  (e.g. if  $N_{X|\mathbb{P}^n}$  is the direct sum of two vector bundles of rank  $\geq 2$ ), then  $\text{Num}(X) \cong \mathbb{Z}$ .*
- (3) *Assume that  $N_{X|\mathbb{P}^n}$  is a direct sum of line bundles. Then  $H^1(\mathcal{O}_X) = 0$ , and if  $d \geq 3$ ,  $\text{Num}(X) \cong \mathbb{Z}$ .*

*Proof.* (1) Assume that there is an exact sequence of the form

$$(3.1) \quad 0 \rightarrow E_1 \rightarrow N_{X|\mathbb{P}^{2d-1}} \rightarrow E_2 \rightarrow 0,$$

with  $E_1$  and  $E_2$  ample vector bundles on  $X$  of rank  $\geq 1$  (in particular,  $E_1$  and  $E_2$  are both of rank  $\leq d - 2$ ). Then the cohomology sequence of the dual of (3.1) is

$$H^1(E_2^\vee) \rightarrow H^1(N_{X|\mathbb{P}^{2d-1}}^\vee) \rightarrow H^1(E_1^\vee)$$

and yields  $H^2(N_{X|\mathbb{P}^{2d-1}}^\vee) = 0$ . Then by Theorem 2.1, (3),  $\text{Num}(X) \cong \mathbb{Z}$ . Now, the Fulton-Hansen connectedness theorem (see [12]) implies that  $\text{Pic}^\tau(X) = 0$ , i.e.  $\text{Num}(X) = \text{Pic}(X)$ , whence  $\text{Pic}(X) \cong \mathbb{Z}$ . On the other hand, the results of Barth-Larsen [20] or of Faltings [9] (see also [2], Theorem 10.3 and Proposition 10.10) imply that  $\mathcal{O}_X(1)$  is not divisible in  $\text{Pic}(X)$ , whence  $\text{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)]$ .

The proof of part (2) is completely similar. In fact, assume that there exists an exact sequence

$$0 \rightarrow E_1 \rightarrow N_{X|\mathbb{P}^{2d}} \rightarrow E_2 \rightarrow 0,$$

with  $E_1$  and  $E_2$  ample vector bundles of rank  $\geq 2$ . Since  $\text{rank}(E_1) + \text{rank}(E_2) = d$ , it follows that  $\text{rank}(E_1), \text{rank}(E_2) \leq d - 2$ . Therefore, by Le Potier’s vanishing theorem,  $H^2(E_1^\vee) = H^2(E_2^\vee) = 0$ . Thus the cohomology sequence

$$H^2(E_2^\vee) \rightarrow H^2(N_{X|\mathbb{P}^{2d}}^\vee) \rightarrow H^2(E_1^\vee)$$

yields  $H^2(N_{X|\mathbb{P}^{2d}}^\vee) = 0$ . Then the conclusion follows from Theorem 2.1, (3).

(2) The hypotheses and the Kodaira vanishing theorem imply  $H^1(N_{X|\mathbb{P}^n}^\vee) = 0$  if  $d \geq 2$  and also  $H^2(N_{X|\mathbb{P}^n}^\vee) = 0$  if  $d \geq 3$ . Thus by Theorem 2.1 we get  $H^1(\mathcal{O}_X) = 0$  if  $d \geq 2$  and  $\text{Num}(X) \cong \mathbb{Z}$  if  $d \geq 3$ .  $\square$

Here are some corollaries of Theorem 3.2:

**Corollary 3.3.** *The normal bundle  $N$  of the Segre embedding  $i: \mathbb{P}^{d-1} \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{2d-1}$  ( $d \geq 3$ ) satisfies condition  $\mathcal{A}_1$ . In particular,  $N$  is indecomposable.*

*Proof.* The proof is a direct consequence of Theorem 3.2, (1).  $\square$

**Corollary 3.4.** *The normal bundle of  $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  in  $\mathbb{P}^7$  (via the Segre embedding) is not a direct sum of line bundles.*

*Proof.* The proof is a direct consequence of Theorem 3.2, (3).  $\square$

**Corollary 3.5.** *Let  $N$  be the normal bundle of the Segre embedding  $i: \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ . Then  $N$  satisfies condition  $\mathcal{A}_2$ . Furthermore, there exists an exact sequence of the form*

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 1) \rightarrow N \rightarrow N' \rightarrow 0,$$

where  $N'$  is the normal bundle of the isomorphic image of  $\mathbb{P}^2 \times \mathbb{P}^2$  under the projection of  $\pi_P: \mathbb{P}^8 \setminus \{P\} \rightarrow \mathbb{P}^7$  from a general point  $P \in \mathbb{P}^8$ . In particular,  $N$  does not satisfy  $\mathcal{A}_1$ .

*Proof.* The first part follows from Theorem 3.2, (2). To prove the second part we use the known fact that  $X := i(\mathbb{P}^2 \times \mathbb{P}^2)$  is a Severi variety in  $\mathbb{P}^8$ ; i.e., the projection of  $\mathbb{P}^8$  from a general point of  $\mathbb{P}^8$  maps  $X$  biregularly onto a submanifold  $X'$  of  $\mathbb{P}^7$ . Then the conclusion follows from Lemma 1.4.  $\square$

**Corollary 3.6.** *Let  $X$  be a submanifold of  $\mathbb{P}^n$  of dimension  $d \geq \frac{n}{2}$ , with  $n \geq 5$ . If  $n = 2d$  assume that  $X$  is simply connected and  $\mathcal{O}_X(1)$  is not divisible in  $\text{Pic}(X)$ . If the normal bundle  $N_{X|\mathbb{P}^n}$  is a direct sum of line bundles, then  $X$  is a complete intersection in  $\mathbb{P}^n$ .*

*Proof.* Since  $d \geq 3$  and  $N_{X|\mathbb{P}^n}$  is a direct sum of line bundles, Theorem 3.2, (3) implies that  $\text{Num}(X) = \mathbb{Z}$ . If  $d > \frac{n}{2}$ , then by the Barth-Larsen theorem,  $X$  is simply connected and  $\mathcal{O}_X(1)$  is not divisible in  $\text{Pic}(X)$ . If instead  $n = 2d$ , we have these statements by the hypotheses. It follows that  $\text{Num}(X) = \text{Pic}(X) = \mathbb{Z}[\mathcal{O}_X(1)]$ . Therefore in all cases the hypothesis that  $N_{X|\mathbb{P}^n}$  is a direct sum of line bundles translates into  $N_{X|\mathbb{P}^n} \cong \bigoplus_{i=1}^{n-d} \mathcal{O}_X(a_i)$ , with  $a_i \geq 1$  and  $n - d \leq \frac{n}{2}$ . Then the conclusion follows from Theorem 1.2 of Faltings.  $\square$

*Remarks 3.7.* i) I am indebted to G. Ottaviani for calling my attention to the fact that if  $d > \frac{n}{2}$ , Corollary 3.6 was first proved by Michael Schneider in [24]. Although also based on Faltings' criterion of complete intersection (Theorem 1.2 above), Schneider's proof is however different from ours because it uses the methods of [13] together with another result of Faltings [9] according to which every submanifold of  $\mathbb{P}^n$  of dimension  $> \frac{n}{2}$  satisfies the effective Grothendieck-Lefschetz condition  $\text{Leff}(\mathbb{P}^n, X)$ .

ii) Basili and Peskine proved that every nonsingular surface in  $\mathbb{P}^4$  whose normal bundle is decomposable is a complete intersection (see [4]). Their proof is based heavily on the methods developed in [8]. For a related result see Méguin [21]. Partial results on the normal bundle of two-codimensional submanifolds of  $\mathbb{P}^n$  (with  $n \geq 5$ ) can be found in Ellia, Franco and Gruson [7].

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