SEMI-COMPACTNESS
OF POSITIVE DUNFORD–PETTIS OPERATORS
ON BANACH LATTICES

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ABSTRACT. We investigate Banach lattices on which each positive Dunford–Pettis operator is semi-compact and the converse. As an interesting consequence, we obtain Theorem 2.7 of Aliprantis–Burkinshaw and an element of Theorem 1 of Wickstead.

1. Introduction and notation

An operator $T$ from a Banach space $E$ into a Banach lattice $F$ is said to be semi-compact if for each $\varepsilon > 0$, there exists some $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$ where $B_H$ is the closed unit ball of $H = E$ or $F$ and $F^+ = \{x \in F : 0 \leq x\}$.

An operator $T$ between two Banach spaces $E$ and $F$ is said to be Dunford–Pettis if the image of each weakly compact subset of $E$ is a compact subset of $F$.

In contrast to compact operators, the class of semi-compact (resp. Dunford–Pettis) operators does not satisfy the analogue of Schauder’s Theorem. However, the class of semi-compact operators satisfies the domination problem (Theorem 18.20 of [3]), but the class of Dunford–Pettis operators fails to satisfy this property, as was proved in [1], [7] and [9].

Finally, a semi-compact operator is not necessarily Dunford–Pettis and, conversely, a Dunford–Pettis operator is not necessarily semi-compact.

In [4] and [5] we studied the compactness of the class of positive Dunford–Pettis operators. And in [6] we characterized Banach lattices on which each positive Dunford–Pettis operator is weakly compact. Also, it follows from Aliprantis and Burkinshaw ([1], Theorem 3.4) that if $E$ and $F$ are two Banach lattices such that $E$ has an order-continuous norm, then each regular Dunford–Pettis operator is AM-compact. Our objective in this paper is to continue the investigation of Banach lattices on which each positive Dunford–Pettis operator is semi-compact and the converse. We will characterize Banach lattices for which each positive Dunford–Pettis operator is semi-compact, and we will give some interesting consequences. Next, we will prove a sufficient condition under which a positive semi-compact operator is weakly compact. More precisely, we will show that if the norm of a Banach lattice $F$ is order continuous, then each positive semi-compact operator...
from a Banach lattice $E$ into $F$ is weakly compact. As consequence, we shall obtain some conditions for which the class of Dunford–Pettis operators, the class of semi-compact operators, the class of weakly compact operators, and the class of compact operators coincide. Finally, whenever $E$ is an order $\sigma$-complete Banach lattice, we will establish that if each positive semi-compact operator from $E$ into $E$ is weakly compact (resp. Dunford–Pettis), then the norm of $E$ is order continuous.

To state our results, we need to fix some notation and recall some definitions. A vector lattice $E$ is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A subspace $F$ of a vector lattice $E$ is said to be a sublattice if for every pair of elements $a, b$ of $F$ the supremum of $a$ and $b$, taken in $E$, belongs to $F$. A subset $B$ of a vector lattice $E$ is said to be solid if it follows from $|y| \leq |x|$ with $x \in B$ and $y \in E$ that $y \in B$. An order ideal of $E$ is a solid subspace. Let $E$ be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of $E$ is said to be order bounded if it is included in some order interval. A Banach lattice is a Banach space $(E, \|\|)$ such that $E$ is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If $E$ is a Banach lattice, its topological dual $E'$, endowed with the dual norm, is also a Banach lattice. For more detail about Banach lattices, the reader is referred to the book of Zaanen [10].

2. Major results

We will use the term operator $T : E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in $F$ whenever $x \geq 0$ in $E$. An operator $T : E \to F$ is regular if $T = T_1 - T_2$ where $T_1$ and $T_2$ are positive operators from $E$ into $F$. It is well known that each positive linear mapping on a Banach lattice is continuous.

Also, a norm $\|\|$ of a Banach lattice $E$ is order continuous if for each net $(x_\alpha)$ such that $x_\alpha \downarrow 0$ in $E$, the sequence $(x_\alpha)$ converges to 0 for the norm $\|\|$ where the notation $x_\alpha \downarrow 0$ means that $(x_\alpha)$ is decreasing, its infimum exists, and $\inf(x_\alpha) = 0$. For example, the norm of the Banach lattice $l^1$ is order continuous but the norm of the Banach lattice $l^\infty$ is not.

To prove the next theorem, we need the following lemma:

**Lemma 2.1.** Let $E, F$ be Banach lattices and let $T$ be a positive Dunford–Pettis operator from $E$ into $F$. If the topological dual $E'$ of $E$ has an order continuous norm, then for each $\varepsilon > 0$, there exists some $y \in E^+$ such that

$$T(B_E \cap E^+) \subset \varepsilon B_F + T([0, y]),$$

where $B_H$ is the closed unit ball of $H = E, F$.

**Proof.** It follows from the proof of Theorem 2.7 and Theorem 2.8 of [1].

Recall that a nonzero element $x$ of a vector lattice $E$ is discrete if the order ideal generated by $x$ equals the subspace generated by $x$. The vector lattice $E$ is discrete if it admits a complete disjoint system of discrete elements. For example, the Banach lattice $l^1$ is discrete but $C([0, 1])$ is not discrete.

A Dunford–Pettis operator is not necessarily semi-compact. In fact, the identity operator $Id_c : c \to c$ is semi-compact but it is not Dunford–Pettis where $c$ is the Banach lattice of all convergent sequences. The following theorem gives a
sufficient and necessary condition for which a regular Dunford–Pettis operator is semi-compact. In fact,

**Theorem 2.2.** Let $E$ be a Banach lattice. Then the following assertions are equivalent:

1. $E'$ has an order continuous norm.
2. Each positive Dunford–Pettis operator from $E$ into $F$ is semi-compact for each Banach lattice $F$.
3. Each positive Dunford–Pettis operator from $E$ into $E$ is semi-compact.

**Proof.** For the implication $1 \Rightarrow 2$. Let $F$ be a Banach lattice and let $T$ be a positive Dunford–Pettis operator from $E$ into $F$. It follows from Lemma 2.1 that for each $\varepsilon > 0$, there exists some $y \in E^+$ such that

$$T(B_E^+ \cap [0,y]) \subset \varepsilon B_F^+ + T([0,y]),$$

where $B_H^+ = B_H \cap E^+$ for $H = E,F$.

Since

$$T([0,y]) \subset [0,T(y)],$$

we obtain

$$T(B_E^+) \subset \varepsilon B_F^+ + [0,T(y)].$$

This proves the result.

For the implication $3 \Rightarrow 1$. If the norm of $E'$ is not order continuous, then it follows from the proof of Theorem 1 of Wickstead [9], that $E$ contains a sublattice that is isomorphic to $l^1$ and there exists a positive projection $P$ from $E$ into $l^1$.

Consider the operator product

$$i \circ P : E \rightarrow l^1 \rightarrow E,$$

where $i$ is the inclusion operator of $l^1$ in $E$. It is clear that $i \circ P$ is a positive Dunford–Pettis operator. We have to prove that $i \circ P$ is not semi-compact. If not, its restriction to $l^1$ is a semi-compact operator, and then the restriction of the operator $P \circ (i \circ P)$ to $l^1$, which coincides with the operator identity $Id_{l^1}$ of $l^1$, is semi-compact; i.e. for each $\varepsilon > 0$, there exists some $y \in (l^1)^+$ such that

$$B_{l^1} \subset \varepsilon B_{l^1} + [-y,y].$$

Since the Banach lattice $l^1$ is discrete and has an order continuous norm, it follows from Theorem 3.22 of [2] that the order interval $[-y,y]$ is compact in $l^1$. Hence, the closed unit ball $B_{l^1}$ is precompact. This presents a contradiction.

The implication $2 \Rightarrow 3$ is trivial.

**Remark 2.3.** If we fix the Banach lattice $F$, the implication $2 \Rightarrow 1$ of Theorem 2.2. is false. In fact, if we take $F$ of finite dimension and $E = l^1$, it is clear that each positive Dunford–Pettis operator from $l^1$ into $F$ is semi-compact, but it is well known that the norm of the topological dual $E' = l^\infty$ is not order continuous.

Recall from Zaanen [10] that a regular operator $T$ from a vector lattice $E$ into a Banach lattice $F$ is said to be AM-compact if it carries each order-bounded subset of $E$ onto a relatively compact subset of $F$.

As a consequence of Theorem 2.2 of [6] and the above theorem, we obtain
Corollary 2.4. Let $E$ be a Banach lattice. Then the following statements are equivalent:
1. Each positive Dunford–Pettis operator from $E$ into $E$ is weakly compact.
2. Each positive Dunford–Pettis operator from $E$ into $E$ is semi-compact.
3. For each positive Dunford–Pettis operator $T$ from $E$ into $E$, the second power operator $T^2$ is compact.
4. For each pair of operators $S$ and $T$ from $E$ into $E$ such that $0 \leq S \leq T$ and $T$ is Dunford–Pettis, the operator $S$ is weakly compact.
5. The norm of the topological dual $E'$ is order continuous.

The following result is a consequence of a theorem of Dodds and Fremlin ([10], Theorem 125.5) and Theorem 2.2:

Corollary 2.5. Let $E$ and $F$ be two Banach lattices such that $E'$ and $F$ have order continuous norms. Let $T$ be a positive operator from $E$ into $F$. Then the following assertions are equivalent:
1. $T$ is compact.
2. $T$ is Dunford–Pettis and AM-compact.
3. $T$ is semi-compact and AM-compact.

Now by combining our Corollary 2.5 and Theorem 3.5 of Aliprantis and Burkinshaw [1], we obtain Theorem 2.7 of [1] as a consequence, which was the basic result of our papers [4] and [5].

Corollary 2.6. Let $E$ be a Banach lattice such that $E$ and its topological dual $E'$ have order continuous norms. Then each positive Dunford–Pettis operator from $E$ into $E$ is compact.

Proof. Let $T$ be a positive Dunford–Pettis operator from $E$ into $E$. Since the norm of $E$ is order continuous, it follows from Theorem 3.5 of [1] that $T$ is AM-compact. Finally, Corollary 2.5 implies that $T$ is compact.

Let $T$ be a positive operator from a Banach lattice $E$ into a Banach lattice $F$. If $T'$ is the adjoint operator from $F'$ into $E'$ defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$, it is clear that $T'$ is positive.

Recall that a Banach space $E$ has the Dunford–Pettis property if and only if each weakly compact operator on $E$, taking its value in another Banach space, is Dunford–Pettis.

Now, to give a sufficient condition under which each positive semi-compact operator is Dunford–Pettis, we study the compactness of a positive semi-compact operator.

Theorem 2.7. Let $E$ and $F$ be Banach lattices. Then each positive semi-compact operator from $E$ into $F$ is compact if one of the following statements is valid:
1. $F$ is discrete and its norm is order continuous.
2. $E'$ is a discrete order continuous norm, and $F$ has an order continuous norm.
3. The norms of $E$, $E'$ and $F$ are order continuous, and $E$ has the Dunford–Pettis property.

Proof. 1. Let $T$ be a positive semi-compact operator from $E$ into $F$. By Theorem 3.22 of [1], for each $y \in F^+$, the order interval $[0,y]$ is norm compact in $F$. On the other hand, for each $\varepsilon > 0$, there exists some $x \in F^+$ such that
$$T(B_E) \subset \varepsilon B_F + [-x,x].$$
This proves that $T(B_E)$ is norm relatively compact in $F$ and hence $T$ is compact.

2. Let $T$ be a positive semi-compact operator from $E$ into $F$. Since the norms of $E'$ and $F$ are order continuous, then the adjoint operator $T'$ from $F'$ into $E'$ is semi-compact (Theorem 125.6 of [10]). As the Banach lattice $E'$ is discrete and its norm is order continuous, then condition 1 implies that $T' : F' \to E'$ is compact, and therefore $T$ is compact.

3. Let $T$ be a positive semi-compact operator from $E$ into $F$. Since $F$ has an order continuous norm, it follows from Theorem 2.2 that $T$ is weakly compact. Now, the Dunford–Pettis property of $E$ implies that $T$ is Dunford–Pettis. On the other hand, $E$ and $E'$ have order continuous norms, so Theorem 2.7 of Aliprantis and Burkinshaw [1] implies that $T$ is compact.

□

As an immediate consequence of Theorem 2.6, we obtain a result of Wickstead ([8], Proposition 2.3 or [9], Theorem 1):

**Corollary 2.8.** Let $E$ and $F$ be Banach lattices, and let $S$ and $T$ be positive operators from $E$ into $F$ such that $0 \leq S \leq T$ and $T$ is compact. If $F$ is discrete and its norm is order continuous, then $S$ is compact.

Recall that a semi-compact operator is not necessarily Dunford–Pettis. For example, the identity operator $\text{Id}_{l^1} : l^1 \to l^1$ is Dunford–Pettis but it is not semi-compact. Another consequence of Theorem 2.7 is the following:

**Corollary 2.9.** Let $E$ and $F$ be Banach lattices. If $F$ is discrete and its norm is order continuous, then each positive semi-compact operator from $E$ into $F$ is Dunford–Pettis.

Recall that a semi-compact operator is not necessarily weakly compact, and conversely a weakly compact operator is not necessarily semi-compact. For example, the identity operator $\text{Id}_{l^2} : l^2 \to l^2$ is weakly compact but it is not semi-compact (if not, since $l^2$ is discrete and its norm is order continuous, it follows from Theorem 2.7 (1) that $\text{Id}_{l^2}$ is compact) and conversely, the identity operator $\text{Id}_c : c \to c$ is semi-compact but it is not weakly compact.

The following proposition gives a sufficient condition under which a positive weakly compact operator is semi-compact:

**Proposition 2.10.** Let $E$ and $F$ be two Banach lattices. If $E$ has the Dunford–Pettis property and its topological dual $E'$ has an order continuous norm, then each positive weakly compact operator from $E$ into $F$ is semi-compact.

**Proof.** Let $T$ be a positive weakly compact operator from $E$ into $F$. Since $E$ admits the Dunford–Pettis property, then $T$ is Dunford–Pettis. Now the semi-compactness of $T$ follows from the order continuousness of the norm of the topological dual $E'$ (Theorem 2.2).

Conversely, we give a sufficient condition under which a positive semi-compact operator is weakly compact.

**Theorem 2.11.** Let $E$ and $F$ be two Banach lattices. If the norm of $F$ is order continuous, then each positive semi-compact operator from $E$ into $F$ is weakly compact.
Proof. Let $T$ be a positive semi-compact operator from $E$ into $F$. Then for each $\varepsilon > 0$, there exists some $y \in F^+$ such that
\[
T(B_E \cap E^+) \subset \varepsilon B_F + [0, y],
\]
where $B_H$ is the unit ball of $H = E, F$. Since $F$ has an order continuous norm, then the order interval $[0, y]$ is weakly compact, and hence $T(B_E \cap E^+)$ is weakly precompact. To prove that $T(B_E \cap E^+)$ is weakly relatively compact, it is sufficient to show that the closure of $T(B_E \cap E^+)$, for the topology $\sigma (F, F')$, is weakly complete. To show this, we use the same proof as Theorem 2.2 of [6]. In fact, let $(T(x_i))_i$ be a Cauchy net for the topology $\sigma (F, F')$, where $(x_i)$ is a net in $B_E \cap E^+$. Since $T(B_E \cap E^+)$ is relatively compact for $\sigma (F'', F')$ in the topological bidual $F''$, the sequence $(T(x_i))_i$ converges to some $\Psi \in F''$ for $\sigma (F'', F')$. Let $m \in \mathbb{N}^*$; it follows from (1), the existence of $y^m \in F^+$, $z^m_i \in B_F$ and $w^m_i \in [0, y^m]$ such that
\[
T(x_i) = \frac{1}{m} z^m + w^m_i.
\]
On the other hand, since $[0, y^m]$ is relatively weakly compact, there exists an accumulation point $a_m \in F$ of the net $(w^m_i)_i$.

If we fix $f \in F'$ with $\|f\| \leq 1$, then there exists some $i_0 \in I$ such that for each $i > i_0$, we have
\[
|f \circ T(x_i) - \Psi(f)| < \frac{1}{m},
\]
and hence
\[
|f(w^m_i) - \Psi(f)| < \frac{2}{m}.
\]
Since $a_m$ is an accumulation point of the sequence $(w^m_i)_i$, there exists some $i > i_0$ such that
\[
|f(w^m_i) - f(a_m)| < \frac{1}{m}.
\]
This implies that
\[
|\Psi(f) - f(a_m)| < \frac{3}{m},
\]
and so
\[
\|\Psi - a_m\|_{F''} \leq \frac{3}{m},
\]
where $\|\cdot\|_{F''}$ is the norm of $F''$. It follows that $(a_m)$ converges in norm to $\Psi$, and hence $\Psi \in F$. This proves that the closure of $T(B_E \cap E^+)$ for $\sigma (F, F')$ in $F$ coincides with its closure in $\hat{F}$ where $\hat{F}$ is the completion of $F$ for $\sigma (F, F')$. Hence, $T(B_E \cap E^+)^{(F, F')}$ is complete.

A Banach lattice $E$ is said to be an AM-space if for each $x, y \in E$ such that $\inf \{x, y\} = 0$, we have $\|x + y\| = \max\{|\|x\|, \|y\|\}$. It is an AL-space if its topological dual $E'$ is an AM-space. For example, the Banach lattice $l^1$ is an AL-space and the Banach lattice $l^\infty$ is an AM-space.

As consequence of Theorem 2.11, we have:

**Corollary 2.12.** Let $E$ be an AM-space with unit and $F$ be a Banach lattice with an order continuous norm. Then each positive operator from $E$ into $F$ is weakly compact. In particular, each positive operator from $E$ into $l^1$ is compact.
Proof. Let $T : E \rightarrow F$ be a positive operator. As $E$ is an AM-space with unit, it follows that $T$ is semi-compact. Now, since $F$ has an order continuous norm, Theorem 2.11 implies that $T$ is weakly compact.

Finally, let $S : E \rightarrow l^1$ be a positive operator where $E$ is an AM-space with unit. Since $l^1$ has an order continuous norm, the operator $S$ is weakly compact. On the other hand, $S = Id_{l^1} \circ S$, where $Id_{l^1} : l^1 \rightarrow l^1$ is the identity operator of $l^1$ which is Dunford–Pettis; hence, $S$ is compact.

Remark 2.13. If $E$ is a Banach lattice with a discrete topological dual $E'$, then there exists a positive semi-compact operator which is not weakly compact. In fact, if we take $E = F = c$, then $E'$ is discrete, and the identity operator $Id_c$ of $E$ is semi-compact but it is not weakly compact.

The following consequence of Theorem 2.11 gives another sufficient condition under which each positive semi-compact operator is Dunford–Pettis.

Corollary 2.14. Let $E$ and $F$ be two Banach lattices. If $E$ has the Dunford–Pettis property and the norm of $F$ is order continuous, then each positive semi-compact operator is Dunford–Pettis.

Proof. Let $T$ be a positive semi-compact operator from $E$ into $F$. Since the norm of $F$ is order continuous, it follows from Theorem 2.11 that $T$ is weakly compact. As $E$ has the Dunford–Pettis property, the operator $T$ is Dunford–Pettis.

Another consequence of Theorem 2.11 is the following:

Corollary 2.15. Let $E$ and $F$ be two Banach lattices such that $E'$ and $F$ have order continuous norms and $E$ has the Dunford–Pettis property. Let $T : E \rightarrow F$ be a positive operator. Then the following assertions are equivalent:

1. $T$ is Dunford–Pettis.
2. $T$ is semi-compact.
3. $T$ is weakly compact.

In particular, if we take $E = F$, under the same conditions as Corollary 2.15, we have the following result:

Corollary 2.16. Let $E$ be a Banach lattice having the Dunford–Pettis property such that $E'$ and $E$ have order continuous norms. Let $T : E \rightarrow E$ be a positive operator. Then the following assertions are equivalent:

1. $T$ is Dunford–Pettis.
2. $T$ is semi-compact.
3. $T$ is weakly compact.
4. $T$ is compact.

Remark 2.17. Let $E$ be a Banach lattice having the Dunford–Pettis property. If the topological dual $E'$ is discrete with an order continuous norm, there exists a positive semi-compact operator which is not Dunford–Pettis. In fact, the identity operator $Id_c$ of $c$ is a positive semi-compact operator which is not Dunford–Pettis. However, the topological dual $c'$ is discrete, $c$ has the Dunford–Pettis property, and $c'$ has an order continuous norm.

Whenever $E = F$ and $E$ is order $\sigma$-complete (i.e. every majorized countable nonempty subset of $E$ has a supremum), we obtain the following converse of Theorem 2.11:
Theorem 2.18. Let $E$ be an order $\sigma$-complete Banach lattice. If each positive semi-compact operator from $E$ into $E$ is weakly compact (resp. Dunford–Pettis), then the norm of $E$ is order continuous.

Proof. Assume that the norm of $E$ is not order continuous. Since $E$ is order $\sigma$-complete, it follows from the proof of Theorem 1 of Wickstead [9] that $E$ contains a sublattice which is isomorphic to $l^\infty$ and there exists a positive projection $P : E \rightarrow l^\infty$.

If we take the operator $i \circ P : E \rightarrow l^\infty \rightarrow E$, it is clear that $i \circ P$ is a semi-compact operator which is not weakly compact (resp. Dunford–Pettis). If not, the restriction of the operator $P \circ (i \circ P)$ to $l^\infty$, which coincides with the identity operator $\text{Id}_{l^\infty}$ of $l^\infty$, would be weakly compact (resp. Dunford–Pettis). This presents a contradiction.

Recall that Wickstead characterized Banach lattices which satisfy the problem of domination for the class of positive weakly compact operators ([8], Theorem 17.10). We observe that by the same proof as Theorem 17.10 of Wickstead [8], we obtain the following result:

Theorem 2.19. Let $E$ and $F$ be two Banach lattices. If every positive operator from $E$ into $F$ dominated by a compact operator is weakly compact, then either $E'$ or $F$ has an order continuous norm.

Proof. If the norms of $F$ and $E'$ are not order continuous, Wickstead constructed in the proof of [8], Theorem 17.10 two positive operators $S, T : E \rightarrow F$ such that $0 \leq S \leq T$ and $T$ is compact but $S$ is not weakly compact. This proves the result.

Whenever $E \neq F$, by combining Theorem 2.19 with Theorem 18.20 of [3] we obtain

Theorem 2.20. Let $E$ and $F$ be two Banach lattices. If each positive semi-compact operator from $E$ into $F$ is weakly compact, then one of the following assertions is valid:

1. $F$ has an order continuous norm.
2. $E'$ has an order continuous norm.

Proof. In fact, let $S, T : E \rightarrow F$ be two operators such that $0 \leq S \leq T$ and $T$ is compact. Since $T$ is semi-compact, it follows that $S$ is semi-compact (Theorem 18.20 of [3]). Hence $S$ is weakly compact and the result comes from Theorem 2.19.

Also, recall that Wickstead ([9], Theorem 2) studied the converse of the domination problem of the class of positive Dunford–Pettis operators. We remark that by the same proof as Theorem 2 of [9], we can establish the following result:

Theorem 2.21. For two Banach lattices $E$ and $F$, the following assertions are equivalent:

1. Every positive operator from $E$ into $F$ dominated by a compact operator is Dunford–Pettis.
2. The norm of $F$ is order continuous or the lattice operations of $E$ are weakly sequentially continuous.
Proof. If the norm of $F$ is not order continuous and the lattice operations of $E$ are not weakly sequentially continuous, Wickstead constructed in the proof of Theorem 2 of [9] two positive operators $S, T : E \to F$ such that $0 \leq S \leq T$ and $T$ is compact but $S$ is not Dunford–Pettis. □

A combination of Theorem 2.21 and Theorem 18.20 of [3] gives

**Theorem 2.22.** Let $E$ and $F$ be Banach lattices. If each positive semi-compact operator from $E$ into $F$ is Dunford–Pettis, then one of the following assertions is valid:

1. The norm of $F$ is order continuous.
2. The lattice operations of $E$ are weakly sequentially continuous.

**Proof.** Let $S$ and $T$ be two positive operators from $E$ into $F$ such that $0 \leq S \leq T$ and $T$ is compact. Since $T$ is semi-compact, $S$ is semi-compact ([3], Theorem 18.20) and then Dunford–Pettis. Now, the result follows from Theorem 2.21. □

**Remark 2.23.** Let $E$ and $F$ be two Banach lattices such that one of the following properties is valid:

1. The topological dual $E'$ of $E$ is discrete.
2. $F$ is discrete.
3. The norm of $E'$ is order continuous.

Then we can always find a positive semi-compact operator from $E$ into $F$ which is not Dunford–Pettis. In fact, the identity operator $I_d : c \to c$ is semi-compact which is not Dunford–Pettis but the above three conditions are satisfied by the Banach lattice $c$.

**Remark 2.24.** There exist Banach lattices $E$ and $F$ and there exists a positive Dunford–Pettis operator from $E$ into $F$ which is semi-compact but not necessarily weakly compact. In fact, we take the Banach lattice $E = F = l^1 \oplus l^\infty$. Since the norms of $E$ and $E'$ are not order continuous, it follows, from Theorem 1 of Wickstead [9], the existence of two operators $S$ and $T$ from $E$ into $E$, such that $0 \leq S \leq T$ and $T$ is compact but $S$ is not compact. On the other hand, the lattice operations of $E$ are weakly sequentially continuous (i.e. the sequence $(\|x_n\|)$ converges to 0 for the weak topology $\sigma(E, E')$ whenever the sequence $(x_n)$ converges to 0 for $\sigma(E, E')$), then an application of Theorem 2 of Wickstead [9] implies that $S$ is Dunford–Pettis. Now, as the norms of $E$ and $E'$ are not order continuous, the operator $S$ is not necessarily weakly compact ([8], Theorem 2.2).

## References


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