PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 6, June 2008, Pages 2007–2018
S 0002-9939(08)09043-6
Article electronically published on February 12, 2008

ON THE CONVERGENCE IN CAPACITY ON COMPACT KAHLER MANIFOLDS AND ITS APPLICATIONS

PHAM HOANG HIEP

(Communicated by Mei-Chi Shaw)

ABSTRACT. The main aim of the present note is to study the convergence in $C_{X,\omega}$ on a compact Kahler mainfold X. The obtained results are used to study global extremal functions and describe the ω-pluripolar hull of an ω-pluripolar subset in X.

INTRODUCTION

The convergence in the capacity C_n on domains in $\mathbf{C}^n$ introduced by Bedford and Taylor (see [BT2]) was investigated by Xing and Cegrell (see [Xi1], [Xi2], [Ce3]). Recently Kołodziej (see [Ko2]) introduced the capacity $C_{X,\omega}$ on a compact Kahler manifold X. Next Guedj and Zeriahi studied it in [GZ]. They proved that $C_{X,\omega}$ is locally equivalent to C_n. The main aim of the present note is to study the convergence in $C_{X,\omega}$ on X. The obtained results are used to study global extremal functions and describe the ω-pluripolar hull of an ω-pluripolar subset in X. In section 2, we introduce a characterization of the convergence in $C_{X,\omega}$ of a sequence of ω-plurisubharmonic functions on X. Next we prove under some conditions that the convergence in $C_{X,\omega}$ on X implies the one in $C_{S,\omega|_S}$ where S is a smooth hypersurface in X. By applying this result, in section 3 we prove that if E is an ω-pluripolar set in $X\backslash S$ where S is a smooth hypersurface in X, then $E^*_X \cap S$ is also ω_S-pluripolar in S, where E^*_X denotes the pluripolar hull of E.

For the general definition of the complex Monge-Ampère operator we refer the reader to the papers [BT1], [BT2], [Ce1], [Ce2].

1. PRELIMINARIES

1.1. Let X be a compact Kahler manifold with a fundamental form $\omega = \omega_X$ with $\int_X \omega^n = 1$. An upper semicontinuous function $\varphi :\ X \to [-\infty, +\infty)$ is called ω-plurisubharmonic (ω-psh) if $\omega + dd^c\varphi \geq 0$. By $\mathrm{PSH}(X,\omega)$ (resp $\mathrm{PSH}^-(X,\omega)$) we denote the set of ω-psh (resp. negative ω-psh) functions on X.

Received by the editors September 30, 2006 and, in revised form, December 11, 2006.
2000 *Mathematics Subject Classification.* Primary 32W20; Secondary 32Q15.
Key words and phrases. Complex Monge-Ampère operator, ω-plurisubharmonic functions, compact Kahler manifold.
This work is supported by the National Research Program for Natural Sciences, Vietnam.

1.2. In [Ko2], Kołodziej introduced the capacity $C_{X,\omega}$ on X by

$$C_X(E) = C_{X,\omega}(E) = \sup\{\int_E \omega_\varphi^n : \ \varphi \in \mathrm{PSH}(X,\omega),\ -1 \le \varphi \le 0\}$$

where $\omega_\varphi^n = (\omega + dd^c\varphi)^n$ and $n = \dim X$.

In [GZ], Guedj and Zeriahi proved that C_X is a Choquet capacity on X and

$$C_X(E) = \int_X (-h^*_{E,\omega})\omega^n_{h^*_{E,\omega}}$$

where $h^*_{E,\omega}$ denotes the upper semicontinuous regularization of $h_{E,\omega}$ given by

$$h_{E,\omega}(z) = \sup\{\varphi(z) : \ \varphi \in \mathrm{PSH}^-(X,\omega),\ \varphi|_E \le -1\}.$$

1.3. Let $u_j, u \in \mathrm{PSH}(X,\omega)$. We say that $\{u_j\}$ converges to u in C_X if

$$C_X(\{|u_j - u| > \delta\}) \to 0$$

as $j \to \infty$, for all $\delta > 0$.

1.4. Let S be a smooth hypersurface in X. For each $z \in S$ we find a neighbourhood U of z and a strictly psh function φ on U such that $\omega = dd^c\varphi$. Define $\omega|_S = dd^c\varphi$ on $U \cap S$. Then ω_S is a fundamental form on S. Obviously if $u \in \mathrm{PSH}(X,\omega)$, then $u|_S \in \mathrm{PSH}(S,\omega_S)$.

1.5. Let $E \subset X$. We say that E is ω-pluripolar if there exists $\varphi \in \mathrm{PSH}(X,\omega)$, $\varphi \not\equiv -\infty$ such that $E \subset \{\varphi = -\infty\}$. In [GZ] the authors proved that E is ω-pluripolar if and only if E is locally pluripolar. Define

$$E^*_X = \bigcap\{u = -\infty : \ u \in \mathrm{PSH}(X,\omega),\ u = -\infty \text{ on } E\}.$$

The set E^*_X is called the ω-pluripolar hull of E in X.

2. A characterization of convergence in C_X

In this section we prove the following.

2.1. Theorem. *Let $u_j, u \in PSH(X,\omega)$ be uniformly bounded. Then the following two are equivalent:*

i) *$u_j \to u$ in C_X;*

ii) *$\overline{\lim}_{j\to\infty} u_j \le u$ and $\lim_{j\to\infty}\int_X (u_j - u)\omega^n_{u_j} = 0$.*

Proof. Set

$$M = \max(1, \sup_{j \ge 1} ||u_j||_{L^\infty(X)}, ||u||_{L^\infty(X)}) < +\infty.$$

i) $\Rightarrow$ ii). Given $\delta > 0$, we have

$$\begin{aligned} |\int_X (u_j - u)\omega^n_{u_j}| &= |\int_{\{|u_j - u| < \delta\}} (u_j - u)\omega^n_{u_j} + \int_{\{|u_j - u| \ge \delta\}} (u_j - u)\omega^n_{u_j}| \\ &\le \delta \int_X \omega^n_{u_j} + 2M \int_{\{|u_j - u| \ge \delta\}} \omega^n_{u_j} \\ &\le \delta + (2M)^{n+1} C_X(\{|u_j - u| \ge \delta\}). \end{aligned}$$

It follows that

$$\overline{\lim_{j\to\infty}} \left| \int_X (u_j - u)\omega_{u_j}^n \right| \le \delta.$$

Therefore

$$\overline{\lim_{j\to\infty}} \left| \int_X (u_j - u)\omega_{u_j}^n \right| = 0.$$

Since $u_j \to u$ in C_X, it is easy to check that $\overline{\lim}_{j\to\infty} u_j \le u$. □

ii) ⇒ i) In order to prove ii) ⇒ i) we need two lemmas.

2.2. Lemma. *Let $u, v \in PSH \cap L^\infty(X,\omega)$ be bounded. Then*

$$\left| \int_X d(u-v) \wedge d^c(u-v) \wedge \omega_{\varphi_1} \wedge ... \wedge \omega_{\varphi_{n-1}} \right| \le C\Big(\int_X (v-u)(\omega_u^n - \omega_v^n)\Big)^{2^{1-n}}$$

$\forall\ \varphi_1, ..., \varphi_{n-1} \in PSH(X,\omega),\ -1 \le \varphi_1, ..., \varphi_{n-1} \le 0$, where C is a positive constant depending only on n and $||u||_{L^\infty(X)}||v||_{L^\infty(X)}$.

Proof. As in [Bl] we set

$$\begin{aligned} f &= u - v, \\ a &= \int_X (v-u)(\omega_u^n - \omega_v^n) \\ &= \int_X (v-u) dd^c(u-v) \wedge \Big(\sum_{j=0}^{n-1} \omega_u^j \wedge \omega_v^{n-1-j}\Big) \\ &= \int_X df \wedge d^c f \wedge T, \end{aligned}$$

where

$$T = \sum_{j=0}^{n-1} \omega_u^j \wedge \omega_v^{n-1-j}.$$

For each $k = 0, ..., n-1$ we will prove inductively that

$$\int_X df \wedge d^c f \wedge \omega_u^i \wedge \omega_v^j \wedge \omega_{\varphi_1} \wedge ... \wedge \omega_{\varphi_k} \le C a^{2^{-k}} \tag{1}$$

$\forall\ i, j:\ i + j + k = n - 1$.

If $k = 0$, then

$$\int_X df \wedge d^c f \wedge \omega_u^i \wedge \omega_v^j \le \int_X df \wedge d^c f \wedge T = a.$$

Assume that (1) holds for $k - 1$. We prove by induction on t that

$$\int_X df \wedge d^c f \wedge \omega_u^i \wedge \omega_v^j \wedge \omega_{\varphi_1} \wedge ... \wedge \omega_{\varphi_t} \wedge \omega^{k-t} \le C a^{2^{-k}}. \tag{2}$$

For $t = 0$, (2) holds by Theorem 2 in [Bl]. Set

$$S = \omega_{\varphi_1} \wedge ... \wedge \omega_{\varphi_{t-1}} \wedge \omega^{k-t}.$$

We have

$$\int_X df\wedge d^cf\wedge\omega_u^i\wedge\omega_v^j\wedge\omega_{\varphi_t}\wedge S$$
$$=\int_X df\wedge d^cf\wedge\omega_u^i\wedge\omega_v^j\wedge\omega\wedge S+\int_X df\wedge d^cf\wedge\omega_u^i\wedge\omega_v^j\wedge dd^c\varphi_t\wedge S.$$

Since (2) holds for $t-1$, we only prove that

$$|\int_X df\wedge d^cf\wedge\omega_u^i\wedge\omega_v^j\wedge dd^c\varphi_t\wedge S|\leq Ca^{2^{-k}}.$$

Indeed, by integration by parts we have

$$|\int_X df\wedge d^cf\wedge\omega_u^i\wedge\omega_v^j\wedge dd^c\varphi_t\wedge S|$$
$$=|\int_X d^c\varphi_t\wedge df\wedge dd^cf\wedge\omega_u^i\wedge\omega_v^j\wedge S|$$
$$=|\int_X df\wedge d^c\varphi_t\wedge dd^cf\wedge\omega_u^i\wedge\omega_v^j\wedge S|$$
$$\leq|\int_X df\wedge d^c\varphi_t\wedge\omega_u\wedge\omega_u^i\wedge\omega_v^j\wedge S|+|\int_X df\wedge d^c\varphi_t\wedge\omega_v\wedge\omega_u^i\wedge\omega_v^j\wedge S|$$
$$=|\int_X df\wedge d^c\varphi_t\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S|+|\int_X df\wedge d^c\varphi_t\wedge\omega_u^i\wedge\omega_v^{j+1}\wedge S|.$$

By the Schwarz inequality it follows that

$$|\int_X df\wedge d^c\varphi_t\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S|^2$$
$$\leq\int_X df\wedge d^cf\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S\int_X d\varphi_t\wedge d^c\varphi_t\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S$$
$$=\int_X df\wedge d^cf\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S\int_X -\varphi_t dd^c\varphi_t\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S$$
$$\leq\int_X df\wedge d^cf\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S\int_X -\varphi_t\omega_{\varphi_t}\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S$$
$$\leq\int_X df\wedge d^cf\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S\int_X \omega_{\varphi_t}\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S$$
$$=\int_X df\wedge d^cf\wedge\omega_u^{i+1}\wedge\omega_v^j\wedge S$$
$$\leq Ca^{2^{1-k}}$$

(because (1) holds for $k-1$).

Therefore

$$|\int\limits_X df \wedge d^c\varphi_t \wedge \omega_u^{i+1} \wedge \omega_v^j \wedge S| \leq Ca^{2^{-k}}.$$

Similarly

$$|\int\limits_X df \wedge d^c\varphi_t \wedge \omega_u^{i} \wedge \omega_v^{j+1} \wedge S| \leq Ca^{2^{-k}}.$$

□

2.3. Lemma. *Let $u_j, u \in PSH(X,\omega)$ be uniformly bounded. Then the following are equivalent:*

i) $u_j \to u$ *in* C_X,

ii) $\overline{\lim}_{j\to\infty} u_j \leq u$ *and* $\lim\limits_{j\to\infty} \int_X (\tilde{u}_j - u_j)\omega_{u_j}^n = 0$,

where $\tilde{u}_j = \max(u_j, u)$.

Proof. i) ⇒ ii). This is the same as in i) ⇒ ii) of Theorem 2.1.

ii) ⇒ i). Since $\tilde{u}_j \to u$ and $\tilde{u}_j = \max(u_j, u)$, it is easy to see that $\tilde{u}_j \to u$ in C_X. Thus to prove $u_j \to u$ in C_X, it suffices to show that $\tilde{u}_j - u_j \to 0$ in C_X. Indeed, for every $\delta > 0$ we have

$$C_X(\{\tilde{u}_j - u_j > \delta\}) = \sup\{\int\limits_{\{\tilde{u}_j - u_j > \delta\}} \omega_\varphi^n : \varphi \in \mathrm{PSH}(X,\omega),\ -1 \leq \varphi \leq 0\}$$
$$\leq \frac{1}{\delta}\sup\{\int\limits_X (\tilde{u}_j - u_j)\omega_\varphi^n : \varphi \in \mathrm{PSH}(X,\omega),\ -1 \leq \varphi \leq 0\}.$$

In order to prove the lemma we prove by induction on $k = 0, ..., n$ that

$$\sup\{\int\limits_X (\tilde{u}_j - u_j)\omega_\varphi^k \wedge \omega^{n-k} : \varphi \in \mathrm{PSH}(X,\omega),\ -1 \leq \varphi \leq 0\} \to 0 \tag{1}$$

as $j \to \infty$.

We show that (1) holds for $k = 0$. We assume conversely that

$$\sup\{\int\limits_X (\tilde{u}_j - u_j) \wedge \omega^n : \varphi \in \mathrm{PSH}(X,\omega),\ -1 \leq \varphi \leq 0\} \not\to 0$$

as $j \to \infty$. We may assume that

$$\int\limits_X (\tilde{u}_j - u_j)\omega^n \geq \epsilon_0,\ \forall\ j \geq 1 \tag{2}$$

for some $\epsilon_0 > 0$. By [Ho], we also may assume that $u_j \to v \in \mathrm{PSH}(X,\omega)$ as $j \to \infty$ in $L^1(X)$ with $v \leq u$. Since $\tilde{u}_j - u_j \to u - v$ weakly, it follows that $D(\tilde{u}_j - u_j) \to D(u - v)$ weakly as $j \to \infty$ where $Du = (\frac{\partial u}{\partial z_1}, ..., \frac{\partial u}{\partial z_n}, \frac{\partial u}{\partial \bar{z}_1}, ..., \frac{\partial u}{\partial \bar{z}_n})$. From Lemma 2.2 we have

$$\int\limits_X |D(\tilde{u}_j - u_j)|^2\omega^n = |\int\limits_X d(\tilde{u}_j - u_j) \wedge d^c(\tilde{u}_j - u_j)\omega^{n-1}|$$
$$\leq C(\int\limits_X (\tilde{u}_j - u_j)(\omega_{u_j}^n - \omega_{\tilde{u}_j}^n))^{2^{1-n}}$$
$$\leq C(\int\limits_X (\tilde{u}_j - u_j)\omega_{u_j}^n)^{2^{1-n}} \to 0$$

as $j \to \infty$. Combining this with the weak convergence of $D(\tilde{u}_j - u_j)$ to $D(u-v)$ we have $D(u-v) = 0$. Hence $u - v = c \geq 0$ a.e in X. Since u and v are ω-psh, we have $u - v = c$ on X. We show that $c = 0$. Indeed, we have

$$\begin{aligned}\int\limits_X (\tilde{u}_j - u_j)\omega_{u_j}^n &\geq \int\limits_X (u - u_j)\omega_{u_j}^n \\ &= c\int\limits_X \omega^n + \int\limits_X (v - u_j)\omega_{u_j}^n \\ &= c + \int\limits_X (v - u_j)\omega_{u_j}^n.\end{aligned}$$

Given $\epsilon > 0$, by [BT2] we find an open subset G of X with $C_X(G) < \epsilon$ and j_0 such that

$$u_j(z) \leq v(z) + \epsilon,\ \forall\ j \geq j_0,\ \ z \in X \backslash G.$$

It follows that

$$\begin{aligned}\int\limits_X (v - u_j)\omega_{u_j}^n &\geq -M^{n+1}C_X(G) - \epsilon\int\limits_X \omega_{u_j}^n \\ &\geq -M^{n+1}\epsilon - \epsilon\end{aligned}$$

for $j \geq j_0$. Letting $j \to \infty$ and $\epsilon \to 0$ we obtain

$$\overline{\lim_{j\to\infty}} \int\limits_X (v - u_j)\omega_{u_j}^n \geq 0.$$

There from $ii)$ we have

$$0 = \overline{\lim_{j\to\infty}} \int\limits_X (\tilde{u}_j - u_j)\omega_{u_j}^n \geq c \geq 0.$$

Thus $c = 0$ and $u = v$. This means that $\tilde{u}_j$ and $u_j \to u$ in $L^1(X)$, which contradicts (2).

Assume that (1) holds for $k-1$. For each $\varphi \in \mathrm{PSH}(X,\omega),\ -1 \leq \varphi \leq 0$, we have

$$\begin{aligned}\int\limits_X (\tilde{u}_j - u_j)\omega_\varphi^k \wedge \omega^{n-k} &= \int\limits_X (\tilde{u}_j - u_j)\omega_\varphi^{k-1} \wedge \omega^{n-k+1} \\ &\quad + \int\limits_X (\tilde{u}_j - u_j)dd^c\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\ &= \int\limits_X (\tilde{u}_j - u_j)\omega_\varphi^{k-1} \wedge \omega^{n-k+1} \\ &\quad - \int\limits_X d(\tilde{u}_j - u_j) \wedge d^c\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k}.\end{aligned}$$

By the induction hypothesis it remains to prove that

$$\sup\{|\int_X d(\tilde{u}_j - u_j) \wedge d^c\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k}| : \ \varphi \in \mathrm{PSH}(X,\omega), \ -1 \le \varphi \le 0\} \to 0$$

as $j \to \infty$. Indeed, by the Schwarz inequality, we have

$$\begin{aligned}
&|\int_X d(\tilde{u}_j - u_j) \wedge d^c\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k}|^2 \\
&\quad \le \int_X d\varphi \wedge d^c\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \int_X d(\tilde{u}_j - u_j) \wedge d^c(\tilde{u}_j - u_j) \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\
&\quad = \int_X -\varphi dd^c\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \int_X d(\tilde{u}_j - u_j) \wedge d^c(\tilde{u}_j - u_j) \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\
&\quad \le \int_X -\varphi \omega_\varphi^{k} \wedge \omega^{n-k} \int_X d(\tilde{u}_j - u_j) \wedge d^c(\tilde{u}_j - u_j) \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\
&\quad \le \int_X \omega_\varphi^{k} \wedge \omega^{n-k} \int_X d(\tilde{u}_j - u_j) \wedge d^c(\tilde{u}_j - u_j) \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\
&\quad = \int_X d(\tilde{u}_j - u_j) \wedge d^c(\tilde{u}_j - u_j) \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k}
\end{aligned}$$

(by Lemma 2.2)

$$\begin{aligned}
&\quad \le C(\int_X (\tilde{u}_j - u_j)(\omega_{u_j}^n - \omega_{\tilde{u}_j}^n))^{2^{1-n}} \\
&\quad \le C(\int_X (\tilde{u}_j - u_j)\omega_{u_j}^n)^{2^{1-n}} \to 0
\end{aligned}$$

as $j \to \infty$.

Now we can complete the proof of ii) $\Rightarrow$ i) in Theorem 2.1. By Lemma 2.3 it remains to show that

$$\lim_{j\to\infty} \int_X (\tilde{u}_j - u_j)\omega_{u_j}^n = 0.$$

The equality follows from the hypothesis *ii*) and the convergence of $\tilde{u}_j$ to u in C_X. $\square$

2.4. Theorem. *Let X be a compact Kahler manifold and S a smooth hypersurface in X. Let $u_j, u \in PSH(X,\omega)$ be uniformly bounded such that $u_j \to u$ in C_X and* $\operatorname{supp} \omega_{u_j}^n \subset K \Subset X \backslash S$ *for $j \ge 1$. Then $u_j|_S \to u|_S$ in C_S as $j \to \infty$.*

Proof. Let $\{U_i\}_{i=1,\dots,m}$ be an open cover of X satisfying

i) For each $i = 1, ..., m$, there exists a holomorphic function f_i on a neighbourhood of $\bar{U}_i$ such that $S \cap U_i = \{f_i = 0\}$, $f_i'(z) \ne 0$ for $z \in \bar{U}_i$ and $||f_i||_{L^\infty(U_i)} \le 1$.

ii) For each $i = 1, ..., m$ either $U_i \cap K = \emptyset$ or $U_i \cap S = \emptyset$.

Let $\{\varphi_i\}_{i=1,...,m}$ be a C^∞-partition of unity associated with $\{U_i\}_{i=1,...,m}$. Set $\psi_i = \log|f_i|$, $\forall\ i = 1,...,m$ and $\tilde{u}_j = \max(u_j, u)$, $\forall\ j \geq 1$. Since $u_j \to u$ in C_X, we have $\overline{\lim}_{j\to\infty} u_j \leq u$ in X and hence $\overline{\lim}_{j\to\infty} u_j \leq u$ in S. By Lemma 2.3 it remains to show that

$$\overline{\lim_{j\to\infty}} \int\limits_S (\tilde{u}_j - u_j)\omega^{n-1}_{u_j|_S} \leq 0.$$

Indeed, we have by Corollary 4.2 in [BT3],

$$\begin{aligned}
&\int\limits_S (\tilde{u}_j - u_j)\omega^{n-1}_{u_j|_S} \\
&\quad = \sum_{i=1}^m \int\limits_S \varphi_i(\tilde{u}_j - u_j)\omega^{n-1}_{u_j|_S} \\
&\quad = \sum_{i=1}^m \int\limits_{S\cap U_i} \varphi_i(\tilde{u}_j - u_j)\omega^{n-1}_{u_j|_S} \\
&\quad = \frac{1}{2\pi}\sum_{i=1}^m \int\limits_{U_i} \varphi_i(\tilde{u}_j - u_j)dd^c\psi_i \wedge \omega^{n-1}_{u_j} \\
&\quad = \frac{1}{2\pi}\sum_{i=1}^m \int\limits_{X} \varphi_i(\tilde{u}_j - u_j)dd^c\psi_i \wedge \omega^{n-1}_{u_j} \\
&\quad = -\frac{1}{2\pi}\sum_{i=1}^m \int\limits_{X} (\tilde{u}_j - u_j)d\varphi_i \wedge d^c\psi_i \wedge \omega^{n-1}_{u_j} \\
&\quad\quad - \frac{1}{2\pi}\sum_{i=1}^m \int\limits_{X} \varphi_i d(\tilde{u}_j - u_j) \wedge d^c\psi_i \wedge \omega^{n-1}_{u_j} \\
&\quad = -\frac{1}{2\pi}\sum_{i=1}^m \int\limits_{X} (\tilde{u}_j - u_j)d\varphi_i \wedge d^c\psi_i \wedge \omega^{n-1}_{u_j} \\
&\quad\quad + \frac{1}{2\pi}\sum_{i=1}^m \int\limits_{X} \varphi_i d\psi_i \wedge d^c(u_j - \tilde{u}_j) \wedge \omega^{n-1}_{u_j} \\
&\quad = -\frac{1}{2\pi}\sum_{i=1}^m \int\limits_{X} (\tilde{u}_j - u_j)d\varphi_i \wedge d^c\psi_i \wedge \omega^{n-1}_{u_j} \\
&\quad\quad - \frac{1}{2\pi}\sum_{i=1}^m \int\limits_{X} \psi_i d\varphi_i \wedge d^c(u_j - \tilde{u}_j) \wedge \omega^{n-1}_{u_j} \\
&\quad\quad - \frac{1}{2\pi}\sum_{i=1}^m \int\limits_{X} \varphi_i \psi_i dd^c(u_j - \tilde{u}_j) \wedge \omega^{n-1}_{u_j} \\
&\quad = A_j + B_j + C_j.
\end{aligned}$$

For C_j we have

$$\begin{aligned} C_j &= -\frac{1}{2\pi}\sum_{i=1}^{m}\int_X \varphi_i\psi_i dd^c(u_j-\tilde{u}_j)\wedge\omega_{u_j}^{n-1} \\ &= -\frac{1}{2\pi}\sum_{i=1}^{m}\int_X \varphi_i\psi_i(\omega_{u_j}-\omega_{\tilde{u}_j})\wedge\omega_{u_j}^{n-1} \\ &\leq -\frac{1}{2\pi}\sum_{i=1}^{m}\int_X \varphi_i\psi_i\omega_{u_j}^{n} = 0 \end{aligned}$$

(because supp $\omega_{u_j}^n \subset K$ for $j \geq 1$ and either $U_i \cap K = \emptyset$ or $U_i \cap S = \emptyset$ for $i = 1, ..., m$). Next write

$$\begin{aligned} B_j &= -\frac{1}{2\pi}\sum_{i=1}^{m}\int_X \psi_i d\varphi_i\wedge d^c(u_j-\tilde{u}_j)\wedge\omega_{u_j}^{n-1} \\ &= -\frac{1}{2\pi}\sum_{i=1}^{m}\int_X \psi_i d(u_j-\tilde{u}_j)\wedge d^c\varphi_i\wedge\omega_{u_j}^{n-1} \\ &= -\frac{1}{2\pi}\int_X d(u_j-\tilde{u}_j)\wedge(\sum_{i=1}^{m}\psi_i d^c\varphi_i)\wedge\omega_{u_j}^{n-1}. \end{aligned}$$

Obviously $g = \sum_{i=1}^{m}\psi_i d^c\varphi_i$ is smooth. Indeed, let $z \in X$. We can assume that $\{i = 1, ..., m : z \in U_i\} = \{1, ..., k\}$. Take a neighbourhood V of z such that $V \subset U_i$ for $i = 1, ..., k$ and $V \cap$ supp$\varphi_i = \emptyset$ for $i = k+1, ..., m$. On V we have

$$\begin{aligned} \sum_{i=1}^{m}\psi_i d^c\varphi_i &= \sum_{i=2}^{m}(\psi_i-\psi_1)d^c\varphi_i \\ &= \sum_{i=2}^{k}(\psi_i-\psi_1)d^c\varphi_i \\ &= \sum_{i=2}^{k}(\log\frac{|f_i|}{|f_1|})d^c\varphi_i. \end{aligned}$$

Therefore g is smooth. Thus for B_j we have

$$\begin{aligned} |B_j| &= |\int_X d(u_j-\tilde{u}_j)\wedge g\wedge\omega_{u_j}^{n-1}| \\ &= |\int_X (\tilde{u}_j-u_j)dg\wedge\omega_{u_j}^{n-1}| \leq C\int_X(\tilde{u}_j-u_j)\omega\wedge\omega_{u_j}^{n-1}, \end{aligned}$$

where C is a positive constant independent on g. Since $\tilde{u}_j$ and $u_j \to u$ in C_X, it follows that $B_j \to 0$ as $j \to \infty$.

Similarly as above, $h = \sum_{i=1}^{m} d\varphi_i \wedge d^c\psi_i$ is smooth. Thus we can find $C > 0$ such that

$$|A_j| \leq C\int_X(\tilde{u}_j-u_j)\omega\wedge\omega_{u_j}^{n-1} \to 0$$

as $j \to \infty$. □

From Theorem 2.4 we obtain the following.

2.5. Corollary. *Let X and S be as in Theorem 2.4 and $u_j, u \in PSH(X,\omega)$ such that u_j increases to u a.e. on X and* $\operatorname{supp} \omega_{u_j}^n \subset K \Subset X \backslash S$ *for $j \geq 1$. Then $u_j|_S$ increases to $u|_S$ a.e. on S.*

Remark. Corollary 2.5 was proved by Bedford and Taylor in [BT3] for $X = \mathbf{C}P^n$.

3. Some applications

In this section we apply the results obtained in Section 2 to investigate global extremal functions and ω-plurisubharmonic hulls of ω-pluripolar sets in a compact Kahler manifold X.

Given E a subset of X and Q a function on E, define

$$V_{E,Q} = \sup\{\varphi \in \mathrm{PSH}(X,\omega) : \ \varphi \leq Q \text{ on } E\}.$$

$V_{E,Q}$ is called the global extremal function of E with the weight Q. We write $V_E = V_{E,0}$.

3.1. Theorem. *Let X be a compact Kahler manifold and S a smooth hypersurface in X. Let K be a compact set in $X \backslash S$ and Q be a lower semicontinuous function on K. Then*

$$(V_{K,Q}|_S)^* = V_{K,Q}^*|_S.$$

We need the following.

3.2. Lemma. *Let K be a compact set in X and $\{Q_j\}$ be a sequence of lower semicontinuous functions on K increasing to Q. Then $\{V_{K,Q_j}\}$ increases to $V_{K,Q}$.*

Proof. Let $\varphi \in \mathrm{PSH}(X,\omega)$, $\varphi \leq Q$ on K. Since $\varphi - Q_j \searrow \varphi - Q \leq 0$ on K, by Dini's theorem for every $\epsilon > 0$ there exists j_0 such that $\varphi - Q_j \leq \epsilon$ on K for $j \geq j_0$. This implies that $\varphi - \epsilon \leq V_{K,Q_j}$ for $j \geq j_0$. It follows that $V_{K,Q} \leq \lim_{j\to\infty} V_{K,Q_j}$. Therefore $\lim_{j\to\infty} V_{K,Q_j} = V_{K,Q}$ because obviously $\lim_{j\to\infty} V_{K,Q_j} \leq V_{K,Q}$.

Now we continue the proof of Theorem 3.1. Take a compact ϵ-neighbourhood E of K with $E \subset X \backslash S$ and a sequence Q_j of continuous function on E such that $Q_j \nearrow Q$, where we define $Q = +\infty$ on $E \backslash K$. As in [Si], V_{E,Q_j} is ω-psh continuous and moreover $\operatorname{supp}\omega_{V_{E,Q_j}}^n \subset E \Subset X \backslash S$ for $j \geq 1$. By Lemma 3.2, V_{E,Q_j} increases to $V_{E,Q}^*$ a.e. on X. Corollary 2.5 implies that V_{E,Q_j} increases to $V_{E,Q}^*$ a.e. on S. Therefore we have

$$V_{E,Q} = \lim_{j\to\infty} V_{E,Q_j} = V_{E,Q}^*$$

a.e. on S. It follows that

$$(V_{E,Q}|_S)^* \geq V_{E,Q}^*|_S$$

a.e. on S. Since both functions are ω_S-psh on S we have

$$(V_{E,Q}|_S)^* \geq V_{E,Q}^*|_S.$$

Therefore

$$(V_{E,Q}|_S)^* = V_{E,Q}^*|_S$$

because obviously

$$(V_{E,Q}|_S)^* \leq V_{E,Q}^*|_S.$$

Let $\mathcal{L}(\mathbf{C}^n)$ be the family of plurisubharmonic functions on $\mathbf{C}^n$ that satisfy

$$\varphi(z) \leq \frac{1}{2}\log(1+|z|^2) + C_\varphi, \ z \in \mathbf{C}^n.$$

We consider a 1-to-1 correspondence between $\mathrm{PSH}(\mathbf{C}P^n, \omega_{\mathbf{C}P^n})$ and the homogeneous Lelong class

$$\mathcal{H}(\mathbf{C}^{n+1}) = \{\varphi \in \mathcal{L}(\mathbf{C}^{n+1}) : \ \varphi(tz) = \varphi(z) + \log|t|, \ z \in \mathbf{C}^{n+1}, \ t \in \mathbf{C}\},$$

which is given by the natural mapping

$$\varphi \in \mathcal{H}(\mathbf{C}^{n+1}) \to \tilde{\varphi}(z) = \varphi(z) - \log|z|, \ z \in \mathbf{C}^{n+1}.$$

From the 1-to-1 mapping and Theorem 3.1 we generalize Theorem 1.1 in [Ko1]. □

3.3. Corollary. *Let K be a compact subset in $\mathbf{C}^n$ and Q be a lower semicontinuous function on K. Then*

$$\overline{\lim_{(t,\xi)\to(0,z)}} \psi_{1\times K,Q}(t,\xi) = \overline{\lim_{\xi\to z}} \psi_{1\times K,Q}(0,\xi), \ z \in \mathbf{C}^n$$

where

$$\psi_{1\times K,Q}(t,z) = \sup\{\varphi(t,z) : \ \varphi \in \mathcal{H}(\mathbf{C}^{n+1}), \ \varphi(1,z) \le Q(z) \ \text{on } K\}.$$

3.4. Theorem. *Let X be a compact Kahler manifold and S a smooth hypersurface in X. Let E be an ω-pluripolar subset in $X\backslash S$. Then $E^*_X \cap S$ is also ω_S-pluripolar in S.*

Proof. Take $v \in \mathrm{PSH}(X, \omega)$, $v \not\equiv -\infty$ such that $E \subset \{v = -\infty\}$ and $v \le -1$. Let Ω_j be an increasing sequence of smooth domains exhausting $X\backslash S$. For each $\epsilon > 0$ and $j \ge 1$, set

$$u_{\epsilon,j} = \sup\{\varphi \in \mathrm{PSH}(X,\omega) : \ \varphi \le \max(\epsilon v, -2^j) \text{ on } \Omega_j\}.$$

It is easy to see that for each $j \ge 1$,

$$\max(\epsilon v, -2^j) \le u_{\epsilon,j} \le V_{\Omega_j}, \ \mathrm{supp}\ \omega^n_{u_{\epsilon,j}} \subset \bar{\Omega}_j$$

and $u_{\epsilon,j} \nearrow V_{\Omega_j}$ a.e. on X as $\epsilon \to 0$. By Corollary 2.5 it follows that $u_{\epsilon,j} \nearrow V_{\Omega_j} \ge 0$ on $S\backslash F_j$ as $\epsilon \to 0$, where F_j is an ω_S-pluripolar set in S. Take $z_0 \in S\backslash(\bigcup_{j=1}^\infty F_j)$ and $\epsilon_j > 0$ such that

$$u_{\epsilon,j}(z_0) \ge -\frac{1}{2^j}$$

for $j \ge 1$. Set

$$u = \sum_{j=1}^{\infty} \frac{u_{\epsilon_j,j}}{2^j}.$$

Then u is ω-psh on X satisfying $u = -\infty$ on E. Moreover $u(z_0) \ge -1$. Thus $E^*_X \cap S$ is ω_S-pluripolar in S. The theorem is proved. □

Acknowledgments

The author is grateful to Professor Nguyen Van Khue for suggesting the problem and for many helpful discussions during the preparation of this work. The author is also indebted to the referee for his useful comments that helped to improve the paper.

References

[Bl] Z. Blocki, *Uniqueness and stability for the complex Monge-Ampère equation on compact Kahler manifolds*, Indiana Univ. Math. J. **52** (2003), no. 6, 1697-1701. MR2021054 (2004m:32073)

[BT1] E. Bedford and B. A. Taylor, *The Dirichlet problem for the complex Monge-Ampère operator*, Invent. Math. **37** (1976), 1-44. MR0445006 (56:3351)

[BT2] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), 1-40. MR674165 (84d:32024)

[BT3] E. Bedford and B. A. Taylor, *Plurisubhurmonic functions with logarithmic singularities*, Ann. Inst. Fourier **38** (1988), 133-171. MR978244 (90f:32016)

[BT4] E. Bedford and B. A. Taylor, *Uniqueness for the complex Monge-Ampère equation for functions of logarithmic growth*, Indiana Univ. Math. J. **38** (1989), 455-469. MR997391 (90i:32025)

[Ce1] U. Cegrell, *Pluricomplex energy*, Acta Mathematica **180** (1998), 187-217. MR1638768 (99h:32016)

[Ce2] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), 159-179. MR2069125 (2005d:32062)

[Ce3] U. Cegrell, *Convergence in capacity*, Technical report, Issac Newton Institute for Mathematical Sciences, 2001.

[GZ] V. Guedj and A. Zeriahi, *Intrinsic capacities on compact Kahler manifolds*, J. Geom. Anal. **15** (2005), no. 4, 607-639. MR2203165 (2006j:32041)

[Ho] L. Hörmander, *Notions of Convexity*, Progress in Mathematics **127**, Birkhäuser, Boston, 1994. MR1301332 (95k:00002)

[Ko1] S. Kołodziej, *Capacities associated to the Siciak extremal function*, Ann. Polon. Math. **XLIX** (1989), 279-290. MR997520 (90h:32039)

[Ko2] S. Kołodziej, *The Monge-Ampère equation on compact Kahler manifolds*, Indiana Univ. Math. J. **52** (2003), 667-686. MR1986892 (2004i:32062)

[Si] J. Siciak, *Extremal plurisubharmonic functions in* $\mathbf{C}^n$, Ann. Polon. Math. **XXXIX** (1981), 175-210. MR617459 (83e:32018)

[Xi1] Y. Xing, *Continuity of the complex Monge-Ampère operator*, Proc. Amer. Math. Soc. **124** (1996), 457-467. MR1322940 (96d:32015)

[Xi2] Y. Xing, *Complex Monge-Ampère measures of pluriharmonic functions with bounded values near the boundary*, Canad. J. Math. **52** (2000), 1085-1100. MR1782339 (2001h:32070)

Department of Mathematics, University of Education (Dai hoc Su Pham Ha Noi), CauGiay, Hanoi, Vietnam

E-mail address: phhiep_vn@yahoo.com