ON THE CONVERGENCE IN CAPACITY ON COMPACT KÄHLER MANIFOLDS AND ITS APPLICATIONS

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Abstract. The main aim of the present note is to study the convergence in $C_{X,\omega}$ on a compact Kähler manifold $X$. The obtained results are used to study global extremal functions and describe the $\omega$-pluripolar hull of an $\omega$-pluripolar subset in $X$.

Introduction

The convergence in the capacity $C_n$ on domains in $\mathbb{C}^n$ introduced by Bedford and Taylor (see [BT2]) was investigated by Xing and Cegrell (see [Xi1], [Xi2], [Ce3]). Recently Kołodziej (see [Ko2]) introduced the capacity $C_{X,\omega}$ on a compact Kähler manifold $X$. Next Guedj and Zeriahi studied it in [GZ]. They proved that $C_{X,\omega}$ is locally equivalent to $C_n$. The main aim of the present note is to study the convergence in $C_{X,\omega}$ on $X$. The obtained results are used to study global extremal functions and describe the $\omega$-pluripolar hull of an $\omega$-pluripolar subset in $X$.

In section 2, we introduce a characterization of the convergence in $C_{X,\omega}$ of a sequence of $\omega$-plurisubharmonic functions on $X$. Next we prove under some conditions that the convergence in $C_{X,\omega}$ on $X$ implies the one in $C_{S,\omega|_S}$ where $S$ is a smooth hypersurface in $X$. By applying this result, in section 3 we prove that if $E$ is an $\omega$-pluripolar set in $X\setminus S$ where $S$ is a smooth hypersurface in $X$, then $E_X^* \cap S$ is also $\omega_S$-pluripolar in $S$, where $E_X^*$ denotes the pluripolar hull of $E$.

For the general definition of the complex Monge-Ampère operator we refer the reader to the papers [BT1], [BT2], [Ce1], [Ce2].

1. Preliminaries

1.1. Let $X$ be a compact Kähler manifold with a fundamental form $\omega = \omega_X$ with $\int_X \omega^n = 1$. An upper semicontinuous function $\varphi : X \to [-\infty, +\infty)$ is called $\omega$-plurisubharmonic ($\omega$-psh) if $\omega + dd^c\varphi \geq 0$. By PSH$(X,\omega)$ (resp PSH$^-(X,\omega)$) we denote the set of $\omega$-psh (resp. negative $\omega$-psh) functions on $X$.

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1.2. In [Ko2], Kolodziej introduced the capacity \( C_{X,\omega} \) on \( X \) by
\[
C_X(E) = C_{X,\omega}(E) = \sup \left\{ \int_E \omega^n : \varphi \in \text{PSH}(X, \omega), -1 \leq \varphi \leq 0 \right\}
\]
where \( \omega^n = (\omega + dd^c \varphi)^n \) and \( n = \dim X \).

In [GZ], Guedj and Zeriahi proved that \( C_X \) is a Choquet capacity on \( X \) and
\[
C_X(E) = \int_X (h_{E,\omega}^*)^n \omega^n_{h_{E,\omega}}
\]
where \( h_{E,\omega}^* \) denotes the upper semicontinuous regularization of \( h_{E,\omega} \) given by
\[
h_{E,\omega}(z) = \sup \{ \varphi(z) : \varphi \in \text{PSH}^-(X, \omega), \varphi|_E \leq -1 \}.
\]

1.3. Let \( u_j, u \in \text{PSH}(X, \omega) \). We say that \( \{u_j\} \) converges to \( u \) in \( C_X \) if
\[
C_X(\{|u_j - u| > \delta\}) \to 0
\]
as \( j \to \infty \), for all \( \delta > 0 \).

1.4. Let \( S \) be a smooth hypersurface in \( X \). For each \( z \in S \) we find a neighbourhood \( U \) of \( z \) and a strictly psh function \( \varphi \) on \( U \) such that \( \omega = dd^c \varphi \). Define \( \omega|_S = dd^c \varphi \) on \( U \cap S \). Then \( \omega|_S \) is a fundamental form on \( S \). Obviously if \( u \in \text{PSH}(X, \omega) \), then \( u|_S \in \text{PSH}(S, \omega|_S) \).

1.5. Let \( E \subset X \). We say that \( E \) is \( \omega \)-pluripolar if there exists \( \varphi \in \text{PSH}(X, \omega) \), \( \varphi \neq -\infty \) such that \( E \subset \{ \varphi = -\infty \} \). In [GZ] the authors proved that \( E \) is \( \omega \)-pluripolar if and only if \( E \) is locally pluripolar. Define
\[
E_X^\omega = \bigcap \{ u = -\infty : u \in \text{PSH}(X, \omega), u = -\infty \text{ on } E \}.
\]
The set \( E_X^\omega \) is called the \( \omega \)-pluripolar hull of \( E \) in \( X \).

2. A CHARACTERIZATION OF CONVERGENCE IN \( C_X \)

In this section we prove the following.

2.1. Theorem. Let \( u_j, u \in \text{PSH}(X, \omega) \) be uniformly bounded. Then the following two are equivalent:

i) \( u_j \to u \) in \( C_X \);

ii) \( \lim_{j \to \infty} u_j \leq u \) and \( \lim_{j \to \infty} \int_X (u_j - u) \omega^n_{u_j} = 0 \).

Proof. Set
\[
M = \max(1, \sup_{j \geq 1} \|u_j\|_{L^\infty(X)}, \|u\|_{L^\infty(X)}) < +\infty.
\]

i) \( \Rightarrow \) ii). Given \( \delta > 0 \), we have
\[
\left| \int_X (u_j - u) \omega^n_{u_j} \right| = \left| \int_{\{|u_j - u| < \delta\}} (u_j - u) \omega^n_{u_j} + \int_{\{|u_j - u| \geq \delta\}} (u_j - u) \omega^n_{u_j} \right| 
\]
\[
\leq \delta \int_X \omega^n_{u_j} + 2M \int_{\{|u_j - u| \geq \delta\}} \omega^n_{u_j} 
\]
\[
\leq \delta + (2M)^n C_X(\{|u_j - u| \geq \delta\}) .
\]
It follows that
\[ \lim_{j \to \infty} \int_X (u_j - u) \omega^n_{u_j} \leq \delta. \]
Therefore
\[ \lim_{j \to \infty} \int_X (u_j - u) \omega^n_{u_j} = 0. \]
Since \( u_j \to u \) in \( C_X \), it is easy to check that \( \lim_{j \to \infty} u_j \leq u \). 
\( \square \)

(ii) \( \Rightarrow \) (i) In order to prove (ii) \( \Rightarrow \) (i) we need two lemmas.

2.2. Lemma. Let \( u, v \in PSH \cap L^\infty(X, \omega) \) be bounded. Then
\[ | \int_X d(u - v) \wedge d^c(u - v) \wedge \omega_{\varphi_1} \wedge \ldots \wedge \omega_{\varphi_{n-1}} | \leq C \left( \int_X (v - u)(\omega^n_u - \omega^n_v) \right)^{2^{n-1}} \]
\( \forall \varphi_1, \ldots, \varphi_{n-1} \in PSH(X, \omega), -1 \leq \varphi_1, \ldots, \varphi_{n-1} \leq 0 \), where \( C \) is a positive constant depending only on \( n \) and \( \|u\|_{L^\infty(X)}\|v\|_{L^\infty(X)} \).

Proof. As in [Bl] we set
\[ f = u - v, \]
\[ a = \int_X (v - u)(\omega^n_u - \omega^n_v) \]
\[ = \int_X (v - u)d^cf(u - v) \wedge \left( \sum_{j=0}^{n-1} \omega^j_u \wedge \omega^{n-1-j}_v \right) \]
\[ = \int_X df \wedge d^c f \wedge T, \]
where
\[ T = \sum_{j=0}^{n-1} \omega^j_u \wedge \omega^{n-1-j}_v. \]

For each \( k = 0, \ldots, n-1 \) we will prove inductively that
\[ \int_X df \wedge d^c f \wedge \omega_i^u \wedge \omega_j^v \wedge \omega_{\varphi_1} \wedge \ldots \wedge \omega_{\varphi_k} \leq Ca^{2^{-k}} \]
\( \forall i, j : i + j + k = n - 1. \)
If \( k = 0 \), then
\[ \int_X df \wedge d^c f \wedge \omega_i^u \wedge \omega_j^v \leq \int_X df \wedge d^c f \wedge T = a. \]
Assume that (1) holds for \( k - 1 \). We prove by induction on \( t \) that
\[ \int_X df \wedge d^c f \wedge \omega_i^u \wedge \omega_j^v \wedge \omega_{\varphi_1} \wedge \ldots \wedge \omega_{\varphi_{t-1}} \wedge \omega^{k-t} \leq Ca^{2^{-k}}. \]
For \( t = 0 \), (2) holds by Theorem 2 in [Bl]. Set
\[ S = \omega_{\varphi_1} \wedge \ldots \wedge \omega_{\varphi_{t-1}} \wedge \omega^{k-t}. \]
We have
\[
\int_X df \wedge d^c f \wedge \omega^i_u \wedge \omega^j_v \wedge S
= \int_X df \wedge d^c f \wedge \omega^i_u \wedge \omega^j_v \wedge \omega \wedge S + \int_X df \wedge d^c f \wedge \omega^i_u \wedge \omega^j_v \wedge dd^c \varphi_t \wedge S.
\]
Since (2) holds for \( t - 1 \), we only prove that
\[
\left| \int_X df \wedge d^c f \wedge \omega^i_u \wedge \omega^j_v \wedge dd^c \varphi_t \wedge S \right| \leq C a^{2^{-k}}.
\]
Indeed, by integration by parts we have
\[
\left| \int_X df \wedge d^c f \wedge \omega^i_u \wedge \omega^j_v \wedge dd^c \varphi_t \wedge S \right|
= \left| \int_X d^c \varphi_t \wedge df \wedge dd^c f \wedge \omega^i_u \wedge \omega^j_v \wedge S \right|
= \left| \int_X df \wedge d^c \varphi_t \wedge dd^c f \wedge \omega^i_u \wedge \omega^j_v \wedge S \right|
\leq \left| \int_X df \wedge d^c \varphi_t \wedge \omega^i_u \wedge \omega^j_v \wedge S \right| + \left| \int_X df \wedge d^c \varphi_t \wedge \omega^i_u \wedge \omega^j_v \wedge S \right|
= \left| \int_X df \wedge d^c \varphi_t \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S \right| + \left| \int_X df \wedge d^c \varphi_t \wedge \omega^i_u \wedge \omega^{j+1}_v \wedge S \right|.
\]
By the Schwarz inequality it follows that
\[
\left| \int_X df \wedge d^c \varphi_t \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S \right|^2
\leq \int_X df \wedge d^c f \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S \int_X d\varphi_t \wedge d^c \varphi_t \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S
= \int_X df \wedge d^c f \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S \int_X -\varphi_t dd^c \varphi_t \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S
\leq \int_X df \wedge d^c f \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S \int_X -\varphi_t \omega \varphi_t \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S
\leq \int_X df \wedge d^c f \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S \int_X \omega \varphi_t \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S
= \int_X df \wedge d^c f \wedge \omega^{i+1}_u \wedge \omega^j_v \wedge S
\leq C a^{2^{-k}}
\] (because (1) holds for \( k - 1 \)).
Therefore
\[
\int_X |df \wedge d^c \varphi_t \wedge \omega^1_u \wedge \omega^1_v \wedge S| \leq Ca^{2^{-k}}.
\]
Similarly
\[
\int_X |df \wedge d^c \varphi_t \wedge \omega^1_u \wedge \omega^1_v \wedge S| \leq Ca^{2^{-k}}.
\]

2.3. Lemma. Let \( u_j, u \in PSH(X, \omega) \) be uniformly bounded. Then the following are equivalent:

i) \( u_j \to u \) in \( C_X \),

ii) \( \lim_{j \to \infty} u_j \leq u \) and \( \lim_{j \to \infty} \int_X (\tilde{u}_j - u_j)\omega^n_{u_j} = 0 \),

where \( \tilde{u}_j = \max(u_j, u) \).

Proof. i) \( \Rightarrow \) ii). This is the same as in i) \( \Rightarrow \) ii) of Theorem 2.1.

ii) \( \Rightarrow \) i). Since \( \tilde{u}_j \to u \) and \( \tilde{u}_j = \max(u_j, u) \), it is easy to see that \( \tilde{u}_j \to u \) in \( C_X \).

Thus to prove \( u_j \to u \) in \( C_X \), it suffices to show that \( \tilde{u}_j - u_j \to 0 \) in \( C_X \). Indeed, for every \( \delta > 0 \) we have
\[
C_X(\{\tilde{u}_j - u_j > \delta\}) = \sup\{ \int \omega^n_\varphi : \varphi \in PSH(X, \omega), -1 \leq \varphi \leq 0 \} \leq \frac{1}{\delta} \sup\{ \int (\tilde{u}_j - u_j)\omega^n_\varphi : \varphi \in PSH(X, \omega), -1 \leq \varphi \leq 0 \}.
\]

In order to prove the lemma we prove by induction on \( k = 0, ..., n \) that
\[
(1) \quad \sup\{ \int_X (\tilde{u}_j - u_j)\omega^n_k \wedge \omega^{n-k} : \varphi \in PSH(X, \omega), -1 \leq \varphi \leq 0 \} \to 0
\]
as \( j \to \infty \).

We show that (1) holds for \( k = 0 \). We assume conversely that
\[
\sup\{ \int_X (\tilde{u}_j - u_j) \wedge \omega^n : \varphi \in PSH(X, \omega), -1 \leq \varphi \leq 0 \} \not\to 0
\]
as \( j \to \infty \). We may assume that
\[
(2) \quad \int_X (\tilde{u}_j - u_j)\omega^n \geq \epsilon_0, \ \forall \ j \geq 1
\]
for some \( \epsilon_0 > 0 \). By [Ho], we also may assume that \( u_j \to v \in PSH(X, \omega) \) as \( j \to \infty \) in \( L^1(X) \) with \( v \leq u \). Since \( \tilde{u}_j - u_j \to u - v \) weakly, it follows that \( D(\tilde{u}_j - u_j) \to D(u - v) \) weakly as \( j \to \infty \) where \( Du = (\partial u / \partial z_1, ..., \partial u / \partial z_n, \partial u / \partial z_1, ..., \partial u / \partial z_n) \).

From Lemma 2.2 we have
\[
\int_X |D(\tilde{u}_j - u_j)|^2 \omega^n = \int_X |d(\tilde{u}_j - u_j) \wedge d^c(\tilde{u}_j - u_j)\omega^{n-1}| \leq C(\int_X (\tilde{u}_j - u_j)(\omega^n_{u_j} - \omega^n_{u_j}))^2 \to 0
\]
as \( j \to \infty \). Combining this with the weak convergence of \( D(\tilde{u}_j - u_j) \) to \( D(u - v) \) we have \( D(u - v) = 0 \). Hence \( u - v = c \geq 0 \) a.e in \( X \). Since \( u \) and \( v \) are \( \omega \)-psh, we have \( u - v = c \) on \( X \). We show that \( c = 0 \). Indeed, we have

\[
\int_X (\tilde{u}_j - u_j) \omega^n_{u_j} \geq \int_X (u - u_j) \omega^n_{u_j} = c \int_X \omega^n + \int_X (v - u_j) \omega^n_{u_j} = c + \int_X (v - u_j) \omega^n_{u_j}.
\]

Given \( \epsilon > 0 \), by [BT2] we find an open subset \( G \) of \( X \) with \( C_X(G) < \epsilon \) and \( j_0 \) such that \( u_j(z) \leq v(z) + \epsilon \), \( \forall j \geq j_0 \), \( z \in X \setminus G \).

It follows that

\[
\int_X (v - u_j) \omega^n_{u_j} \geq -M^{n+1}C_X(G) - \epsilon \int_X \omega^n_{u_j} \geq -M^{n+1} \epsilon - \epsilon
\]

for \( j \geq j_0 \). Letting \( j \to \infty \) and \( \epsilon \to 0 \) we obtain

\[
\lim_{j \to \infty} \int_X (v - u_j) \omega^n_{u_j} \geq 0.
\]

There from \( ii) \) we have

\[
0 = \lim_{j \to \infty} \int_X (\tilde{u}_j - u_j) \omega^n_{u_j} \geq c \geq 0.
\]

Thus \( c = 0 \) and \( u = v \). This means that \( \tilde{u}_j \) and \( u_j \to u \) in \( L^1(X) \), which contradicts (2).

Assume that (1) holds for \( k - 1 \). For each \( \varphi \in \text{PSH}(X, \omega), \ -1 \leq \varphi \leq 0 \), we have

\[
\int_X (\tilde{u}_j - u_j) \omega_{\varphi}^k \wedge \omega^{n-k} = \int_X (\tilde{u}_j - u_j) \omega_{\varphi}^{k-1} \wedge \omega^{n-k+1} + \int_X (\tilde{u}_j - u_j) dd^c \varphi \wedge \omega_{\varphi}^{k-1} \wedge \omega^{n-k} = \int_X (\tilde{u}_j - u_j) \omega_{\varphi}^{k-1} \wedge \omega^{n-k+1} - \int_X d(\tilde{u}_j - u_j) \wedge d^c \varphi \wedge \omega_{\varphi}^{k-1} \wedge \omega^{n-k}.
\]
By the induction hypothesis it remains to prove that

\[
\sup_X \left| \int \left( \tilde{u}_j - u_j \right) \wedge d^c \varphi \wedge \omega^{k-1}_\varphi \wedge \omega^{n-k} \right| : \varphi \in \text{PSH}(X, \omega), \ -1 \leq \varphi \leq 0 \to 0
\]

as \( j \to \infty \). Indeed, by the Schwarz inequality, we have

\[
\left| \int_X \left( \tilde{u}_j - u_j \right) \wedge d^c \varphi \wedge \omega^{k-1}_\varphi \wedge \omega^{n-k} \right|^2 \leq \int_X d\varphi \wedge d^c \varphi \wedge \omega^{k-1}_\varphi \wedge \omega^{n-k} \int_X \left( \tilde{u}_j - u_j \right) \wedge d^c \left( \tilde{u}_j - u_j \right) \wedge \omega^{k-1}_\varphi \wedge \omega^{n-k}
\]

\[
= \int_X -\varphi d^{c\varphi} \wedge \omega^{k-1}_\varphi \wedge \omega^{n-k} \int_X \left( \tilde{u}_j - u_j \right) \wedge d^c \left( \tilde{u}_j - u_j \right) \wedge \omega^{k-1}_\varphi \wedge \omega^{n-k}
\]

\[
\leq \int_X -\omega^{k}_\varphi \wedge \omega^{n-k} \int_X \left( \tilde{u}_j - u_j \right) \wedge d^c \left( \tilde{u}_j - u_j \right) \wedge \omega^{k-1}_\varphi \wedge \omega^{n-k}
\]

\[
\leq \int_X \omega^{k}_\varphi \wedge \omega^{n-k} \int_X \left( \tilde{u}_j - u_j \right) \wedge d^c \left( \tilde{u}_j - u_j \right) \wedge \omega^{k-1}_\varphi \wedge \omega^{n-k}
\]

\[
= \int_X \left( \tilde{u}_j - u_j \right) \wedge d^c \left( \tilde{u}_j - u_j \right) \wedge \omega^{k-1}_\varphi \wedge \omega^{n-k}
\]

(by Lemma 2.2)

\[
\leq C \left( \int_X \left( \tilde{u}_j - u_j \right) (\omega^0_{u_j} - \omega^0_{\tilde{u}_j}) \right)^{2^{1-n}}
\]

\[
\leq C \left( \int_X \left( \tilde{u}_j - u_j \right) \omega^0_{u_j} \right)^{2^{1-n}} \to 0
\]

as \( j \to \infty \).

Now we can complete the proof of ii) \( \Rightarrow \) i) in Theorem 2.1. By Lemma 2.3 it remains to show that

\[
\lim_{j \to \infty} \int_X \left( \tilde{u}_j - u_j \right) \omega^n_{u_j} = 0.
\]

The equality follows from the hypothesis ii) and the convergence of \( \tilde{u}_j \) to \( u \) in \( C_X \). \( \square \)

2.4. Theorem. Let \( X \) be a compact Kahler manifold and \( S \) a smooth hypersurface in \( X \). Let \( u_j, u \in \text{PSH}(X, \omega) \) be uniformly bounded such that \( u_j \to u \) in \( C_X \) and \( \text{supp} \ \omega^n_{u_j} \subset \Omega \subset X \setminus S \) for \( j \geq 1 \). Then \( u_j|_S \to u|_S \) in \( C_S \) as \( j \to \infty \).

Proof. Let \( \{U_i\}_{i=1}^m \) be an open cover of \( X \) satisfying

i) For each \( i = 1, \ldots, m \), there exists a holomorphic function \( f_i \) on a neighbourhood of \( U_i \) such that \( S \cap U_i = \{ f_i = 0 \} \), \( f_i(z) \neq 0 \) for \( z \in U_i \) and \( ||f_i||_{L^\infty(U_i)} \leq 1 \).

ii) For each \( i = 1, \ldots, m \) either \( U_i \cap K = \emptyset \) or \( U_i \cap S = \emptyset \).
Let \( \{ \varphi_i \}_{i=1}^{m} \) be a \( C^\infty \)-partition of unity associated with \( \{ U_i \}_{i=1}^{m} \). Set \( \psi_i = \log |f_i|, \forall \ i = 1, \ldots, m \) and \( \tilde{u}_j = \max(u_j, \bar{u}) \), \( \forall \ j \geq 1 \). Since \( u_j \to u \) in \( C_X \), we have \( \lim_{j \to \infty} u_j \leq u \) in \( X \) and hence \( \lim_{j \to \infty} u_j \leq u \) in \( S \). By Lemma 2.3 it remains to show that

\[
\lim_{j \to \infty} \int_S (\tilde{u}_j - u_j) \omega_{u_j}^{-1} \leq 0.
\]

Indeed, we have by Corollary 4.2 in [BT3],

\[
\int_S (\tilde{u}_j - u_j) \omega_{u_j}^{-1}
= \sum_{i=1}^{m} \int_S \varphi_i (\tilde{u}_j - u_j) \omega_{u_j}^{-1}
= \sum_{i=1}^{m} \int_{S \cap U_i} \varphi_i (\tilde{u}_j - u_j) \omega_{u_j}^{-1}
= \frac{1}{2\pi} \sum_{i=1}^{m} \int_{U_i} \varphi_i (\tilde{u}_j - u_j) dd^c \psi_i \wedge \omega_{u_j}^{-1}
= \frac{1}{2\pi} \sum_{i=1}^{m} \int_{X} \varphi_i (\tilde{u}_j - u_j) dd^c \psi_i \wedge \omega_{u_j}^{-1}
= -\frac{1}{2\pi} \sum_{i=1}^{m} \int_{X} (\tilde{u}_j - u_j) d\varphi_i \wedge d^c \psi_i \wedge \omega_{u_j}^{-1}
+ \frac{1}{2\pi} \sum_{i=1}^{m} \int_{X} \psi_i d\varphi_i \wedge d^c (u_j - \tilde{u}_j) \wedge \omega_{u_j}^{-1}
= A_j + B_j + C_j.
\]
For \( C_j \) we have
\[
C_j = -\frac{1}{2\pi} \sum_{i=1}^{m} \int_{X} \varphi_i \psi_i \, dd^c (u_j - \tilde{u}_j) \wedge \omega_{u_j}^{n-1} \\
= -\frac{1}{2\pi} \sum_{i=1}^{m} \int_{X} \varphi_i \psi_i (\omega_{u_j} - \omega_{\tilde{u}_j}) \wedge \omega_{u_j}^{n-1} \\
\leq -\frac{1}{2\pi} \sum_{i=1}^{m} \int_{X} \varphi_i \psi_i \omega_{u_j}^{n} = 0
\]
(because \( \text{supp} \omega_{u_j}^{n} \subset K \) for \( j \geq 1 \) and either \( U_i \cap K = \emptyset \) or \( U_i \cap S = \emptyset \) for \( i = 1, \ldots, m \)). Next write
\[
B_j = -\frac{1}{2\pi} \sum_{i=1}^{m} \int_{X} \psi_i d\varphi_i \wedge dd^c (u_j - \tilde{u}_j) \wedge \omega_{u_j}^{n-1} \\
= -\frac{1}{2\pi} \sum_{i=1}^{m} \int_{X} \psi_i d(u_j - \tilde{u}_j) \wedge dd^c \varphi_i \wedge \omega_{u_j}^{n-1} \\
= -\frac{1}{2\pi} \int_{X} d(u_j - \tilde{u}_j) \wedge (\sum_{i=1}^{m} \psi_i dd^c \varphi_i) \wedge \omega_{u_j}^{n-1}.
\]
Obviously \( g = \sum_{i=1}^{m} \psi_i dd^c \varphi_i \) is smooth. Indeed, let \( z \in X \). We can assume that \( \{i = 1, \ldots, m : z \in U_i\} = \{1, \ldots, k\} \). Take a neighbourhood \( V \) of \( z \) such that \( V \subset U_i \) for \( i = 1, \ldots, k \) and \( V \cap \text{supp} \varphi_i = \emptyset \) for \( i = k+1, \ldots, m \). On \( V \) we have
\[
\sum_{i=1}^{m} \psi_i dd^c \varphi_i = \sum_{i=2}^{m} (\psi_i - \psi_1) dd^c \varphi_i \\
= \sum_{i=2}^{k} (\psi_i - \psi_1) dd^c \varphi_i \\
= \sum_{i=2}^{k} (\log |f_i|/|f_1|) dd^c \varphi_i.
\]
Therefore \( g \) is smooth. Thus for \( B_j \) we have
\[
|B_j| = \left| \int_{X} d(u_j - \tilde{u}_j) \wedge g \wedge \omega_{u_j}^{n-1} \right| \\
= \left| \int_{X} (\tilde{u}_j - u_j) dg \wedge \omega_{u_j}^{n-1} \right| \leq C \int_{X} (\tilde{u}_j - u_j) \omega \wedge \omega_{u_j}^{n-1},
\]
where \( C \) is a positive constant independent on \( g \). Since \( \tilde{u}_j \) and \( u_j \to u \) in \( C_X \), it follows that \( B_j \to 0 \) as \( j \to \infty \).

Similarly as above, \( h = \sum_{i=1}^{m} d\varphi_i \wedge dd^c \psi_i \) is smooth. Thus we can find \( C > 0 \) such that
\[
|A_j| \leq C \int_{X} (\tilde{u}_j - u_j) \omega \wedge \omega_{u_j}^{n-1} \to 0
\]
as \( j \to \infty \). \qed
Let $X$ and $S$ be as in Theorem 2.4 and $u_j, u \in \text{PSH}(X, \omega)$ such that $u_j$ increases to $u$ a.e. on $X$ and $\text{supp } \omega^{u_j}_n \subset K \in X \setminus S$ for $j \geq 1$. Then $u_j|_S$ increases to $u|_S$ a.e. on $S$.

**Remark.** Corollary 2.5 was proved by Bedford and Taylor in [BT3] for $X = \mathbb{C}P^n$.

### 3. Some applications

In this section we apply the results obtained in Section 2 to investigate global extremal functions and $\omega$-plurisubharmonic hulls of $\omega$-pluripolar sets in a compact Kahler manifold $X$.

Given $E$ a subset of $X$ and $Q$ a function on $E$, define

$$V_{E,Q} = \sup \{ \varphi \in \text{PSH}(X, \omega) : \varphi \leq Q \text{ on } E \}.$$ 

$V_{E,Q}$ is called the global extremal function of $E$ with the weight $Q$. We write $V_E = V_{E,0}$.

**3.1. Theorem.** Let $X$ be a compact Kahler manifold and $S$ a smooth hypersurface in $X$. Let $K$ be a compact set in $X \setminus S$ and $Q$ be a lower semicontinuous function on $K$. Then

$$(V_{K,Q}|_S)^* = V^*_{K,Q}|_S.$$

We need the following.

**3.2. Lemma.** Let $K$ be a compact set in $X$ and $\{Q_j\}$ be a sequence of lower semicontinuous functions on $K$ increasing to $Q$. Then $\{V_{K,Q_j}\}$ increases to $V_{K,Q}$.

**Proof.** Let $\varphi \in \text{PSH}(X, \omega)$, $\varphi \leq Q$ on $K$. Since $\varphi - Q_j \searrow \varphi - Q \leq 0$ on $K$, by Dini’s theorem for every $\epsilon > 0$ there exists $j_0$ such that $\varphi - Q_j \leq \epsilon$ on $K$ for $j \geq j_0$. This implies that $\varphi - \epsilon \leq V_{K,Q_j}$ for $j \geq j_0$. It follows that $V_{K,Q} \leq \lim_{j \to \infty} V_{K,Q_j}$. Therefore $\lim_{j \to \infty} V_{K,Q_j} = V_{K,Q}$ because obviously $\lim_{j \to \infty} V_{K,Q_j} \leq V_{K,Q}$.

Now we continue the proof of Theorem 3.1. Take a compact $\epsilon$-neighbourhood $E$ of $K$ with $E \subset X \setminus S$ and a sequence $Q_j$ of continuous function on $E$ such that $Q_j \searrow Q$, where we define $Q = +\infty$ on $E \setminus K$. As in [Si], $V_{E,Q_j}$ is $\omega$-psh continuous and moreover $\text{supp } \omega^{Q_j}_{E,Q_j} \subset E \subset X \setminus S$ for $j \geq 1$. By Lemma 3.2, $V_{E,Q_j}$ increases to $V^*_{E,Q}$ a.e. on $X$. Corollary 2.5 implies that $V_{E,Q_j}$ increases to $V_{E,Q}$ a.e. on $S$. Therefore we have

$$V_{E,Q} = \lim_{j \to \infty} V_{E,Q_j} = V^*_{E,Q}$$

a.e. on $S$. It follows that

$$(V_{E,Q}|_S)^* \geq V^*_{E,Q}|_S$$

a.e. on $S$. Since both functions are $\omega_S$-psh on $S$ we have

$$(V_{E,Q}|_S)^* \geq V^*_{E,Q}|_S.$$ 

Therefore

$$(V_{E,Q}|_S)^* = V^*_{E,Q}|_S$$

because obviously

$$(V_{E,Q}|_S)^* \leq V^*_{E,Q}|_S.$$ 

Let $\mathcal{L}(\mathbb{C}^n)$ be the family of plurisubharmonic functions on $\mathbb{C}^n$ that satisfy

$$\varphi(z) \leq \frac{1}{2} \log(1 + |z|^2) + C_\varphi, \quad z \in \mathbb{C}^n.$$
We consider a 1-to-1 correspondence between $\text{PSH}(\mathbb{C}P^n, \omega_{\mathbb{C}P^n})$ and the homogeneous Lelong class

$$\mathcal{H}(\mathbb{C}^{n+1}) = \{ \varphi \in \mathcal{L}(\mathbb{C}^{n+1}) : \varphi(tz) = \varphi(z) + \log |t|, \ z \in \mathbb{C}^{n+1}, \ t \in \mathbb{C} \},$$

which is given by the natural mapping

$$\varphi \in \mathcal{H}(\mathbb{C}^{n+1}) \mapsto \tilde{\varphi}(z) = \varphi(z) - \log |z|, \ z \in \mathbb{C}^{n+1}.$$

From the 1-to-1 mapping and Theorem 3.1 we generalize Theorem 1.1 in [Ko1]. □

3.3. Corollary. Let $K$ be a compact subset in $\mathbb{C}^n$ and $Q$ be a lower semicontinuous function on $K$. Then

$$\lim_{(t, \xi) \to (0, z)} \psi_{1 \times K, Q}(t, \xi) = \lim_{\xi \to z} \psi_{1 \times K, Q}(0, \xi), \ z \in \mathbb{C}^n$$

where

$$\psi_{1 \times K, Q}(t, z) = \sup \{ \varphi(t, z) : \varphi \in \mathcal{H}(\mathbb{C}^{n+1}), \varphi(1, z) \leq Q(z) \text{ on } K \}.$$

3.4. Theorem. Let $X$ be a compact Kähler manifold and $S$ a smooth hypersurface in $X$. Let $E$ be an $\omega$-pluripolar subset in $X \setminus S$. Then $E^* \cap S$ is also $\omega_S$-pluripolar in $S$.

Proof. Take $v \in \text{PSH}(X, \omega)$, $v \not\equiv -\infty$ such that $E \subset \{ v = -\infty \}$ and $v \leq -1$. Let $\Omega_j$ be an increasing sequence of smooth domains exhausting $X \setminus S$. For each $\epsilon > 0$ and $j \geq 1$, set

$$u_{\epsilon, j} = \sup \{ \varphi \in \text{PSH}(X, \omega) : \varphi \leq \max(\epsilon v, -2^j) \text{ on } \Omega_j \}.$$

It is easy to see that for each $j \geq 1$,

$$\max(\epsilon v, -2^j) \leq u_{\epsilon, j} \leq V_{\Omega_j}, \ \text{supp } \omega^n_{u_{\epsilon, j}} \subset \bar{\Omega}_j$$

and $u_{\epsilon, j} \not\equiv V_{\Omega_j}$ a.e. on $X$ as $\epsilon \to 0$. By Corollary 2.5 it follows that $u_{\epsilon, j} \not\equiv V_{\Omega_j}$ ≥ 0 on $S \setminus F_j$ as $\epsilon \to 0$, where $F_j$ is an $\omega_S$-pluripolar set in $S$. Take $z_0 \in S \setminus (\bigcup_{j=1}^{\infty} F_j)$ and $\epsilon_j > 0$ such that

$$u_{\epsilon, j}(z_0) \geq -\frac{1}{2^j}$$

for $j \geq 1$. Set

$$u = \sum_{j=1}^{\infty} \frac{u_{\epsilon_j, j}}{2^j}.$$

Then $u$ is $\omega$-psh on $X$ satisfying $u = -\infty$ on $E$. Moreover $u(z_0) \geq -1$. Thus $E^*_X \cap S$ is $\omega_S$-pluripolar in $S$. The theorem is proved. □

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