

ON KREIN'S EXAMPLE

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ABSTRACT. In his 1953 paper [Matem. Sbornik **33** (1953), 597–626] Mark Krein presented an example of a symmetric rank one perturbation of a self-adjoint operator such that for all values of the spectral parameter in the interior of the spectrum, the difference of the corresponding spectral projections is not trace class. In the present note it is shown that in the case in question this difference has simple Lebesgue spectrum filling in the interval $[-1, 1]$ and, therefore, the pair of the spectral projections is generic in the sense of Halmos but not Fredholm.

The spectral shift function plays a very important role in perturbation theory for self-adjoint operators. It was introduced in a special case by I. Lifshitz [8] and in the general case (in the framework of trace class perturbations) by M. Krein in his celebrated 1953 paper [7]. He showed that for a pair of self-adjoint not necessarily bounded operators A_0 and A_1 such that their difference $A_1 - A_0$ is trace class, there exists a unique function $\xi \in L^1(\mathbb{R})$ satisfying the trace formula

$$(1) \quad \operatorname{tr}(\varphi(A_1) - \varphi(A_0)) = \int_{\mathbb{R}} \varphi'(x)\xi(x)dx$$

whenever φ belongs to a class of admissible functions. Surprisingly, the condition $\varphi \in C^1$ alone does not imply the validity of the trace formula (1) (see [9]).

If one goes ahead and formally puts $\varphi = \chi_\mu$ in (1) with χ_μ a characteristic function of the interval $\delta_\mu := (-\infty, \mu)$, one would arrive at the “naive Lifshitz formula”

$$(2) \quad \xi(\mu) = \operatorname{tr}(E_{A_1}(\delta_\mu) - E_{A_0}(\delta_\mu)),$$

where $E_{A_j}(\delta_\mu)$ denotes the spectral projection for the operator A_j , $j = 0, 1$, associated with the semi-infinite interval δ_μ . Note that representation (2) holds for all $\mu \in \Delta$ whenever Δ is a joint spectral gap for the operators A_0 and A_1 .

In general the right hand side of (2) does not make sense: In Section 6 of [7] Krein presented an example of two self-adjoint operators such that their difference $A_1 - A_0$ is of rank one but $E_{A_1}(\delta_\mu) - E_{A_0}(\delta_\mu)$ is not trace class and, hence, the trace in the r.h.s. of (2) is ill-defined (we refer the reader to the review [2] for further discussion).

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In this example Krein introduces a bounded integral operator A_0 in $L^2(0, \infty)$ with the kernel given by

$$A_0(x, y) = \begin{cases} \sinh x e^{-y}, & x \leq y, \\ \sinh y e^{-x}, & x \geq y, \end{cases}$$

and its rank one perturbation A_1 with kernel

$$A_1(x, y) = A_0(x, y) + e^{-x}e^{-y}$$

so that

$$A_1(x, y) = \begin{cases} \cosh x e^{-y}, & x \leq y, \\ \cosh y e^{-x}, & x \geq y. \end{cases}$$

In fact A_j , $j = 0, 1$, are the resolvents of the Dirichlet ($j = 0$) and Neumann ($j = 1$) one-dimensional Laplacian $-d^2/dx^2$ at the spectral point -1 . Obviously, the operators A_j , $j = 0, 1$, have a simple, purely absolutely continuous spectrum filling in the interval $[0, 1]$.

Krein shows that the difference

$$(3) \quad K_\mu := E_{A_1}(\delta_\mu) - E_{A_0}(\delta_\mu), \quad 0 < \mu < 1,$$

of the spectral projections for the operators A_1 and A_0 is the integral operator with the kernel $k_\mu(x + y)$, where

$$k_\mu(x) = \frac{2 \sin \sqrt{\lambda(\mu)} x}{\pi x}$$

with

$$\lambda(\mu) = \frac{1}{\mu} - 1, \quad 0 < \mu < 1.$$

As noted by Krein in [7] the operator K_μ , $0 < \mu < 1$, is not Hilbert-Schmidt, since for any $\lambda > 0$

$$\int_0^\infty \int_0^\infty \frac{\sin^2 \sqrt{\lambda}(x+y)}{(x+y)^2} dx dy = \int_0^\infty \frac{\sin^2 u}{u} du = \infty.$$

Employing the theory of Hankel operators [4] it not hard to show that the operator K_μ , $0 < \mu < 1$, has an absolutely continuous spectrum $[-1, 1]$ of uniform multiplicity one. However, to the best of our knowledge there are no general criteria guaranteeing the absence of singular spectrum for this operator. In particular, the related results of [5], [6] are not applicable to the case in question.

In the present note we perform a detailed spectral analysis of the operator K_μ . Our result is given by the following

Theorem 1. *For any $0 < \mu < 1$ the operator K_μ defined in (3) has a simple purely absolutely continuous spectrum filling in the interval $[-1, 1]$.*

Proof. Observe that K_μ , $0 < \mu < 1$, is unitary equivalent to $K_{\frac{1}{2}}$ with the unitary equivalence given by the scaling transformation

$$U_\lambda : f(x) \mapsto \lambda^{1/4} f(\sqrt{\lambda}x), \quad f \in L^2(0, \infty).$$

That is,

$$K_\mu = U_{\lambda(\mu)} K_{\frac{1}{2}} U_{\lambda(\mu)}^*.$$

Therefore, it suffices to prove the theorem for the operator $K_{\frac{1}{2}}$. For brevity we set $K := K_{\frac{1}{2}}$. Since

$$\frac{2 \sin x}{\pi x} = \frac{1}{2\pi} \int_{-1}^1 2 e^{-itx} dt,$$

the operator K is a Hankel integral operator with a discontinuous symbol

$$s(t) = \begin{cases} 2, & t \in [-1, 1], \\ 0, & t \notin [-1, 1]. \end{cases}$$

In the Hilbert space ℓ^2_+ of complex square-summable one-sided sequences $x = \{x_0, x_1, \dots\}$ we introduce the Hankel operator $H(\phi)$ with the symbol ϕ given by

$$\phi(z) := s \left(i \frac{1-z}{1+z} \right), \quad |z| = 1, \quad z \neq -1,$$

which is the characteristic function of the set $\{z \in \mathbb{C} \mid |z| = 1, \operatorname{Re} z \geq 0\}$. That is,

$$(H(\phi)x)_n = \sum_{k=0}^{\infty} c_{n+k+1} x_k, \quad x = \{x_0, x_1, \dots\} \in \ell^2_+,$$

where c_k are Fourier coefficients of the function ϕ ,

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} \phi(e^{i\theta}) d\theta = \frac{2}{\pi k} \sin(\pi k/2), \quad k \in \mathbb{N}.$$

The operators K and $H(\phi)$ are unitarily equivalent (see, e.g., [10, p. 14]). For any $p \notin \mathbb{N}$ let H_p and \tilde{H}_p be the p -shifted Hilbert matrices on ℓ^2_+ ,

$$(H_p x)_n = \sum_{k=0}^{\infty} (n+k+1-p)^{-1} x_k, \quad (\tilde{H}_p x)_n = \sum_{k=0}^{\infty} (-1)^{n+k} (n+k+1-p)^{-1} x_k.$$

In particular, H_0 is the standard Hilbert matrix.

Denote by \mathcal{L}_+ (respectively \mathcal{L}_-) the set of all sequences in ℓ^2_+ with vanishing odd (respectively even) elements, that is,

$$\begin{aligned} \mathcal{L}_+ &= \{x \in \ell^2_+ \mid x_{2k+1} = 0 \quad \forall k \in \mathbb{N}_0\}, \\ \mathcal{L}_- &= \{x \in \ell^2_+ \mid x_{2k} = 0 \quad \forall k \in \mathbb{N}_0\}. \end{aligned}$$

Let \mathcal{P}_{\pm} be the orthogonal projections in ℓ^2_+ onto \mathcal{L}_{\pm} , respectively.

Observe that $c_k = 0$ for all $k \in 2\mathbb{N}$, $c_k = \frac{2}{\pi k}$ for all $k \in 4\mathbb{N}_0 + 1$, and $c_k = -\frac{2}{\pi k}$ for all $k \in 4\mathbb{N}_0 + 3$. This immediately implies that $\mathcal{P}_+ H(\phi) \mathcal{P}_- = 0$ and $\mathcal{P}_- H(\phi) \mathcal{P}_+ = 0$. Furthermore, for any $x \in \mathcal{L}_+$ we have

$$(\mathcal{P}_+ H(\phi) \mathcal{P}_+ x)_{2n} = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{2(n+k+1/2)} x_{2k}.$$

Thus, $\mathcal{P}_+ H(\phi) \mathcal{P}_+$ is unitarily equivalent to the operator $\pi^{-1} \tilde{H}_{1/2}$ on ℓ^2_+ . Similarly, for any $x \in \mathcal{L}_-$ we have

$$(\mathcal{P}_- H(\phi) \mathcal{P}_- x)_{2n+1} = -\frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{2(n+k+3/2)} x_{2k+1},$$

which implies that $\mathcal{P}_- H(\phi) \mathcal{P}_-$ is unitarily equivalent to $(-\pi^{-1} \tilde{H}_{-1/2})$ on ℓ^2_+ .

Since for any $p \notin \mathbb{N}$ the operators H_p and \tilde{H}_p are unitarily equivalent, we have that the Hankel operator $H(\phi)$ is unitarily equivalent to the orthogonal sum

$(\pi^{-1}H_{1/2}) \oplus (-\pi^{-1}H_{-1/2})$ with respect to the orthogonal decomposition $\ell_+^2 \oplus \ell_+^2$, so is K .

By a result of Rosenblum [11, Theorems 1, 4 and 5] for any $p \leq 1/2$ the operator H_p is unitary equivalent (with the “diagonalization map” given explicitly in Theorem 4 in [11]) to the multiplication operator by $\pi/\sinh(\pi\sqrt{\lambda})$ on $L^2((0, \infty); \rho_p(\lambda)d\lambda)$ with the density

$$\rho_p(\lambda) = \frac{1}{2\pi^2} \sinh(2\pi\sqrt{\lambda}) |\Gamma(1/2 - p - i\sqrt{\lambda})|^2, \quad \lambda > 0,$$

where Γ denotes the gamma function. Thus, the spectrum of the operator H_p , $p \leq 1/2$, is purely absolutely continuous, has uniform multiplicity 1, and fills the interval $[0, \pi]$. Hence, by the argument above, the operator K has a simple purely absolutely continuous spectrum filling in the interval $[-1, 1]$, thereby completing the proof. \square

We conclude our note by a remark of geometric character.

Remark 2. Since for any $\mu \in (0, 1)$ the essential spectrum of K_μ fills in the interval $[-1, 1]$, one concludes that the pair $(E_{A_1}(\delta_\mu), E_{A_0}(\delta_\mu))$ of the spectral projections is not Fredholm in the sense of [1]. Moreover, the absence of the pure point spectrum of K_μ implies, in particular, that

$$\text{Ker}(E_{A_1}(\delta_\mu) - E_{A_0}(\delta_\mu) \pm I) = \{0\},$$

thereby proving that the subspaces $\text{Ran } E_{A_1}(\delta_\mu)$ and $\text{Ran } E_{A_0}(\delta_\mu)$, $\mu \in (0, 1)$, are in a generic position in the sense of Halmos [3].

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