PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 6, June 2008, Pages 2209–2216
S 0002-9939(08)09161-2
Article electronically published on February 7, 2008

A WEIERSTRASS TYPE REPRESENTATION FOR MINIMAL SURFACES IN SOL

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(Communicated by Chuu-Lian Terng)

Dedicated to Professor Takeshi Sasaki on his 60th birthday

Abstract. The normal Gauss map of a minimal surface in the model space Sol of solvegeometry is a harmonic map with respect to a certain singular Riemannian metric on the extended complex plane.

1. Introduction

Since the discovery of a holomorphic quadratic differential (called a *generalized Hopf differential* or an *Abresch-Rosenberg differential*) for CMC surfaces (constant mean curvature surfaces) in 3-dimensional homogeneous Riemannian manifolds with 4-dimensional isometry group, global geometry of constant mean curvature surfaces in such spaces has been extensively studied [1]–[2].

D. A. Berdinskiĭ and I. A. Taĭmanov [4] gave a representation formula for minimal surfaces in 3-dimensional Lie groups in terms of spinors and Dirac operators.

The simply connected homogeneous Riemannian 3-manifolds with 4-dimensional isometry group have a structure of principal fiber bundle with 1-dimensional fiber and constant curvature base. More explicitly, such homogeneous spaces are one of the following spaces: the Heisenberg group Nil, the universal covering $\widetilde{\mathrm{SL}}_2\mathbb{R}$ of the special linear group equipped with naturally reductive metric, the the special unitary group SU(2) equipped with the Berger sphere metric, and the reducible Riemannian symmetric space $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$.

On the other hand, the model spaces of Thurston's 3-dimensional model geometries [10] are space forms, Nil, $\widetilde{\mathrm{SL}}_2\mathbb{R}$ with naturally reductive metric, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$ and the space Sol, the model space of solvegeometry.

Abresch and Rosenberg showed that the existence of a generalized Hopf differential in a simply connected Riemannian 3-manifold is equivalent to the property that the ambient space has at least a 4-dimensional isometry group [2, Theorem 5]. Note that if the dimension of the isometry group of a Riemannian 3-manifold is greater than 3, then the action of the isometry group is transitive.

Thus for the space Sol, one cannot expect an Abresch-Rosenberg-type quadratic differential for CMC surfaces.

Received by the editors September 26, 2006.
2000 *Mathematics Subject Classification.* Primary 53A10, 53C15, 53C30.
Key words and phrases. Solvable Lie groups, minimal surfaces.
The first author was partially supported by Kakenhi 18540068.

Thus, another approach for CMC surface geometry in Sol is expected.

The space Sol belongs to the following two parameter family of simply connected homogeneous Riemannian 3-manifolds:

$$G(\mu_1,\mu_2) = (\mathbb{R}^3(x^1,x^2,x^3), g_{(\mu_1,\mu_2)}),$$

with group structure

$$(x^1,x^2,x^3)\cdot(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3) = (x^1+e^{\mu_1 x^3}\tilde{x}^1, x^2+e^{\mu_2 x^3}\tilde{x}^2, x^3+\tilde{x}^3)$$

and left invariant metric

$$g_{(\mu_1,\mu_2)} = e^{-2\mu_1 x^3}(dx^1)^2 + e^{-2\mu_2 x^3}(dx^2)^2 + (dx^3)^2.$$

This family includes $\mathrm{Sol} = G(1,-1)$ as well as the Euclidean 3-space $\mathbb{E}^3 = G(0,0)$, hyperbolic 3-space $H^3 = G(1,1)$ and $H^2\times\mathbb{R} = G(0,1)$.

In this paper, we study the (normal) Gauss map of minimal surfaces in $G(\mu_1,\mu_2)$. In particular, we shall show that the normal Gauss map of non-vertical minimal surfaces is a harmonic map with respect to an appropriate metric if and only if $\mu_1^2 = \mu_2^2$.

As a consequence, we shall give a Weierstrass-type representation formula for minimal surfaces in Sol.

The results of this article were partially reported at the London Mathematical Society Durham Conference "Methods of Integrable Systems in Geometry" (August, 2006).

2. Solvable Lie group

In this paper, we study the following two-parameter family of homogeneous Riemannian 3-manifolds:

$$\left\{(\mathbb{R}^3(x^1,x^2,x^3), g_{(\mu_1,\mu_2)}) \mid (\mu_1,\mu_2)\in\mathbb{R}^2\right\}, \tag{2.1}$$

where the metrics $g = g_{(\mu_1,\mu_2)}$ are defined by

$$g_{(\mu_1,\mu_2)} := e^{-2\mu_1 x^3}(dx^1)^2 + e^{-2\mu_2 x^3}(dx^2)^2 + (dx^3)^2. \tag{2.2}$$

Each homogeneous space $(\mathbb{R}^3, g_{(\mu_1,\mu_2)})$ is realized as the following solvable matrix Lie group:

$$G(\mu_1,\mu_2) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x^3 \\ 0 & e^{\mu_1 x^3} & 0 & x^1 \\ 0 & 0 & e^{\mu_2 x^3} & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x^1,x^2,x^3\in\mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g}(\mu_1,\mu_2)$ is given explicitly by

$$\mathfrak{g}(\mu_1,\mu_2) = \left\{ \begin{pmatrix} 0 & 0 & 0 & y^3 \\ 0 & \mu_1 y^3 & 0 & y^1 \\ 0 & 0 & \mu_2 y^3 & y^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; y^1,y^2,y^3\in\mathbb{R} \right\}. \tag{2.3}$$

Then we can take the following orthonormal basis $\{E_1,E_2,E_3\}$ of $\mathfrak{g}(\mu_1,\mu_2)$:

$$E_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, E_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, E_3 = \begin{pmatrix} 0&0&0&1\\0&\mu_1&0&0\\0&0&\mu_2&0\\0&0&0&0 \end{pmatrix}.$$

Then the commutation relation of $\mathfrak{g}$ is given by

$$[E_1, E_2] = 0,\ [E_2, E_3] = -\mu_2 E_2,\ [E_3, E_1] = \mu_1 E_1.$$

Left-translating the basis $\{E_1, E_2, E_3\}$, we obtain the following orthonormal frame field:

$$e_1 = e^{\mu_1 x^3} \frac{\partial}{\partial x^1},\ e_2 = e^{\mu_2 x^3} \frac{\partial}{\partial x^2},\ e_3 = \frac{\partial}{\partial x^3}.$$

One can easily check that every $G(\mu_1, \mu_2)$ is a *non-unimodular Lie group* except $\mu_1 + \mu_2 = 0$.

The Levi-Civita connection ∇ of $G(\mu_1, \mu_2)$ is described by

$$\begin{array}{ccc} \nabla_{e_1} e_1 = \mu_1 e_3, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = -\mu_1 e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = \mu_2 e_3, & \nabla_{e_2} e_3 = -\mu_2 e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0. \end{array} \tag{2.4}$$

Example 2.1 (Euclidean 3-space)**.** The Lie group $G(0, 0)$ is isomorphic and isometric to the Euclidean 3-space $\mathbb{E}^3 = (\mathbb{R}^3, +)$.

Example 2.2 (Hyperbolic 3-space)**.** Take $\mu_1 = \mu_2 = c \neq 0$. Then $G(c, c)$ is a warped product model of the hyperbolic 3-space:

$$H^3(-c^2) = (\mathbb{R}^3(x^1, x^2, x^3), e^{-2cx^3}\{(dx^1)^2 + (dx^2)^2\} + (dx^3)^2).$$

Example 2.3 (Riemannian product $H^2(-c^2) \times \mathbb{E}^1$)**.** Take $(\mu_1, \mu_2) = (0, c)$ with $c \neq 0$. Then the resulting homogeneous space is $\mathbb{R}^3$ with metric:

$$(dx^1)^2 + e^{-2cx^3}(dx^2)^2 + (dx^3)^2.$$

Hence $G(0, c)$ is identified with the Riemannian direct product of the Euclidean line $\mathbb{E}^1(x^1)$ and the warped product model

$$(\mathbb{R}^2(x^2, x^3), e^{-2cx^3}(dx^2)^2 + (dx^3)^2\ )$$

of $H^2(-c^2)$. Thus $G(0, c)$ is identified with $\mathbb{E}^1 \times H^2(-c^2)$.

Example 2.4 (Solvmanifold)**.** The model space Sol of the 3-dimensional *solvegeometry* [10] is $G(1, -1)$. The Lie group $G(1, -1)$ is isomorphic to the Minkowski motion group

$$E(1, 1) := \left\{ \left(\begin{array}{ccc} e^{x^3} & 0 & x^1 \\ 0 & e^{-x^3} & x^2 \\ 0 & 0 & 1 \end{array} \right) \middle| \ x^1, x^2, x^3 \in \mathbb{R} \right\}.$$

The full isometry group is $G(1, -1)$ itself.

Example 2.5. Since $[e_1, e_2] = 0$, the distribution D spanned by e_1 and e_2 is involutive. The maximal integral surface M of D through a point (x_0^1, x_0^2, x_0^3) is the plane $x^3 = x_0^3$. One can see that M is flat of constant mean curvature $(\mu_1 + \mu_2)/2$ (see (2.4)).

(1) If $(\mu_1, \mu_2) = (0, 0)$, then M is a totally geodesic plane.
(2) If $\mu_1 = \mu_2 = c \neq 0$, then M is a horosphere in the hyperbolic 3-space $H^3(-c^2)$.
(3) If $\mu_1 = -\mu_2 \neq 0$, then M is a non-totally geodesic minimal surface.

3. Integral representation formula

Let M be a Riemann surface and $(\mathfrak{D}, z)$ be a simply connected coordinate region. The exterior derivative d is decomposed as

$$d = \partial + \bar{\partial},\ \partial = \frac{\partial}{\partial z}dz,\ \bar{\partial} = \frac{\partial}{\partial \bar{z}}d\bar{z},$$

with respect to the conformal structure of M. Take a triplet $\{\omega^1, \omega^2, \omega^3\}$ of (1,0)-forms which satisfies the following differential system:

$$\bar{\partial}\omega^i = \mu_i \overline{\omega^i} \wedge \omega^3,\ i = 1, 2, \tag{3.1}$$

$$\bar{\partial}\omega^3 = \mu_1 \omega^1 \wedge \overline{\omega^1} + \mu_2 \omega^2 \wedge \overline{\omega^2}. \tag{3.2}$$

Proposition 3.1 ([5]). *Let $\{\omega^1, \omega^2, \omega^3\}$ be a solution to (3.1)-(3.2) on a simply connected coordinate region $\mathfrak{D}$. Then*

$$\varphi(z, \bar{z}) = 2\int_{z_0}^{z} \mathrm{Re}\ \left(e^{\mu_1 x^3(z,\bar{z})} \cdot \omega^1, e^{\mu_2 x^3(z,\bar{z})} \cdot \omega^2, \omega^3\right)$$

is a harmonic map of $\mathfrak{D}$ into $G(\mu_1, \mu_2)$. Conversely, any harmonic map of $\mathfrak{D}$ into $G(\mu_1, \mu_2)$ can be represented in this form.

Equivalently, the resulting harmonic map $\varphi(z, \bar{z})$ is defined by the following data:

$$\omega^1 = e^{-\mu_1 x^3} x^1_z dz,\ \omega^2 = e^{-\mu_1 x^3} x^2_z dz,\ \omega^3 = x^3_z dz, \tag{3.3}$$

where the coefficient functions are solutions to

$$x^i_{z\bar{z}} - \mu_i(x^3_z x^i_{\bar{z}} + x^3_{\bar{z}} x^i_z) = 0\ (i = 1, 2), \tag{3.4}$$

$$x^3_{z\bar{z}} + \mu_1 e^{-2\mu_1 x^3} x^1_z x^1_{\bar{z}} + \mu_2 e^{-2\mu_2 x^3} x^2_z x^2_{\bar{z}} = 0. \tag{3.5}$$

Corollary 3.1 ([5]). *Let $\{\omega^1, \omega^2, \omega^3\}$ be a solution to*

$$\bar{\partial}\omega^i = \mu_i \overline{\omega^i} \wedge \omega^3,\ i = 1, 2, \tag{3.6}$$

$$\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 = 0 \tag{3.7}$$

on a simply connected coordinate region $\mathfrak{D}$. Then

$$\varphi(z, \bar{z}) = 2\int_{z_0}^{z} \mathrm{Re}\ \left(e^{\mu_1 x^3(z,\bar{z})} \cdot \omega^1, e^{\mu_2 x^3(z,\bar{z})} \cdot \omega^2, \omega^3\right)$$

is a weakly conformal harmonic map of $\mathfrak{D}$ into $G(\mu_1, \mu_2)$. Moreover $\varphi(z, \bar{z})$ is a minimal immersion if and only if

$$\omega^1 \otimes \overline{\omega^1} + \omega^2 \otimes \overline{\omega^2} + \omega^3 \otimes \overline{\omega^3} \neq 0.$$

4. The normal Gauss map

Let $\varphi : M \to G(\mu_1, \mu_2)$ be a conformal immersion. Take a unit normal vector field N along φ. Then, by the left translation we obtain the following smooth map:

$$\psi := dL_{\varphi}^{-1} \cdot N : M \to S^2 \subset \mathfrak{g}(\mu_1, \mu_2).$$

The resulting map ψ takes value in the unit 2-sphere S^2 in the Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$. Here, via the orthonormal basis $\{E_1, E_2, E_3\}$, we identify $\mathfrak{g}(\mu_1, \mu_2)$ with the Euclidean 3-space $\mathbb{E}^3(u^1, u^2, u^3)$.

The smooth map ψ is called the *normal Gauss map* of φ.

Let $\varphi : \mathfrak{D} \to G(\mu_1, \mu_2)$ be a weakly conformal harmonic map of a simply connected Riemann surface $\mathfrak{D}$ determined by the data $(\omega^1, \omega^2, \omega^3)$. Express the data as $\omega^i = \phi^i dz$. Then the induced metric I of φ is

$$I = 2(\sum_{i=1}^{3} |\phi^i|^2) dz d\bar{z}.$$

Moreover these three coefficient functions satisfy

$$\frac{\partial \phi^3}{\partial \bar{z}} = -\sum_{i=1}^{2} \mu_i |\phi^i|^2, \quad \frac{\partial \phi^i}{\partial \bar{z}} = \mu_i \, \overline{\phi^i} \, \phi^3, \ i = 1, 2,$$

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0. \tag{4.1}$$

The harmonic map φ is a minimal immersion if and only if

$$|\phi^1|^2 + |\phi^2|^2 + |\phi^3|^2 \neq 0. \tag{4.2}$$

Here we would like to remark that ϕ^3 is identically zero if and only if φ is a *vertical plane* $x^3 =$ constant. (See Example 2.5). As we saw in Example 2.5, the vertical plane φ is minimal if and only if $\mu_1 + \mu_2 = 0$.

Hereafter we assume that ϕ^3 is not identically zero. Then we can introduce two mappings f and g by

$$f := \phi^1 - \sqrt{-1}\phi^2, \quad g := \frac{\phi^3}{\phi^1 - \sqrt{-1}\phi^2}. \tag{4.3}$$

By definition, f and g take values in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Using these two $\overline{\mathbb{C}}$-valued functions, φ is rewritten as

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^{z} \mathrm{Re} \left(e^{\mu_1 x^3} \frac{1}{2} f(1 - g^2), e^{\mu_2 x^3} \frac{\sqrt{-1}}{2} f(1 + g^2), fg \right) dz.$$

The normal Gauss map is computed as

$$\psi(z, \bar{z}) = \frac{1}{1 + |g|^2} \left(2\mathrm{Re}\,(g) E_1 + 2\mathrm{Im}\,(g) E_2 + (|g|^2 - 1) E_3 \right).$$

Under the stereographic projection $\mathcal{P} : S^2 \setminus \{\infty\} \subset \mathfrak{g}(\mu_1, \mu_2) \to \mathbb{C} := \mathbb{R}E_1 + \mathbb{R}E_2$, the map ψ is identified with the $\overline{\mathbb{C}}$-valued function g. Based on this fundamental observation, we call the function g the *normal Gauss map* of φ. The harmonicity together with the integrability (3.4)–(3.5) are equivalent to the following system for f and g:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} |f|^2 g \{\mu_1 (1 - \bar{g}^2) - \mu_2 (1 + \bar{g}^2)\}, \tag{4.4}$$

$$\frac{\partial g}{\partial \bar{z}} = -\frac{1}{4} \{\mu_1 (1 + g^2)(1 - \bar{g}^2) + \mu_2 (1 - g^2)(1 + \bar{g}^2)\} \bar{f}. \tag{4.5}$$

Theorem 4.1 ([6]). *Let f and g be $\overline{\mathbb{C}}$-valued functions which are solutions to the system* (4.4)–(4.5). *Then*

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^{z} \mathrm{Re} \left(e^{\mu_1 x^3} \frac{1}{2} f(1 - g^2), e^{\mu_2 x^3} \frac{\sqrt{-1}}{2} f(1 + g^2), fg \right) dz \tag{4.6}$$

is a weakly conformal harmonic map of $\mathfrak{D}$ into $G(\mu_1, \mu_2)$.

Example 4.1. Assume that $\mu_1 \neq 0$. Take the following two $\overline{\mathbb{C}}$-valued functions:

$$f = \frac{\sqrt{-1}}{\mu_1(z+\bar{z})},\ g = -\sqrt{-1}.$$

Then f and g are solutions to (4.4)–(4.5). By the integral representation formula, we can see that the minimal surface determined by the data (f, g) is a plane $x^2 =$ constant. Note that this plane is totally geodesic in $G(1, -1)$.

From (4.4)–(4.5), we can eliminate f and deduce the following PDE for g:

$$\begin{aligned} g_{z\bar{z}} &- \frac{2g\{\mu_1(1-\bar{g}^2) - \mu_2(1+\bar{g}^2)\}g_z g_{\bar{z}}}{\mu_1(1+g^2)(1-\bar{g}^2) + \mu_2(1-g^2)(1+\bar{g}^2)} \\ &+ \frac{4\bar{g}(1-g^4)(\mu_1^2-\mu_2^2)|g_{\bar{z}}|^2}{(\mu_1^2+\mu_2^2)|1-g^4|^2 + \mu_1\mu_2\{(1+g^2)^2(1-\bar{g}^2)^2 + (1+\bar{g}^2)^2(1-g^2)^2\}} \\ &= 0. \end{aligned} \tag{4.7}$$

Theorem 4.2. *Equation* (4.7) *is the harmonic map equation for a map* $g : \mathfrak{D} \longrightarrow \overline{\mathbb{C}}(w, \bar{w})$ *if and only if* $\mu_1^2 = \mu_2^2$.

(1) *If* $\mu_1 = \mu_2 \neq 0$*, then equation* (4.7) *becomes*

$$\frac{\partial^2 g}{\partial z \partial \bar{z}} + \frac{2|g|^2\bar{g}}{1-|g|^4}\frac{\partial g}{\partial z}\frac{\partial g}{\partial \bar{z}} = 0. \tag{4.8}$$

The differential equation (4.8) *is the harmonic map equation for a map* g *from* $\mathfrak{D}$ *into* $\left(\bar{\mathbb{C}}(w, \bar{w}), \frac{dwd\bar{w}}{|1-|w|^4|}\right)$. *The singular metric* $\frac{dwd\bar{w}}{|1-|w|^4|}$ *is called the Kokubu metric* ([3], [8]).

(2) *If* $\mu_1 = -\mu_2 \neq 0$*, then* (4.7) *becomes*

$$\frac{\partial^2 g}{\partial z \partial \bar{z}} - \frac{2g}{g^2-\bar{g}^2}\frac{\partial g}{\partial z}\frac{\partial g}{\partial \bar{z}} = 0. \tag{4.9}$$

The differential equation (4.9) *is the harmonic map equation for a map* g *from* $\mathfrak{D}$ *into* $\left(\overline{\mathbb{C}}(w, \bar{w}), \frac{dwd\bar{w}}{|w^2-\bar{w}^2|}\right)$.

Proof. Consider a possibly singular Riemannian metric $\lambda^2 dwd\bar{w}$ on the extended complex plane $\overline{\mathbb{C}}(w, \bar{w})$. Denote by Γ^w_{ww} the Christoffel symbol of the metric with respect to $(w, \bar{w})$. Then for a map $g : M \longrightarrow \bar{\mathbb{C}}(w, \bar{w})$, the *tension field* $\tau(g)$ of g is given by

$$\tau(g) = 4\lambda^{-2}\left(g_{z\bar{z}} + \Gamma^w_{ww} g_z g_{\bar{z}}\right). \tag{4.10}$$

By comparing the equations (4.7) and $\tau(g) = 0$, one can readily see that (4.7) is a harmonic map equation if and only if $\mu_1^2 = \mu_2^2$.

In order to find a suitable metric on $\bar{\mathbb{C}}(w, \bar{w})$ with which (4.7) is a harmonic map equation, one simply needs to solve the first order PDE:

$$\begin{cases} \Gamma^w_{ww} = \dfrac{2|w|^2\bar{w}}{1-|w|^4} & \text{if } \mu_1 = \mu_2 \neq 0, \\ \Gamma^w_{ww} = -\dfrac{2w}{w^2-\bar{w}^2} & \text{if } \mu_1 = -\mu_2 \neq 0, \end{cases}$$

whose solutions are $\lambda^2 = 1/|1-|w|^4|$ and $\lambda^2 = 1/|w^2-\bar{w}^2|$, respectively. □

Corollary 4.1. *Let* $g : \mathfrak{D} \to \left(\overline{\mathbb{C}}(w, \bar{w}), \frac{dwd\bar{w}}{|w^2-\bar{w}^2|}\right)$ *be a harmonic map. Define a function* f *on* $\mathfrak{D}$ *by*

$$f = \frac{2\bar{g}_z}{g^2 - \bar{g}^2}.$$

Then

$$\varphi(z, \bar{z}) = 2\int_{z_0}^{z} \mathrm{Re}\left(e^{x^3}\frac{1}{2}f(1-g^2), e^{-x^3}\frac{\sqrt{-1}}{2}f(1+g^2), fg\right)dz$$

is a weakly conformal harmonic map of $\mathfrak{D}$ *into* Sol.

Remark 1. Direct computation shows the following formulas:

(1) The sectional curvature of $(\overline{\mathbb{C}}(w, \overline{w}), dwd\bar{w}/|1-|w|^4|)$ is $-8|w|^2/|1-|w|^4|$.
(2) The sectional curvature of $(\overline{\mathbb{C}}(w, \overline{w}), dwd\bar{w}/|w^2-\bar{w}^2|)$ is $-8|w|^2/|w^2-\bar{w}^2|$.

Remark 2. The normal Gauss map of a non-vertical minimal surface in the Heisenberg group Nil is a harmonic map into the hyperbolic 2-space. See [7].

Aiyama and Akutagawa [3] studied the Dirichlet problem at infinity for proper harmonic maps from the unit disc to the extended complex plane equipped with the Kokubu metric. To close this paper we propose the following problem:

Problem 4.1. Study the Dirichlet problem at infinity for harmonic maps into the extended complex plane with metric $dwd\bar{w}/|w^2 - \bar{w}^2|$ and apply it for the construction of minimal surfaces in Sol.

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