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NORMALIZATION OF MONOMIAL IDEALS AND HILBERT FUNCTIONS

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Abstract. We study the normalization of a monomial ideal, and show how to compute its Hilbert function (using Ehrhart polynomials) if the ideal is zero dimensional. A positive lower bound for the second coefficient of the Hilbert polynomial is shown.

1. Introduction

Let $R = k[x_1,\dots,x_d]$ be a polynomial ring over a field k and let I be a monomial ideal of R minimally generated by $x^{v_1},\dots,x^{v_q}$. As usual for $a = (a_i)$ in $\mathbb{N}^d$ we set $x^a = x_1^{a_1}\cdots x_d^{a_d}$. If $\mathcal{R}$ is the Rees algebra of I, $\mathcal{R} = R[It]$, we call its integral closure $\overline{\mathcal{R}}$ the *normalization* of I. This algebra has for components the integral closures of the powers of I:

$$\mathcal{R} = R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R \oplus \overline{I}t \oplus \cdots \oplus \overline{I^i}t^i \oplus \cdots = \overline{\mathcal{R}}.$$

In our situation $\mathcal{R} \subset \overline{\mathcal{R}}$ is a finite extension. By a result of Vasconcelos [14, Theorem 7.58], the I-filtration $\mathcal{F} = \{\overline{I^i}\}_{i=0}^{\infty}$ stabilizes for $i \geq d$, i.e., $\overline{I^i} = I\,\overline{I^{i-1}}$ for $i \geq d$. We complement this result by showing that if $\deg(x^{v_i}) = r$ for all i, then $\mathcal{F}$ stabilizes for i greater than or equal to the minimum of $\operatorname{rank}(v_1,\dots,v_q)$ and $d - \lfloor d/r \rfloor + 1$ (Proposition 2.1 and Corollary 2.7).

If $\dim(R/I) = 0$, we are interested in studying the *Hilbert function* of $\mathcal{F}$:

$$f(n) = \ell_R(R/\overline{I^n}) = c_d n^d + c_{d-1}n^{d-1} + \cdots + c_1 n + c_0 \quad (c_i \in \mathbb{Q};\ n \gg 0).$$

We will express $f(n)$ as a difference of two Ehrhart polynomials (Proposition 3.6) and show a lower bound for c_{d-1} (Proposition 3.16). In particular we obtain an efficient way of computing the Hilbert function of $\mathcal{F}$ using integer programming methods. As an application we show that $e_0(d-1) - 2e_1 \geq d-1$, where e_i is the *ith* Hilbert coefficient of f. For monomial ideals this improves the inequality $e_0(d-1) \geq 2e_1$ given by Polini, Ulrich and Vasconcelos [12, Theorem 3.2] that holds for an arbitrary $\mathfrak{m}$-primary ideal I of a regular local ring $(R,\mathfrak{m})$. This inequality turns out to be useful to bound the length of divisorial chains for classes of Rees algebras [12, Corollary 3.5].

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In the sequel we use [4, 13] as references for standard terminology and notation on commutative algebra and polyhedral geometry. We denote the set of non-negative real (resp. integer, rational) numbers by $\mathbb{R}_+$ (resp. $\mathbb{N}$, $\mathbb{Q}_+$).

2. Normalization of monomial ideals

To avoid repetition, we continue to use the notation and definitions used in the Introduction.

Proposition 2.1. *Let r_0 be the rank of the matrix $(v_1,\ldots,v_q)$. If $v_1,\ldots,v_q$ lie in a hyperplane of $\mathbb{R}^d$ not containing the origin, then $\overline{I^b}=I\overline{I^{b-1}}$ for $b\geq r_0$.*

Proof. Let $\mathbb{Q}_+\mathcal{A}'$ be the cone in $\mathbb{Q}^{d+1}$ generated by the set

$$\mathcal{A}'=\{(v_1,1),\ldots,(v_q,1),e_1,\ldots,e_d\},$$

where e_i is the ith unit vector in $\mathbb{Q}^{d+1}$. Assume $b\geq r_0$. Notice that we invariably have $I\overline{I^{b-1}}\subset\overline{I^b}$. To show the reverse inclusion take $x^\alpha\in\overline{I^b}$, i.e., $x^{m\alpha}\in I^{bm}$ for some $0\neq m\in\mathbb{N}$. Hence $(\alpha,b)\in\mathbb{Q}_+\mathcal{A}'$. Applying Carathéodory's theorem for cones [13, Corollary 7.1i], we can write

$$(\alpha,b)=\lambda_1(v_{i_1},1)+\cdots+\lambda_r(v_{i_r},1)+\mu_1e_{j_1}+\cdots+\mu_se_{j_s}\quad(\lambda_\ell,\mu_k\in\mathbb{Q}_+),$$

where $\{(v_{i_1},1),\ldots,(v_{i_r},1),e_{j_1},\ldots,e_{j_s}\}$ is a linearly independent set contained in $\mathcal{A}'$. Notice that $v_{i_1},\ldots,v_{i_r}$ are also linearly independent because they lie in a hyperplane not containing the origin. Hence $r\leq r_0$. Since $b=\lambda_1+\cdots+\lambda_r$, we obtain that $\lambda_\ell\geq 1$ for some ℓ, say $\ell=1$. Then

$$\alpha=v_{i_1}+(\lambda_1-1)v_{i_1}+\lambda_2v_{i_2}+\cdots+\lambda_rv_{i_r}+\mu_1e_{j_1}+\cdots+\mu_se_{j_s},$$

and consequently $x^\alpha\in I\overline{I^{b-1}}$. □

Remark 2.2. In this proof we may replace $\mathbb{Q}$ by $\mathbb{R}$ because according to [15, p. 219] we have the equality $\mathbb{Z}^{d+1}\cap\mathbb{R}_+\mathcal{A}'=\mathbb{Z}^{d+1}\cap\mathbb{Q}_+\mathcal{A}'$.

Lemma 2.3. *Let $\mathcal{A}'=\{e_1,\ldots,e_d\}\cup\{(a_1,\ldots,a_d,1)|\,a_i\in\mathbb{N};\ \sum_i a_i=r\}$, where $r\geq 2$ is an integer. Then the irreducible representation of the cone $\mathbb{R}_+\mathcal{A}'$, as an intersection of closed halfspaces, is given by*

$$\mathbb{R}_+\mathcal{A}'=H_{e_1}^+\cap\cdots\cap H_{e_d}^+\cap H_{e_{d+1}}^+\cap H_a^+,$$

where $a=(1,\ldots,1,-r)$ and $H_a^+=\{x\in\mathbb{R}^{d+1}\,|\,\langle x,a\rangle\geq 0\}$.

Proof. We set $\mathcal{A}=\{e_1,\ldots,e_d,re_1+e_{d+1},\ldots,re_d+e_{d+1}\}$ and $N=\{e_1,\ldots,e_{d+1},a\}$. The cone $\mathbb{R}_+\mathcal{A}'$ has dimension $d+1$ and one has the equality $\mathbb{R}_+\mathcal{A}'=\mathbb{R}_+\mathcal{A}$. Thus it suffices to prove that F is a facet of $\mathbb{R}_+\mathcal{A}$ if and only if $F=H_b\cap\mathbb{R}_+\mathcal{A}$ for some $b\in N$. Let $1\leq i\leq d+1$. Consider the following sets of vectors:

$$\Gamma_i=\{e_1,e_2,\ldots,\widehat{e_i},\ldots,e_d,e_{d+1}\}\ \text{ and }\ \Gamma=\{re_1+e_{d+1},\ldots,re_d+e_{d+1}\},$$

where $\widehat{e_i}$ means to omit e_i from the list. Since Γ_i and Γ are linearly independent, we obtain that $F=H_b\cap\mathbb{R}_+\mathcal{A}$ is a facet for $b\in N$, i.e., $\dim(F)=d$ and $\mathbb{R}_+\mathcal{A}\subset H_b^+$. Conversely let F be a facet of the cone $\mathbb{R}_+\mathcal{A}$. There are linearly independent vectors $\alpha_1,\ldots,\alpha_d\in\mathcal{A}$ and $0\neq b=(b_1,\ldots,b_{d+1})\in\mathbb{R}^{d+1}$ such that

(i) $F=\mathbb{R}_+\mathcal{A}\cap H_b$,
(ii) $\mathbb{R}\alpha_1+\cdots+\mathbb{R}\alpha_d=H_b$, and
(iii) $\mathbb{R}_+\mathcal{A}\subset H_b^+$.

Since $e_1, \dots, e_d$ are in $\mathcal{A}$, by (iii) one has

$$\langle e_i, b\rangle = b_i \geq 0 \quad \text{for} \quad i = 1, \dots, d. \tag{2.1}$$

Set $\mathcal{B} = \{\alpha_1, \dots, \alpha_d\}$ and consider the matrix M whose rows are the vectors in $\mathcal{B}$. To finish the proof we need only show that there exists $c \in N$ such that $H_b = H_c$. Consider the following cases. Case (1): If the ith column of M is zero for some $1 \leq i \leq d+1$, we set $c = e_i$. Case (2): If $\mathcal{B} = \{re_1 + e_{d+1}, \dots, re_d + e_{d+1}\}$, then we set $c = a$. Case (3): Now we assume that

$$\mathcal{B} = \{e_{i_1}, \dots, e_{i_s}, re_{j_1} + e_{d+1}, \dots, re_{j_t} + e_{d+1}\},$$

where $s, t > 0$, $s + t = d$, $1 \leq i_1 < \cdots < i_s \leq d$, $1 \leq j_1 < \cdots < j_t \leq d$, and M has all its columns different from zero. Since $b_{i_1} = 0$, using

$$\begin{aligned} \langle re_{i_1} + e_{d+1}, b\rangle &= b_{d+1} \geq 0, \\ \langle re_{j_1} + e_{d+1}, b\rangle &= rb_{j_1} + b_{d+1} = 0, \end{aligned}$$

and equation (2.1), we obtain $b_{d+1} = 0$. Then $e_{d+1} \in H_b$. It follows readily that H_b is generated, as a vector space, by the set $\{e_1, e_2, \dots, \widehat{e_i}, \dots, e_d, e_{d+1}\}$ for some $1 \leq i \leq d$. Thus in this case we set $c = e_i$. □

Let $r \geq 2$ be an integer. For the rest of this section we make two assumptions: (i) $R[t]$ has the grading δ induced by setting $\delta(x_i) = 1$ and $\delta(t) = 1 - r$, and (ii) $\deg(x^{v_i}) = r$ for all i. Thus $\mathcal{R} = R[It]$ becomes a standard graded k-algebra. In this case $\mathcal{R}$ and $\overline{\mathcal{R}}$ have rational Hilbert series. The degree as a rational function of the Hilbert series of $\mathcal{R}$, denoted by $a(\mathcal{R})$, is called the *a-invariant* of $\mathcal{R}$.

For use below recall that the *rth Veronese ideal* of R is the ideal generated by all monomials of R of degree r. If x^a is a monomial we set $\log(x^a) = a$.

Proposition 2.4. *Let J be the rth Veronese ideal of R and let $S = R[Jt]$ be its Rees algebra.* (a) *If $r \geq d$, then $a(S) = -2$.* (b) *If $2 \leq r < d$ and $d = qr + s$, where $0 \leq s < r$, then*

$$a(S) = \begin{cases} -(q+2) & \text{if} \quad s \geq 2, \\ -(q+1) & \text{if} \quad s = 0 \text{ or } s = 1. \end{cases}$$

Proof. Let $\mathcal{A}'$ be as in Lemma 2.3. As S is normal, according to a formula of Danilov-Stanley [4], the canonical module ω_S of S can be expressed as

$$\omega_S = (\{x^a t^b \,|\, (a,b) \in \mathbb{N}\mathcal{A}' \cap (\mathbb{R}_+\mathcal{A}')^{\mathrm{o}}\}) = (\{x^a t^b \,|\, (a,b) \in \mathbb{Z}^{d+1} \cap (\mathbb{R}_+\mathcal{A}')^{\mathrm{o}}\}), \tag{2.2}$$

where $(\mathbb{R}_+\mathcal{A}')^{\mathrm{o}}$ denotes the relative interior of $\mathbb{R}_+\mathcal{A}'$ and $\mathbb{N}\mathcal{A}'$ is the subsemigroup of $\mathbb{N}^{d+1}$ generated by $\mathcal{A}'$. In our situation recall that $a(S) = -\min\{i \,|\, (\omega_S)_i \neq 0\}$.

Let $m \in \omega_S$. We can write $m = x^a(x^b t^c)$, where $x^b t^c = (f_1 t) \cdots (f_c t)$ and f_i is a monomial of degree r for all i. Notice that $\delta(m) = |a| + c$, where $a = (a_i)$ and $|a| = a_1 + \cdots + a_d$. Since $\log(m) = (a + b, c)$ is in the interior of the cone $\mathbb{R}_+\mathcal{A}'$, using Lemma 2.3 one has $c \geq 1$, $a_i + b_i \geq 1$ for all i, and $|a| + |b| \geq rc + 1$. As $|b| = rc$, altogether we get

$$|a| + |b| \geq d \quad \text{and} \quad |a| \geq 1. \tag{2.3}$$

In particular $\delta(m) \geq 2$. This shows the inequality $a(S) \leq -2$ because m was an arbitrary monomial in ω_S. To prove (a) notice that by Lemma 2.3 the monomial $m_1 = x_1^{r-d+2} x_2 \cdots x_d t$ is in ω_S (see the argument below) and $\delta(m_1) = 2$. Hence $a(S) = -2$. To prove (b) there are three cases to consider. We only show the case $s \geq 2$; the cases $s = 1$ and $s = 0$ can be shown similarly.

Case $s \geq 2$: First we show that $\delta(m) \geq q+2$. If $c > q$, then from equation (2.3) we get $\delta(m) \geq q+2$. Assume $c \leq q$. One has the inequality

$$r(q-c)+s \geq (q-c)+2. \tag{2.4}$$

From equation (2.3) one has $|a|+|b| = |a|+rc \geq d = rq+s$. Consequently

$$\delta(m) = |a|+c \geq r(q-c)+s+c. \tag{2.5}$$

Hence from equations (2.4) and (2.5) we get $\delta(m) \geq q+2$. Therefore one has the inequality $a(S) \leq -(q+2)$; to show equality it suffices to prove that the monomial

$$m_2 = x_1^2 x_2^2 \cdots x_{r-s+1}^2 x_{r-s+2} \cdots x_d t^{q+1}$$

is in ω_S and has degree $q+2$. An easy calculation shows that $\delta(m_2) = q+2$. Finally let us see that m_2 is in ω_S via Lemma 2.3. That the entries of $\log(m_2)$ satisfy $X_i > 0$ for all i is clear. The inequality

$$X_1+X_2+\cdots+X_d > rX_{d+1}, \tag{2.6}$$

after making X_i equal to the ith entry of $\log(m_2)$, transforms into

$$2(r-s+1)+(d-(r-s+1)) > r(q+1),$$

but the left hand side is $r(q+1)+1$, hence $\log(m_2)$ satisfies equation (2.6). Hence $\log(m_2)$ is in the interior of $\mathbb{R}_+\mathcal{A}'$, i.e., $m_2 \in \omega_S$. □

For some other explicit formulae of a-invariants see [2]. The next result sharpens [6, Theorem 3.3] for the class of ideals generated by monomials of the same degree.

Proposition 2.5. *If $2 \leq r < d$, then the normalization $\overline{\mathcal{R}}$ of I is generated as an $\mathcal{R}$-module by elements $g \in R[t]$ of t-degree at most $d - \lfloor d/r \rfloor$.*

Proof. Set $f_i = x^{v_i}$ for $i = 1,\ldots,q$. Consider the subsemigroup C of $\mathbb{N}^{d+1}$ generated by the set $\{e_1,\ldots,e_d,(v_1,1),\ldots,(v_q,1)\}$ and the subgroup $\mathbb{Z}C$ generated by C. Since $\mathbb{Z}C = \mathbb{Z}^{d+1}$, the normalization of I can be expressed as

$$\overline{\mathcal{R}} = k[\{x^a t^b \,|\, (a,b) \in \mathbb{Z}^{d+1} \cap \mathbb{R}_+C\}].$$

Let $m = x^a t^b$ be a monomial of $\overline{\mathcal{R}}$ with $(a,b) \neq 0$. We claim that $\delta(m) \geq b$. To show this inequality write

$$(a,b) = \lambda_1 e_1 + \cdots + \lambda_d e_d + \mu_1 \log(f_1 t) + \cdots + \mu_q \log(f_q t),$$

where $\lambda_i \geq 0, \mu_j \geq 0$ for all i,j. Hence

$$|a| = \lambda_1 + \cdots + \lambda_d + (\mu_1 + \cdots + \mu_q)r \quad \text{and} \quad b = \mu_1 + \cdots + \mu_q.$$

Consequently $\delta(m) = |a| + (1-r)b = (\lambda_1 + \cdots + \lambda_d) + b \geq b$. We may assume that k is infinite. There is a Noether normalization $A = k[z_1,\ldots,z_{d+1}] \overset{\varphi}{\hookrightarrow} \mathcal{R}$ such that $z_1,\ldots,z_{d+1} \in \mathcal{R}_1$. If ψ is the inclusion from $\mathcal{R}$ to $\overline{\mathcal{R}}$, note that $A \xrightarrow{\psi\varphi} \overline{\mathcal{R}}$ is a Noether normalization. By [9], the ring $\overline{\mathcal{R}}$ is Cohen-Macaulay. Hence $\overline{\mathcal{R}}$ is a free A-module that according to [15, Proposition 2.2.14] can be written as

$$\overline{\mathcal{R}} = Am_1 \oplus \cdots \oplus Am_n, \tag{2.7}$$

where $m_i = x^{\beta_i} t^{b_i}$. Set $h_i = |\{j \,|\, \delta(m_j) = i\}|$. Using the fact that the length is additive we obtain the following expression for the Hilbert series of $\overline{\mathcal{R}}$:

$$H(\overline{\mathcal{R}}, z) = \sum_{i=0}^{n} \frac{z^{\delta(m_i)}}{(1-z)^{d+1}} = \frac{h_0 + h_1 z + \cdots + h_s z^s}{(1-z)^{d+1}}.$$

Recall that $a(\overline{\mathcal{R}}) = -\min\{i \,|\, (\omega_{\overline{\mathcal{R}}})_i \neq 0\}$, where $\omega_{\overline{\mathcal{R}}}$ is the canonical module of $\overline{\mathcal{R}}$. Let J be the rth Veronese ideal of R and let $S = R[Jt]$ be its Rees algebra. Notice that $\overline{\mathcal{R}} \subset S$ because $R[It] \subset S$ and S is normal. Since $\dim(\overline{\mathcal{R}}) = \dim(S) = d+1$, from the Danilov-Stanley formula (see equation (2.2)) it is seen that $a(\overline{\mathcal{R}}) \leq a(S)$; see the proof of [6, Proposition 3.5]. Therefore using Proposition 2.4 we get

$$a(\overline{\mathcal{R}}) = s - (d+1) \leq a(R[Jt]) \leq -\lfloor d/r \rfloor - 1,$$

and $s \leq d - \lfloor d/r \rfloor$. Altogether if $m_i = x^{\beta_i} t^{b_i}$, one has

$$b_i \leq \delta(m_i) \leq s \leq d + 1 + a(R[Jt]) \leq d - \lfloor d/r \rfloor. \tag{2.8}$$

Therefore b_i, the t-degree of m_i, is less than or equal to $d - \lfloor d/r \rfloor$, as required. □

Proposition 2.6. *$\overline{I^b} = I\overline{I^{b-1}}$ for $b \geq d + 2 + a(R[Jt])$.*

Proof. It suffices to prove the inclusion $\overline{I^b} \subset I\overline{I^{b-1}}$. Let $x^a \in \overline{I^b}$, i.e., $m = x^a t^b \in \overline{\mathcal{R}}$. From equation (2.7) and noticing that $A \subset \mathcal{R}$, we can write $m = (x^\gamma t^c) m_i$ for some i, where $m_i = x^{\beta_i} t^{b_i}$ and $x^\gamma \in I^c$. Using equation (2.8) gives $c \geq 1$. Thus $x^a \in I^c \overline{I^{b_i}}$. To complete the proof notice that $I^c \overline{I^{b_i}} = I(I^{c-1}\overline{I^{b_i}}) \subset I\overline{I^{b_i+c-1}} = I\overline{I^{b-1}}$. □

Corollary 2.7. *$\overline{I^b} = I\overline{I^{b-1}}$ for $b \geq d - \lfloor d/r \rfloor + 1$.*

Proof. By Proposition 2.4 one has $a(R[Jt]) \leq -\lfloor d/r \rfloor - 1$. Hence the result follows applying Proposition 2.6. □

3. Zero dimensional monomial ideals and Hilbert functions

Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k, with $d \geq 2$, and let I be a zero dimensional monomial ideal of R minimally generated by $x^{v_1}, \ldots, x^{v_q}$. Here we will study the integral closure of the powers of I and its Hilbert function.

We may assume that $v_i = a_i e_i$ for $1 \leq i \leq d$, where $a_1, \ldots, a_d$ are positive integers and e_i is the ith unit vector of $\mathbb{Q}^d$. Set $\alpha_0 = (1/a_1, \ldots, 1/a_d)$. We may also assume that $\{v_{d+1}, \ldots, v_s\}$ is the set of v_i such that $\langle v_i, \alpha_0 \rangle < 1$, and $\{v_{s+1}, \ldots, v_q\}$ is the set of v_i such that $i > d$ and $\langle v_i, \alpha_0 \rangle \geq 1$. Consider the convex polytopes in $\mathbb{Q}^d$:

$$P := \operatorname{conv}(v_1, \ldots, v_s), \quad S := \operatorname{conv}(0, v_1, \ldots, v_d) = \{x \,|\, x \geq 0;\ \langle x, \alpha_0 \rangle \leq 1\},$$

and the rational convex polyhedron $Q := \mathbb{Q}^d_+ + \operatorname{conv}(v_1, \ldots, v_q)$.

Proposition 3.1 ([8, Proposition 1.1])**.** *$\overline{I^n} = (\{x^a |\, a \in nQ \cap \mathbb{Z}^d\})$ for $0 \neq n \in \mathbb{N}$.*

Let us give a simpler expression for Q. From the equality

$$\mathbb{Q}^d_+ + \operatorname{conv}(v_1, \ldots, v_d) = \{x \,|\, x \geq 0;\ \langle x, \alpha_0 \rangle \geq 1\},$$

we get that $v_i \in \mathbb{Q}^d_+ + P$ for $i = 1, \ldots, q$. Using the finite basis theorem for polyhedra [13, Corollary 7.1b] we have that $\mathbb{Q}^d_+ + P$ is a convex set. Hence $Q \subset \mathbb{Q}^d_+ + P$, and consequently we obtain the equality

$$Q = \mathbb{Q}^d_+ + P. \tag{3.1}$$

Corollary 3.2. *If $\langle v_i, \alpha_0 \rangle \geq 1$ for all i, then $\overline{I^n} = \overline{(x_1^{a_1}, \ldots, x_d^{a_d})^n}$ for $n \geq 1$.*

Proof. It follows at once from Proposition 3.1 and equation (3.1). Notice that in this case $P = \operatorname{conv}(v_1, \ldots, v_d)$. □

The *Hilbert function* of the filtration $\mathcal{F} = \{\overline{I^n}\}_{n=0}^{\infty}$ is defined as

$$f(n) = \ell(R/\overline{I^n}) = \dim_k(R/\overline{I^n}); \quad n \in \mathbb{N} \setminus \{0\}; \quad f(0) = 0.$$

As usual $\ell(R/\overline{I^n})$ denotes the length of $R/\overline{I^n}$ as an R-module. For simplicity we call f the *Hilbert function* of $\mathcal{F}$.

Corollary 3.3. *$f(n) = \ell(R/\overline{I^n}) = |\mathbb{N}^d \setminus nQ|$ for $n \geq 1$.*

Proof. The length of $R/\overline{I^n}$ equals the dimension of $R/\overline{I^n}$ as a k-vector space. By Proposition 3.1 the set $\mathcal{B} = \{x^c | \, c \notin nQ\}$ is precisely the set of standard monomials of $R/\overline{I^n}$. Thus $\mathcal{B}$ yields a k-vector space basis of $R/\overline{I^n}$, and the equality follows. □

The Hilbert function of $\mathcal{F}$ is a polynomial function of degree d:

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0 \qquad (n \gg 0),$$

where $c_0, \ldots, c_d \in \mathbb{Q}$ and $c_d \neq 0$. The polynomial $c_d x^d + \cdots + c_0$ is called the *Hilbert polynomial* of $\mathcal{F}$. One has the equality $d! c_d = e(I) = e(\overline{I})$, where $e(I)$ is the multiplicity of I; see [8]. We will express $f(n)$ as a difference of two Ehrhart polynomials and then show a positive lower bound for c_{d-1}.

The *Ehrhart function* of P is the numerical function $\chi_P \colon \mathbb{N} \to \mathbb{N}$ given by $\chi_P(n) = |\mathbb{Z}^d \cap nP|$. This is a polynomial function of degree $d_1 = \dim(P)$:

$$\chi_P(n) = b_{d_1} n^{d_1} + \cdots + b_1 n + b_0 \quad (n \gg 0),$$

where $b_i \in \mathbb{Q}$ for all i. The polynomial $E_P(x) = b_{d_1} x^{d_1} + \cdots + b_1 x + b_0$ is called the *Ehrhart polynomial* of P. In general some of the coefficients of $E_P(x)$ may be negative. It is unknown whether the coefficients are non-negative if the vertices of P have $\{0,1\}$-entries.

Remark 3.4. Some well known properties of E_P are (see [4]):

(1) $b_{d_1} = \mathrm{vol}(P)$, where $\mathrm{vol}(P)$ denotes the relative volume of P.
(2) $b_{d_1 - 1} = (\sum_{i=1}^{s} \mathrm{vol}(F_i))/2$, where $F_1, \ldots, F_s$ are the facets of P.
(3) $\chi_P(n) = E_P(n)$ for all integers $n \geq 0$. In particular $E_P(0) = 1$.
(4) *Reciprocity law of Ehrhart*: $E_P^{\circ}(n) = (-1)^d E_P(-n) \;\; \forall\, n \geq 1$, where $E_P^{\circ}(n) = |\mathbb{Z}^d \cap (nP)^{\circ}|$ and $(nP)^{\circ}$ is the relative interior of nP.

Lemma 3.5. *$P = S \cap Q$.*

Proof. Clearly $P \subset S \cap Q$. Conversely let $z = (z_i) \in S \cap Q$. Assume that $z \notin P$. By the separating hyperplane theorem [10, Theorem 3.23], there are $0 \neq b = (b_i) \in \mathbb{R}^d$ and $c \in \mathbb{R}$ such that $\langle b, v_i \rangle \leq c$ for $i = 1, \ldots, s$ and $\langle b, z \rangle > c$. Assume $c > 0$. Since $b_i a_i \leq c$ for all i and $\langle \alpha_0, z \rangle \leq 1$, we get $\langle b, z \rangle \leq \langle \alpha_0, z \rangle c \leq c$, a contradiction. If $c = 0$, then $b_i \leq 0$ for all i and $\langle b, z \rangle \leq 0$, a contradiction. If $c < 0$, we write $z = \delta + p$, for some $\delta \in \mathbb{Q}_+^d$ and $p \in P$. Then $c < \langle b, z \rangle = \langle b, \delta \rangle + \langle b, p \rangle \leq \langle b, \delta \rangle + c$. Thus $0 < \langle b, \delta \rangle$, a contradiction, because $b_i \leq 0$ for all i. □

Proposition 3.6. *$f(n) = E_S(n) - E_P(n)$ for $n \in \mathbb{N}$. In particular*

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0 \text{ for } n \in \mathbb{N} \text{ and } c_0 = 0.$$

Proof. Since $E_P(0) = E_S(0) = 1$, we get the equality at $n = 0$. Assume $n \geq 1$. Using Lemma 3.5, we get the decomposition $Q = (\mathbb{Q}_+^d \setminus S) \cup P$. Hence

$$nQ = (\mathbb{Q}_+^d \setminus nS) \cup nP \implies \mathbb{N}^d \setminus nQ = [\mathbb{N}^d \cap (nS)] \setminus [\mathbb{N}^d \cap (nP)].$$

Therefore by Corollary 3.3 we obtain $f(n) = E_S(n) - E_P(n)$. □

Example 3.7. Let $I = (x_1^4, x_2^5, x_3^6, x_1x_2x_3^2)$. Notice that

$$P = \mathrm{conv}((4,0,0),(0,5,0),(0,0,6),(1,1,2)).$$

Using *Normaliz* [5], to compute the Ehrhart polynomials of S and P, we get

$$f(n) = E_S(n) - E_P(n) = (1 + 6n + 19n^2 + 20n^3)$$
$$-(1 + (1/6)n + (3/2)n^2 + (13/3)n^3) = (35/6)n + (35/2)n^2 + (47/3)n^3.$$

Theorem 3.8 ([14, Theorem 7.58])**.** $\overline{I^b} = I\overline{I^{b-1}}$ *for* $b \geq d$.

Remark 3.9. We can use polynomial interpolation together with Theorem 3.8 and Proposition 3.6 to determine $c_1, \ldots, c_d$; see Example 3.10.

Example 3.10. Let $I = (x_1^{10}, x_2^8, x_3^5)$. Using *CoCoA* [7] we obtain that the values of f at $n = 0,1,2,3$ are $0, 112, 704, 2176$. By polynomial interpolation we get

$$f(n) = \ell(R/\overline{I^n}) = (200/3)n^3 + 40x^2 + (16/3)n, \quad \forall\, n \geq 0.$$

Lemma 3.11. *Let* $\alpha = (\alpha_i)$ *and* $\beta = (\beta_i)$ *be two vectors in* $\mathbb{Q}_+^d$ *such that* $\alpha_i = \beta_i$ *for* $i = 1, \ldots, d-1$, $\beta_d > \alpha_d$ *and* $\langle \beta, \alpha_0 \rangle < 1$. *Then*

(a) $\beta \in \mathrm{conv}(v_1, \ldots, v_d, \alpha)$.
(b) *If* $\alpha_i > 0$ *for* $i = 1, \ldots, d-1$, *then* $\beta \in \mathrm{conv}(v_1, \ldots, v_d, \alpha)^{\mathrm{o}}$.
(c) *If* $\alpha_i > 0$ *for* $i = 1, \ldots, d-1$ *and* $\alpha \in P$, *then* $\beta \in P^{\mathrm{o}}$.

Proof. (a) To see that β is a convex combination of $v_1, \ldots, v_d, \alpha$ we set:

$$\begin{aligned} s &= \sum_{i=1}^d \alpha_i/a_i = \langle \alpha_0, \alpha \rangle < 1, \quad \mu = 1 - \left[\frac{\beta_d - \alpha_d}{a_d(1-s)}\right] > 0, \\ \lambda_i &= (1-\mu)\alpha_i/a_i \geq 0, \quad i = 1, \ldots, d-1, \\ \lambda_d &= (\beta_d - \mu\alpha_d)/a_d = ((\beta_d - \alpha_d)/a_d) + \alpha_d(1-\mu)/a_d > 0. \end{aligned}$$

Then $\beta = \lambda_1 v_1 + \cdots + \lambda_d v_d + \mu\alpha$ and $\lambda_1 + \cdots + \lambda_d + \mu = 1$, as required.

(b) Set $V = \{v_1, \ldots, v_d, \alpha\}$ and $\Delta = \mathrm{conv}(V)$. Since V is affinely independent, Δ is a d-simplex. From [3, Theorem 7.3], the facets of Δ are precisely those sets of the form $\mathrm{conv}(W)$, where W is a subset of V having d points. If β is not in the interior of Δ, then β must lie in its boundary by (a). Therefore β lies in some facet of Δ, which rapidly yields a contradiction.

(c) By part (b) we get $\beta \in \mathrm{conv}(v_1, \ldots, v_d, \alpha)^{\mathrm{o}} \subset P^{\mathrm{o}}$, as required. □

Notation. The relative boundary of P will be denoted by ∂P.

Lemma 3.12. *If* $\alpha \in \partial P \setminus \mathrm{conv}(v_1, \ldots, v_d)$ *and* $\alpha_i > 0$ *for* $i = 1, \ldots, d$, *then the vector* $\alpha' = (\alpha_1, \ldots, \alpha_{d-1}, 0)$ *is not in* P.

Proof. Notice that $\langle \alpha, \alpha_0 \rangle < 1$. If $\alpha' \in P$, then by Lemma 3.11(c) we obtain $\alpha \in P^{\mathrm{o}}$, a contradiction. Thus $\alpha' \notin P$. □

For use below we set

$$\begin{aligned} K_i &= \{(a_i) \in S \mid a_i = 0\} = \mathrm{conv}(\{v_1, \ldots, v_d, 0\} \setminus \{v_i\}); \quad 1 \leq i \leq d, \\ H &= \mathrm{conv}(v_1, \ldots, v_d); \quad K = (\bigcup_{i=1}^d K_i) \setminus H; \quad L = \partial P \setminus H, \quad \text{if } H \subsetneq P. \end{aligned}$$

Consider the map $\psi\colon L \to K$ given by

$$\psi(\alpha) = \begin{cases} \alpha, & \text{if } \alpha_i = 0 \text{ for some } 1 \le i \le d, \\ (\alpha_1, \ldots, \alpha_{d-1}, 0), & \text{if } \alpha_i > 0 \text{ for all } 1 \le i \le d. \end{cases}$$

Take $\alpha \in L$. Then $\langle \alpha, \alpha_0 \rangle < 1$. Since $\partial P \subset P \subset S$ it is seen that $\psi(\alpha) \in K$. Indeed if $\alpha_i = 0$ for some i, then $\psi(\alpha) = \alpha \in K_i \setminus H$. If $\alpha_i > 0$ for all i, then α is a convex combination of $v_1, \ldots, v_d, 0$. Hence $\psi(\alpha)$ is a convex combination of $v_1, \ldots, v_{d-1}, 0$ and $\psi(\alpha) \in K_d \setminus H$.

Lemma 3.13. *ψ is injective.*

Proof. Let $\alpha = (\alpha_i), \beta = (\beta_i) \in L$. Assume $\psi(\alpha) = \psi(\beta)$. If $\alpha_i = 0$ for some i and $\beta_j = 0$ for some j, then clearly $\alpha = \beta$. If $\beta_i > 0$ for $i = 1, \ldots, d$ and $\alpha_j = 0$ for some j, then $\alpha_d = 0$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, d-1$. By Lemma 3.12 we can readily see that this case cannot occur. If $\alpha_i \beta_i > 0$ for all i; then $\alpha = \beta$ by Lemma 3.11(c). $\square$

Let us introduce some more notation. We set

$$\begin{aligned} \mathcal{A}_i &= \{v_j \,|\, 1 \le j \le s;\ x_i \notin \mathrm{supp}(x^{v_j})\}; \\ P_i &= \mathrm{conv}(\mathcal{A}_i); \quad H_i = \mathrm{conv}(\{v_1, \ldots, v_d\} \setminus \{v_i\}) \subset P_i \subset K_i. \end{aligned}$$

Lemma 3.14. *$\partial P \cap K_i = P_i$ for $i = 1, \ldots, d$.*

Proof. For simplicity of notation assume $i = 1$. Let $\alpha = (\alpha_i) \in \partial P \cap K_1$, then $\alpha \in P$ and $\alpha_1 = 0$. Since α is a convex combination of $v_1, \ldots, v_s$ it follows rapidly that α is a convex combination of $\mathcal{A}_1$, i.e., $\alpha \in P_1$. Conversely let $\alpha \in P_1$. Clearly $\alpha \in K_1 \cap P$ because $\mathcal{A}_1 \subset K_1 \cap P$. Assume that $\alpha \notin \partial P$. Then $\alpha \in P^{\mathrm{o}}$. If $\dim(P) = d-1$, we have that $P = \mathrm{conv}(v_1, \ldots, v_d)$ and P is a simplex. Thus by [3, Theorem 7.3], the facets of P are $F_1, \ldots, F_d$, where $F_i = \mathrm{conv}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d)$. The relative boundary of P is equal to $F_1 \cup \cdots \cup F_d$. Hence $\alpha \notin F_i$ for all i, and we can write $\alpha = \lambda_1 v_1 + \cdots + \lambda_d v_d$, where $\sum_{i=1}^{d} \lambda_i = 1$ and $0 < \lambda_i < 1$ for $i = 1, \ldots, d$. Thus we get $\alpha_i > 0$ for $i = 1, \ldots, d$, a contradiction. If $\dim(P) = d$, then $\alpha \in P^{\mathrm{o}} \subset S^{\mathrm{o}}$. As in the previous case, but now using the fact that S is a d-simplex, we get $\alpha_i > 0$ for all i, a contradiction. Hence $\alpha \in \partial P$. $\square$

Lemma 3.15 ([1, p. 38])**.** *Let $A_1, \ldots, A_t$ be finite subsets of a set S; then*

$$\left|\bigcup_{i=1}^{t} A_i\right| = \sum_{i=1}^{t} |A_i| - \sum_{i<j}^{t} |A_i \cap A_j| + \sum_{i<j<k}^{t} |A_i \cap A_j \cap A_k| \mp \cdots + (-1)^{t-1} \left|\bigcap_{i=1}^{t} A_i\right|.$$

Proposition 3.16. *Let I_i be the ideal obtained from I by making $x_i = 0$ and let $e(I_i)$ be its multiplicity. If $c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$ is the Hilbert polynomial of the filtration $\mathcal{F} = \{\overline{I^n}\}_{n=0}^{\infty}$, then*

$$2c_{d-1} \ge \sum_{i=1}^{d-1} \frac{e(I_i)}{(d-1)!}.$$

Proof. Case (I): $\dim(P) = d$. Let $E_S(x) = a_d x^d + \cdots + a_1 x + 1$ (resp. $E_P(x) = b_d x^d + \cdots + b_1 x + 1$) be the Ehrhart polynomial of S (resp. P). By Proposition 3.6, we have the equality $c_i = a_i - b_i$ for all i. From the decompositions

$$P = P^{\mathrm{o}} \cup \partial P, \quad S = S^{\mathrm{o}} \cup \partial S, \quad \partial S = K \cup H, \quad \partial P = L \cup H,$$

and using the reciprocity law of Ehrhart (Remark 3.4) we get:

$$\begin{aligned} f(n) &= E_S(n) - E_P(n) \\ &= E_S^{\circ}(n) + |\partial(nS)\cap\mathbb{Z}^d| - (E_P^{\circ}(n) + |\partial(nP)\cap\mathbb{Z}^d|) \\ &= (-1)^d E_S(-n) - (-1)^d E_P(-n) + |nK\cap\mathbb{Z}^d| - |nL\cap\mathbb{Z}^d| \end{aligned}$$

for $0 \neq n \in \mathbb{N}$. Therefore, after simplifying this equality, we obtain:

$$2(c_{d-1}n^{d-1} + c_{d-3}n^{d-3} + \text{terms of lower degree}) = |nK\cap\mathbb{Z}^d| - |nL\cap\mathbb{Z}^d| = g(n).$$

By the comments just before Lemma 3.13, we have the inclusions

$$\psi(L) \subset M := \left[\left(\bigcup_{i=1}^{d-1}(\partial P\cap K_i)\right)\cup K_d\right]\setminus H \subset K := \left(\bigcup_{i=1}^{d} K_i\right)\setminus H.$$

Using Lemma 3.14, we obtain

$$M = \left(\bigcup_{i=1}^{d-1}(P_i\setminus H_i)\right)\cup(K_d\setminus H_d) \quad\text{and}\quad K = \bigcup_{i=1}^{d}(K_i\setminus H_i).$$

Set $h(n) = |nK\cap\mathbb{Z}^d| - |nM\cap\mathbb{Z}^d|$. Since $P_i\setminus H_i \subset P_i$, $K_i\setminus H_i\subset K_i$ for all i and because $P_i\cap P_j$, $K_i\cap K_j$ are polytopes of dimension at most $d-2$ for $i\neq j$, by the inclusion-exclusion principle (Lemma 3.15) we obtain

$$\begin{aligned} h(n) = \sum_{i=1}^{d}&|n(K_i\setminus H_i)\cap\mathbb{Z}^d| - \sum_{i=1}^{d-1}|n(P_i\setminus H_i)\cap\mathbb{Z}^d| \\ &-|n(K_d\setminus H_d)\cap\mathbb{Z}^d| + p(n) = \sum_{i=1}^{d-1}(E_{K_i}(n) - E_{P_i}(n)) + p(n) \qquad (n\gg 0), \end{aligned}$$

where $|p(n)|$ is bounded by a polynomial function $P(n)$ of degree at most $d-2$. In particular $\lim_{n\to\infty}(p(n)/n^{d-1}) = 0$. By Lemma 3.13, the map $\overline{\psi}\colon nL\to n\psi(L)$ given by $\overline{\psi}(n\alpha) = n\psi(\alpha)$ is injective. Hence

$$\overline{\psi}(nL\cap\mathbb{Z}^d)\subset n\psi(L)\cap\mathbb{Z}^d\subset nM\cap\mathbb{Z}^d \;\Rightarrow\; |nL\cap\mathbb{Z}^d|\leq|nM\cap\mathbb{Z}^d|.$$

Consequently $g(n) = |nK\cap\mathbb{Z}^d| - |nL\cap\mathbb{Z}^d| \geq h(n)$. Altogether we get

$$2c_{d-1} = \lim_{n\to\infty}\frac{g(n)}{n^{d-1}} \geq \lim_{n\to\infty}\frac{h(n)}{n^{d-1}} = \lim_{n\to\infty}\left(\frac{\sum_{i=1}^{d-1}(E_{K_i}(n)-E_{P_i}(n))}{n^{d-1}} + \frac{p(n)}{n^{d-1}}\right).$$

Therefore the required inequality follows by observing that the polynomial function $f_i(n) = E_{K_i}(n) - E_{P_i}(n)$ is the Hilbert function of I_i. Thus $f_i(n)$ has degree $d-1$ and its leading coefficient is equal to $e(I_i)/(d-1)!$.

Case (II): $\dim(P) = d-1$. Let $E_S(x) = a_dx^d + \cdots + a_1x + 1$ (resp. $E_P(x) = b_{d-1}x^{d-1} + \cdots + b_1x + 1$) be the Ehrhart polynomial of S (resp. P). There is an injective map from nP to nK_d induced by $\alpha\mapsto(\alpha_1,\ldots,\alpha_{d-1},0)$. Hence

$$\mathrm{vol}(P) = \lim_{n\to\infty}\frac{|\mathbb{Z}^d\cap nP|}{n^{d-1}} \leq \lim_{n\to\infty}\frac{|\mathbb{Z}^d\cap nK_d|}{n^{d-1}} = \mathrm{vol}(K_d).$$

The facets of S are $K_1, \ldots, K_d$ and $K_{d+1} := P$. Therefore by Proposition 3.6 and using the formulas for a_{d-1} and b_{d-1} (see Remark 3.4) we conclude that

$$\begin{aligned} c_{d-1} = a_{d-1} - b_{d-1} &= \frac{1}{2}\sum_{i=1}^{d+1} \mathrm{vol}(K_i) - \mathrm{vol}(P) = -\frac{1}{2}\mathrm{vol}(P) + \frac{1}{2}\sum_{i=1}^{d} \mathrm{vol}(K_i) \\ &\geq \frac{1}{2}\sum_{i=1}^{d-1} \mathrm{vol}(K_i) = \frac{1}{2}\sum_{i=1}^{d-1} \frac{e(I_i)}{(d-1)!}. \end{aligned}$$

□

Let $e_0, e_1, \ldots, e_d$ be the Hilbert coefficients of $\mathcal{F}$. Recall that we have

$$f(n) = e_0\binom{n+d-1}{d} - e_1\binom{n+d-2}{d-1} + \cdots + (-1)^{d-1}e_{d-1}\binom{n}{1} + (-1)^d e_d,$$

where $e_0 = e(I)$ is the multiplicity of I and $c_d = e_0/d!$. Notice that $e_d = 0$ because $f(0) = 0$, and $e_i \geq 0$ for all i; this follows from [11].

Corollary 3.17. $e_0(d-1) - 2e_1 \geq e(I_1) + \cdots + e(I_{d-1}) \geq d - 1$.

Proof. From the equality $c_{d-1} = \frac{1}{d!}\left[e_0\binom{d}{2} - de_1\right]$ and using Proposition 3.16 we obtain the desired inequality. □

Example 3.18. Let $\mathfrak{m} = (x_1, \ldots, x_d)$ and let $I = \mathfrak{m}^k$. Then

$$f(n) = \binom{kn+d-1}{d} = \frac{k^d}{d!}n^d + \frac{k^{d-1}}{(d-2)!2}n^{d-1} + \text{ terms of lower degree},$$

$e_0 = k^d$, $e_1 = (d-1)(k^d - k^{d-1})/2$, and we have equality in Proposition 3.16.

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