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FUNCTIONAL RELATIONS AND SPECIAL VALUES OF MORDELL-TORNHEIM TRIPLE ZETA AND L-FUNCTIONS

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Abstract. In this paper, we prove the existence of meromorphic continuation of certain triple zeta-functions of Lerch's type. Based on this result, we prove some functional relations for triple zeta and L-functions of the Mordell-Tornheim type. Using these functional relations, we prove new explicit evaluation formulas for special values of these functions. These can be regarded as triple analogues of known results for double zeta and L-functions.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

In this paper, we study triple zeta and L-functions defined by

$$\zeta_{MT,3}(s_1, s_2, s_3, s_4\,; \alpha, \beta, \gamma, \delta) := \sum_{l,m,n=1}^{\infty} \frac{e^{2\pi i l\alpha} e^{2\pi i m\beta} e^{2\pi i n\gamma} e^{2\pi i(l+m+n)\delta}}{l^{s_1} m^{s_2} n^{s_3} (l+m+n)^{s_4}}, \tag{1.1}$$

$$L_{MT,3}(s_1, s_2, s_3, s_4\,; \varphi, \chi, \psi, \omega) := \sum_{l,m,n=1}^{\infty} \frac{\varphi(l)\chi(m)\psi(n)\omega(l+m+n)}{l^{s_1} m^{s_2} n^{s_3} (l+m+n)^{s_4}}, \tag{1.2}$$

where $i = \sqrt{-1}$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\varphi, \chi, \psi, \omega$ are primitive Dirichlet characters. These are absolutely convergent when $\Re s_j \geq 1$ $(1 \leq j \leq 4)$. These are triple analogues of Tornheim's double series $T(a, b, c) = \sum_{m,n\geq 1} m^{-a} n^{-b} (m+n)^{-c}$ ([11]) which was also studied independently by Mordell ([6]).

The first-named author ([3]) analytically studied $T(s_1, s_2, s_3)$ for $(s_1, s_2, s_3) \in \mathbb{C}^3$, and more generally defined a multiple analogue of $T(s_1, s_2, s_3)$, which is called the Mordell-Tornheim multiple zeta-function denoted by $\zeta_{MT,r}(s_1, \ldots, s_r, s_{r+1})$ ([4]). In particular, $\zeta_{MT,2}(s_1, s_2, s_3) = T(s_1, s_2, s_3)$ and $\zeta_{MT,3}(s_1, s_2, s_3, s_4) = \zeta_{MT,3}(s_1, s_2, s_3, s_4\,; 1, 1, 1, 1)$.

The double series $T(a, b, c)$ and its χ-analogue have been extensively studied. For instance, evaluation formulas for special values of them have been studied by many authors, and functional relations for them have been obtained in [7, 8, 17, 18]. However, there are very few papers in which properties of $\zeta_{MT,r}$ for $r \geq 3$ have been

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studied. Recently Ochiai and the authors studied $\zeta_{MT,3}(s_1, s_2, s_3, s_4)$ and relevant functions, and proved some functional relations for them ([5]).

In this paper we aim to study analytic properties of (1.1) and (1.2), and to prove certain functional relations for them. The results in this paper can be regarded as triple analogues of the known results for double zeta and L-functions proved in the papers quoted above and [9, 10, 13, 14, 15], and also be regarded as χ-analogues of the results in [5]. Therefore this paper is a step toward the general theory of functional relations for Mordell-Tornheim r-ple zeta and L-functions including known results of r-ple zeta values given in [16].

2. Analytic continuations of certain triple zeta and L-functions

The aim of this section is to prove the following proposition by using the same method as in [3, 4].

Proposition 2.1. *The triple zeta-function* (1.1) *and the triple L-function* (1.2) *can be continued meromorphically to the whole $\mathbb{C}^4$ space.*

Proof. Since

$$e^{2\pi il\alpha}e^{2\pi im\beta}e^{2\pi in\gamma}e^{2\pi i(l+m+n)\delta} = e^{2\pi il(\alpha+\delta)}e^{2\pi im(\beta+\delta)}e^{2\pi in(\gamma+\delta)},$$

we may assume $\delta = 0$ without loss of generality in the proof of (1.1). We use the notation

$$\zeta_{MT,2}(s_1, s_2, s_3\,; \alpha, \beta, \gamma) := \sum_{m_1,m_2=1}^{\infty} \frac{e^{2\pi im_1\alpha}e^{2\pi im_2\beta}e^{2\pi i(m_1+m_2)\gamma}}{m_1^{s_1}\ m_2^{s_2}(m_1+m_2)^{s_3}}.$$

First we assume $\Re s_j > 1$ $(1 \le j \le 4)$. We recall the Mellin-Barnes formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz, \tag{2.1}$$

where s, λ are complex numbers with $\Re s > 0$, $|\arg\lambda| < \pi$, $\lambda \ne 0$, c is real with $-\Re s < c < 0$, and the path (c) of integration is the vertical line $\Re z = c$. Since

$$(l+m+n)^{-s_4} = (l+m)^{-s_4}\cdot\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)}\left(\frac{n}{l+m}\right)^z dz,$$

where $-\Re s_4 < c < 0$, (1.1) with $\delta = 0$ can be written as

$$\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)}\zeta_{MT,2}(s_1, s_2, s_4+z; \alpha, \beta, 0)\phi(s_3-z, \gamma)dz,$$

where $\phi(s,\gamma) = \sum_{n\ge1} e^{2\pi in\gamma}n^{-s}$. Furthermore, by using

$$(l+m)^{-s_4} = l^{-s_4}\cdot\frac{1}{2\pi i}\int_{(c')}\frac{\Gamma(s_4+z')\Gamma(-z')}{\Gamma(s_4)}\left(\frac{m}{l}\right)^{z'} dz',$$

we have

$$\begin{aligned}\zeta_{MT,2}(s_1, s_2, s_4; \alpha, \beta, 0) = \frac{1}{2\pi i}\int_{(c')}&\frac{\Gamma(s_4+z')\Gamma(-z')}{\Gamma(s_4)}\\ &\times \phi(s_1+s_4+z', \alpha)\phi(s_2-z', \beta)dz',\end{aligned} \tag{2.2}$$

where $-\Re s_4 < c' < 0$. We set $\nu(x) := 1$ (resp. 0) if $x \in \mathbb{Z}$ (resp. $x \notin \mathbb{Z}$). Then we can determine the poles of the integrand on the right-hand side of (2.2) as

$$z' = -s_4 - k,\ z' = k\ \ (k \in \mathbb{N}_0)\,;\ \nu(\alpha)(z' = 1 - s_1 - s_4),\ \ \nu(\beta)(z' = s_2 - 1).$$

In fact, the two former expressions come from the Γ-factors, and the latter two expressions come from the ϕ-factors.

Now we shift the path $\Re z' = c'$ to $\Re z' = M - \varepsilon$ for a sufficiently large $M \in \mathbb{N}$ and a sufficiently small positive $\varepsilon \in \mathbb{R}$. Then the relevant poles through this shifting are $z' = s_2 - 1$ and $z' = k$ $(0 \le k \le M-1)$. Counting their residues, we can obtain

$$\begin{aligned}
\zeta_{MT,2}(s_1, s_2, s_4; \alpha, \beta, 0) &= \nu(\beta) \frac{\Gamma(s_2 + s_4 - 1)\Gamma(1 - s_2)}{\Gamma(s_4)} \phi(s_1 + s_2 + s_4 - 1, \alpha) \\
&+ \sum_{k=0}^{M-1} \binom{-s_4}{k} \phi(s_1 + s_4 + k, \alpha)\phi(s_2 - k, \beta) \\
&+ \frac{1}{2\pi i} \int_{(M-\varepsilon)} \frac{\Gamma(s_4 + z')\Gamma(-z')}{\Gamma(s_4)} \phi(s_1 + s_4 + z', \alpha)\phi(s_2 - z', \beta) dz'.
\end{aligned}$$

Similar to our previous results [3, 4, 5], this gives the meromorphic continuation of $\zeta_{MT,2}(s_1, s_2, s_4; \alpha, \beta, 0)$ and determines its singularities as

$$\nu(\alpha)(s_1 + s_4 = 1 - k),\ \nu(\beta)(s_2 + s_4 = 1 - k),\ \nu(\alpha)\nu(\beta)(s_1 + s_2 + s_4 = 2).$$

Hence singularities of $\zeta_{MT,2}(s_1, s_2, s_4 + z; \alpha, \beta, 0)\phi(s_3 - z, \gamma)$ are

$$\begin{aligned}
&\nu(\alpha)(s_1 + s_4 + z = 1 - k),\ \nu(\beta)(s_2 + s_4 + z = 1 - k), \\
&\nu(\alpha)\nu(\beta)(s_1 + s_2 + s_4 + z = 2),\ \nu(\gamma)(s_3 - z = 1).
\end{aligned}$$

These expressions can be rewritten as $z = -s_1 - s_4 + 1 - k$, $z = -s_2 - s_4 + 1 - k$, $z = 2 - s_1 - s_2 - s_4$, $z = s_3 - 1$. Therefore, when we shift the path $\Re z = c$ to $\Re z = M - \varepsilon$ for a sufficiently large $M \in \mathbb{N}$ and a sufficiently small positive $\varepsilon \in \mathbb{R}$, we have only to consider the poles $z = k$ and $z = s_3 - 1$ in the case $\nu(\gamma) = 1$. Hence we have

$$\begin{aligned}
&\zeta_{MT,3}(s_1, s_2, s_3, s_4\,; \alpha, \beta, \gamma, 0) \\
&\quad = -\frac{\Gamma(s_3 + s_4 - 1)\Gamma(1 - s_3)}{\Gamma(s_4)} \times \mathrm{Res}_{z = s_3 - 1}(\zeta_{MT,2}(s_1, s_2, s_4 + z; \alpha, \beta, 0)\phi(s_3 - z, \gamma)) \\
&\quad + \sum_{k=0}^{M-1} \binom{-s_4}{k} \zeta_{MT,2}(s_1, s_2, s_4 + k; \alpha, \beta, 0)\phi(s_3 - k, \gamma) \\
&\quad + \frac{1}{2\pi i} \int_{(M-\varepsilon)} \frac{\Gamma(s_4 + z)\Gamma(-z)}{\Gamma(s_4)} \zeta_{MT,2}(s_1, s_2, s_4 + z; \alpha, \beta, 0)\phi(s_3 - z, \gamma) dz.
\end{aligned}$$

This gives the meromorphic continuation of (1.1).

As for (1.2), we denote the conductors of $\varphi, \chi, \psi, \omega$ by f_1, f_2, f_3, f_4, respectively. It follows from [19, Lemma 4.7] that (1.2) can be written as

$$\begin{aligned}
&\frac{1}{\tau(\overline{\varphi})\tau(\overline{\chi})\tau(\overline{\psi})\tau(\overline{\omega})} \sum_{j_1=1}^{f_1} \sum_{j_2=1}^{f_2} \sum_{j_3=1}^{f_3} \sum_{j_4=1}^{f_4} \overline{\varphi}(j_1)\overline{\chi}(j_2)\overline{\psi}(j_3)\overline{\omega}(j_4) \\
&\quad \times \sum_{l,m,n=1}^{\infty} \frac{e^{2\pi i l(j_1/f_1 + j_4/f_4)} e^{2\pi i m(j_2/f_2 + j_4/f_4)} e^{2\pi i n(j_3/f_3 + j_4/f_4)}}{l^{s_1} m^{s_2} n^{s_3} (l + m + n)^{s_4}},
\end{aligned}$$

where $\tau(\,\cdot\,)$ is the Gauss sum. Hence, from the assertion for (1.1), we obtain the meromorphic continuation of (1.2). This completes the proof of Proposition 2.1. $\square$

3. Functional relations for triple zeta and L-functions

In this section, we use the same method as in [7]. We denote by $\{B_j(x)\}$ the Bernoulli polynomials defined by $te^{xt}/(e^t-1)=\sum_{j\geq 0} B_j(x)t^j/j!$ ($|t|<2\pi$). It is known (see [1, p. 266, (22) and p. 267, (24)]) that

$$B_{2j} := B_{2j}(0) = (-1)^{j+1}2(2j)!(2\pi)^{-2j}\zeta(2j) \qquad (j\in\mathbb{N}_0), \tag{3.1}$$

$$B_j(x-[x]) = -\frac{j!}{(2\pi i)^j}\lim_{K\to\infty}\sum_{\substack{k=-K\\ k\neq 0}}^{K}\frac{e^{2\pi ikx}}{k^j} \qquad (j\in\mathbb{N}), \tag{3.2}$$

where $\zeta(s)$ is the Riemann zeta-function and $[\cdot]$ is the integer part. Especially, we can see $\zeta(0)=-1/2$ by (3.1). For $k\in\mathbb{Z}$, $j\in\mathbb{N}$ we have

$$\int_0^1 e^{-2\pi ikx}B_j(x)\,dx = \begin{cases} 0 & (k=0),\\ -(2\pi ik)^{-j}j! & (k\neq 0).\end{cases} \tag{3.3}$$

In fact the case of $k=0$ is obvious, and in the case of $k\neq 0$, we have (3.3) by using (3.2). Next we quote [1, p. 276, 19(b)], for $p,q\geq 1$, which is

(3.4)

$$B_p(x)B_q(x) = \sum_{k=0}^{\max(p,q)/2}\left\{p\binom{q}{2k}+q\binom{p}{2k}\right\}\frac{B_{2k}B_{p+q-2k}(x)}{p+q-2k} - (-1)^p\frac{p!q!}{(p+q)!}B_{p+q}.$$

Definition 3.1. Let $\zeta(s)$ and $L(s,\chi)$ be the Riemann zeta-function and the Dirichlet L-function, respectively. Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ and φ,χ,ψ,ω be Dirichlet characters. For $s_1,s_2,s_3,s_4\in\mathbb{C}$ with $\Re s_j\geq 1$ ($1\leq j\leq 4$), we define

$$L_{MT,2}(s_1,s_2,s_3\,;\varphi,\chi,\psi) := \sum_{m_1,m_2=1}^{\infty}\frac{\varphi(m_1)\chi(m_2)\psi(m_1+m_2)}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}},$$

$$S_j(s_1,s_2,s_3,s_4\,;\alpha,\beta,\gamma,\delta) := \sum_{\substack{m_1,m_2,m_3,m_4=1\\ m_1+m_j=m_k+m_l}}^{\infty}\frac{e^{2\pi im_1\alpha}e^{2\pi im_2\beta}e^{2\pi im_3\gamma}e^{2\pi im_4\delta}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}m_4^{s_4}},$$

$$T_j(s_1,s_2,s_3,s_4\,;\varphi,\chi,\psi,\omega) := \sum_{\substack{m_1,m_2,m_3,m_4=1\\ m_1+m_j=m_k+m_l}}^{\infty}\frac{\varphi(m_1)\chi(m_2)\psi(m_3)\omega(m_4)}{m_1^{s_1}m_2^{s_2}m_3^{s_3}m_4^{s_4}},$$

where $\{j,k,l\}=\{2,3,4\}$. Note that these functions are absolutely convergent in this region. In fact, putting $m_1+m_2=m_3+m_4=n$, we can see that

$$\sum_{\substack{m_1,m_2,m_3,m_4=1\\ m_1+m_2=m_3+m_4}}^{\infty}\frac{1}{m_1m_2m_3m_4} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\frac{1}{m(n-m)}\right)^2$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{m=1}^{n-1}\left(\frac{1}{m}+\frac{1}{n-m}\right)\right)^2 = 8\sum_{1\leq l<m<n}\frac{1}{lmn^2} + 4\sum_{1\leq m<n}\frac{1}{m^2n^2} < \infty.$$

For $a, b \in \mathbb{N}$ and $s_3, s_4 \in \mathbb{C}$ with $\Re s_j \geq 1$ $(j = 3, 4)$, we define

$$\begin{aligned} &M_1(a, b, s_3, s_4\,; \gamma, \delta) \\ &\quad := \frac{2}{a!\, b!} \sum_{k=0}^{\max(a,b)/2} \left\{ a \binom{b}{2k} + b \binom{a}{2k} \right\} (a + b - 2k - 1)!(2k)! \\ &\qquad \times \zeta(2k) \zeta_{MT,2}(s_3, s_4, a + b - 2k\,; \gamma, \delta, 1), \end{aligned} \tag{3.5}$$

$$\begin{aligned} &M_2(a, b, s_3, s_4\,; \gamma, \delta) \\ &\quad := \frac{2}{a!\, b!} \sum_{k=0}^{\max(a,b)/2} \left\{ a \binom{b}{2k} + b \binom{a}{2k} \right\} (a + b - 2k - 1)!(2k)! \\ &\qquad \times \zeta(2k) \zeta_{MT,2}(a + b - 2k, s_4, s_3\,; 1, \delta, \gamma) \\ &\qquad + \frac{2(-1)^{a+b}}{a!\, b!} \sum_{k=0}^{\max(a,b)/2} \left\{ a \binom{b}{2k} + b \binom{a}{2k} \right\} (a + b - 2k - 1)!(2k)! \\ &\qquad \times \zeta(2k) \zeta_{MT,2}(a + b - 2k, s_3, s_4\,; 1, \gamma, \delta) \\ &\qquad + (-1)^{a+1} \frac{(2\pi i)^{a+b} B_{a+b}}{(a + b)!} \phi(s_3 + s_4, \gamma + \delta). \end{aligned} \tag{3.6}$$

Similarly, for any ψ, ω, we define $N_1(a, b, s_3, s_4\,; \psi, \omega)$ as a function given by replacing $\zeta_{MT,2}(s_3, s_4, a + b - 2k\,; \gamma, \delta, 1)$ by $L_{MT,2}(s_3, s_4, a + b - 2k\,; \psi, \omega, \mathbf{1})$ on the right-hand side of (3.5), where $\mathbf{1}$ is the trivial character. Furthermore we define $N_2(a, b, s_3, s_4\,; \psi, \omega)$ as that given by replacing $\zeta_{MT,2}(a + b - 2k, s_4, s_3\,; 1, \delta, \gamma)$, $\zeta_{MT,2}(a+b-2k, s_3, s_4\,; 1, \gamma, \delta)$ and $\phi(s_3+s_4, \gamma+\delta)$ by $L_{MT,2}(a+b-2k, s_4, s_3\,; \mathbf{1}, \omega, \psi)$, $L_{MT,2}(a + b - 2k, s_3, s_4\,; \mathbf{1}, \psi, \omega)$ and $L(s_3 + s_4, \psi\omega)$ on the right-hand side of (3.6), respectively.

Note that, similar to the proof of Proposition 2.1, by using the Mellin-Barnes formula (2.1), we can prove the existence of meromorphic continuation of $\zeta_{MT,2}(s_1, s_2, s_3; \alpha, \beta, \gamma)$ and $L_{MT,2}(s_1, s_2, s_3; \varphi, \chi, \psi)$, and furthermore of M_1, M_2, N_1, N_2 in Definition 3.1.

Lemma 3.2. *For $a, b \in \mathbb{N}$, $\gamma, \delta \in \mathbb{R}$ and $s_3, s_4 \in \mathbb{C}$ with $\Re s_3 \geq 1$, $\Re s_4 \geq 1$, we have*

$$\begin{aligned} &(-1)^b \zeta_{MT,3}(b, s_3, s_4, a\,; 1, \gamma, \delta, 1) \\ &\qquad + (-1)^a \zeta_{MT,3}(s_3, s_4, a, b\,; \gamma, \delta, 1, 1) \\ &\qquad + S_2(a, b, s_3, s_4\,; 1, 1, \gamma, \delta) \\ &\quad = M_1(a, b, s_3, s_4\,; \gamma, \delta) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\zeta_{MT,3}(s_4, a, b, s_3\,; \delta, 1, 1, \gamma) \\ &\qquad + (-1)^a S_3(a, b, s_3, s_4\,; 1, 1, \gamma, \delta) \\ &\qquad + (-1)^b S_4(a, b, s_3, s_4\,; 1, 1, \gamma, \delta) \\ &\qquad + (-1)^{a+b} \zeta_{MT,3}(a, b, s_3, s_4\,; 1, 1, \gamma, \delta) \\ &\quad = M_2(a, b, s_3, s_4\,; \gamma, \delta). \end{aligned} \tag{3.8}$$

Proof. We have

$$\lim_{K\to\infty}\int_0^1 \sum_{k=1}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{l=1}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi im(x+\gamma)}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi in(x+\delta)}}{n^{s_4}}dx=0,$$

$$\begin{aligned}
&\lim_{K\to\infty}\int_0^1 \sum_{k=-K}^{-1}\frac{e^{2\pi ikx}}{k^a}\sum_{l=1}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi im(x+\gamma)}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi in(x+\delta)}}{n^{s_4}}dx\\
&\quad=(-1)^a\zeta_{MT,3}(b,s_3,s_4,a\,;1,\gamma,\delta,1),\\
&\lim_{K\to\infty}\int_0^1 \sum_{k=1}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{l=-K}^{-1}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi im(x+\gamma)}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi in(x+\delta)}}{n^{s_4}}dx\\
&\quad=(-1)^b\zeta_{MT,3}(s_3,s_4,a,b\,;\gamma,\delta,1,1),\\
&\lim_{K\to\infty}\int_0^1 \sum_{k=-K}^{-1}\frac{e^{2\pi ikx}}{k^a}\sum_{l=-K}^{-1}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi im(x+\gamma)}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi in(x+\delta)}}{n^{s_4}}dx\\
&\quad=(-1)^{a+b}S_2(a,b,s_3,s_4\,;1,1,\gamma,\delta)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{K\to\infty}\int_0^1 \sum_{k=1}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{l=1}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{-2\pi im(x-\gamma)}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi in(x+\delta)}}{n^{s_4}}dx\\
&\quad=\zeta_{MT,3}(s_4,a,b,s_3\,;\delta,1,1,\gamma),\\
&\lim_{K\to\infty}\int_0^1 \sum_{k=-K}^{-1}\frac{e^{2\pi ikx}}{k^a}\sum_{l=1}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{-2\pi im(x-\gamma)}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi in(x+\delta)}}{n^{s_4}}dx\\
&\quad=(-1)^a S_3(a,b,s_3,s_4\,;1,1,\gamma,\delta),\\
&\lim_{K\to\infty}\int_0^1 \sum_{k=1}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{l=-K}^{-1}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{-2\pi im(x-\gamma)}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi in(x+\delta)}}{n^{s_4}}dx\\
&\quad=(-1)^b S_4(a,b,s_3,s_4\,;1,1,\gamma,\delta),\\
&\lim_{K\to\infty}\int_0^1 \sum_{k=-K}^{-1}\frac{e^{2\pi ikx}}{k^a}\sum_{l=-K}^{-1}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{-2\pi im(x-\gamma)}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi in(x+\delta)}}{n^{s_4}}dx\\
&\quad=(-1)^{a+b}\zeta_{MT,3}(a,b,s_3,s_4\,;1,1,\gamma,\delta).
\end{aligned}$$

Hence we see that, for example, for $\Re s_3>1$ and $\Re s_4>1$,

$$\begin{aligned}
&(-1)^a\zeta_{MT,3}(b,s_3,s_4,a\,;1,\gamma,\delta,1)+(-1)^b\zeta_{MT,3}(s_3,s_4,a,b\,;\gamma,\delta,1,1)\\
&\quad+(-1)^{a+b}S_2(a,b,s_3,s_4\,;1,1,\gamma,\delta)\\
&\quad=\int_0^1\lim_{K\to\infty}\sum_{\substack{k=-K\\k\neq0}}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{\substack{l=-K\\l\neq0}}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi im(x+\gamma)}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi in(x+\delta)}}{n^{s_4}}dx.
\end{aligned}$$

Changing the order of limitation and integration is justified by bounded convergency. Multiplying by $(-1)^{a+b}$ on both sides, and using (3.1)-(3.4), we obtain (3.7), and similarly obtain (3.8). Note that these hold for $\Re s_3\geq1$ and $\Re s_4\geq1$ by absolute convergency. $\square$

Next, in the proof of Lemma 3.2, we replace $\sum_{m\geq 1} e^{2\pi i m(x+\gamma)} m^{-s}$ and $\sum_{m\geq 1} e^{2\pi i m(x+\delta)} m^{-s}$ by $\sum_{m\geq 1} \psi(m) e^{2\pi i m x} m^{-s}$ and $\sum_{m\geq 1} \omega(m) e^{2\pi i m x} m^{-s}$, respectively. Then we obtain the following.

Lemma 3.3. *For $a, b \in \mathbb{N}$ and $s_3, s_4 \in \mathbb{C}$ with $\Re s_3 \geq 1$, $\Re s_4 \geq 1$, we have*

$$\begin{aligned}(-1)^b L_{MT,3}(b, s_3, s_4, a\,; \mathbf{1}, \psi, \omega, \mathbf{1}) + (-1)^a L_{MT,3}(s_3, s_4, a, b\,; \psi, \omega, \mathbf{1}, \mathbf{1}) \\ + T_2(a, b, s_3, s_4\,; \mathbf{1}, \mathbf{1}, \psi, \omega) = N_1(a, b, s_3, s_4\,; \psi, \omega)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&L_{MT,3}(s_4, a, b, s_3\,; \omega, \mathbf{1}, \mathbf{1}, \psi) + (-1)^a T_3(a, b, s_3, s_4\,; \mathbf{1}, \mathbf{1}, \psi, \omega) \\ &\quad + (-1)^b T_4(a, b, s_3, s_4\,; \mathbf{1}, \mathbf{1}, \psi, \omega) + (-1)^{a+b} L_{MT,3}(a, b, s_3, s_4\,; \mathbf{1}, \mathbf{1}, \psi, \omega) \\ &= N_2(a, b, s_3, s_4\,; \psi, \omega).\end{aligned} \tag{3.10}$$

Using these results, we can obtain the main theorems.

Theorem 3.4. *For $a, b, c \in \mathbb{N}$ and $\delta \in \mathbb{R}$,*

$$\begin{aligned}&\zeta_{MT,3}(a, b, c, s\,; 1, 1, 1, \delta) - (-1)^{b+c} \zeta_{MT,3}(b, c, s, a\,; 1, 1, \delta, 1) \\ &\qquad - (-1)^{c+a} \zeta_{MT,3}(c, s, a, b\,; 1, \delta, 1, 1) - (-1)^{a+b} \zeta_{MT,3}(s, a, b, c\,; \delta, 1, 1, 1) \\ &\quad = (-1)^{a+b} M_2(a, b, c, s\,; 1, \delta) - (-1)^b M_1(a, c, b, s\,; 1, \delta) \\ &\qquad - (-1)^a M_1(c, b, a, s\,; 1, \delta)\end{aligned} \tag{3.11}$$

holds for $s \in \mathbb{C}$ except for singular points of functions on both sides.

Proof. By the definition, we have

$$S_3(a, b, c, s\,; 1, 1, 1, \delta) = \sum_{\substack{k,l,m,n=1 \\ k+m=l+n}}^{\infty} \frac{e^{2\pi i n\delta}}{k^a m^c l^b n^s} = S_2(a, c, b, s\,; 1, 1, 1, \delta),$$

$$S_4(a, b, c, s\,; 1, 1, 1, \delta) = \sum_{\substack{k,l,m,n=1 \\ m+l=k+n}}^{\infty} \frac{e^{2\pi i n\delta}}{m^c l^b k^a n^s} = S_2(c, b, a, s\,; 1, 1, 1, \delta).$$

Therefore we have

$$\begin{aligned}&S_2(a, c, b, s\,; 1, 1, 1, \delta) = M_1(a, c, b, s\,; 1, 1, 1, \delta) \\ &\quad - (-1)^c \zeta_{MT,3}(c, b, s, a\,; 1, 1, \delta, 1) - (-1)^a \zeta_{MT,3}(b, s, a, c\,; 1, \delta, 1, 1), \\ &S_2(c, b, a, s\,; 1, 1, 1, \delta) = M_1(c, b, a, s\,; 1, 1, 1, \delta) \\ &\quad - (-1)^b \zeta_{MT,3}(b, a, s, c\,; 1, 1, \delta, 1) - (-1)^c \zeta_{MT,3}(a, s, c, b\,; 1, \delta, 1, 1).\end{aligned}$$

Substituting these relations into (3.8), and noting the existence of meromorphic continuation, we obtain the assertion of this theorem. □

Note that the case $\delta = 1$ coincides with the result in [5, Theorem 4.5]. By the same consideration as in the proof of Theorem 3.4 and using Lemma 3.2, we can obtain the following result which can be regarded as a χ-analogue of [5, Theorem 4.5] and as a triple analogue of the recent results about double zeta and L-functions (see [7, 8, 17, 18]).

Theorem 3.5. *For $a, b, c \in \mathbb{N}$,*

$$\begin{aligned} &L_{MT,3}(a,b,c,s\,;\mathbf{1},\mathbf{1},\mathbf{1},\chi) - (-1)^{b+c} L_{MT,3}(b,c,s,a\,;\mathbf{1},\mathbf{1},\chi,\mathbf{1}) \\ &\qquad - (-1)^{c+a} L_{MT,3}(c,s,a,b\,;\mathbf{1},\chi,\mathbf{1},\mathbf{1}) - (-1)^{a+b} L_{MT,3}(s,a,b,c\,;\chi,\mathbf{1},\mathbf{1},\mathbf{1}) \\ &\quad = (-1)^{a+b} N_2(a,b,c,s\,;\mathbf{1},\chi) - (-1)^{b} N_1(a,c,b,s\,;\mathbf{1},\chi) \\ &\qquad - (-1)^{a} N_1(c,b,a,s\,;\mathbf{1},\chi) \end{aligned}$$

holds for $s \in \mathbb{C}$ except for singular points of functions on both sides.

Example 3.6. Putting $(a,b,c,s) = (1,1,1,2)$ in Theorem 3.5, we have

$$\begin{aligned} &L_{MT,3}(1,1,1,2\,;\mathbf{1},\mathbf{1},\mathbf{1},\chi) - 3L_{MT,3}(1,1,2,1\,;\mathbf{1},\mathbf{1},\chi,\mathbf{1}) \\ &\quad = -2L_{MT,2}(2,2,1;\mathbf{1},\chi,\mathbf{1}) - 2L_{MT,2}(2,1,2;\mathbf{1},\mathbf{1},\chi) \\ &\qquad - 4L_{MT,2}(1,2,2;\mathbf{1},\chi,\mathbf{1}) - 2\zeta(2)L(3,\chi). \end{aligned} \tag{3.12}$$

4. Alternating triple zeta values and triple L-values

In this section, we prove new evaluation formulas for alternating triple zeta-functions and for triple L-functions. We first prepare the following lemma which comes from [12, Lemma 8] and is equivalent to [5, Lemma 2.1].

Lemma 4.1. *Let $\phi(s) := \phi(s,1/2) = \sum_{n\geq 1}(-1)^n n^{-s} = \left(2^{1-s}-1\right)\zeta(s)$. For any function $g : \mathbb{N}_0 \to \mathbb{C}$, we put*

$$h(j) := \sum_{\mu=0}^{[(j-1)/2]} g(j-2\mu)\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu+1)!} \qquad (j \in \mathbb{N}).$$

Then we have

$$g(a) = -2 \sum_{\substack{j=1 \\ j\equiv a \ (\mathrm{mod}\ 2)}}^{a} \phi(a-j)h(j) \qquad (a \in \mathbb{N}).$$

Proof. We can easily obtain the assertion by putting $P_m = g(2m+1)$ and $Q_m = h(2m+1)$ (resp. $P_m = g(2m+2)$ and $Q_m = h(2m+2)$) for odd (resp. even) a in [12, Lemma 8]. $\square$

From [5, (3.6) and (3.10)], we have

$$\begin{aligned} &\sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n}\cos((l+m+n)\theta)}{l^{2p}m^{2p}n^{2p}(l+m+n)^{2d}} \\ &\quad + \sum_{j=0}^{p} \phi(2p-2j) \sum_{\nu=0}^{2j} \binom{2d-1+2j-\nu}{2j-\nu} \frac{(-1)^{\nu}\theta^{\nu}}{\nu!} \\ &\qquad \times \left\{ \sum_{\substack{l,m=1 \\ l\neq m}}^{\infty} \frac{(-1)^{l+m}\cos^{(\nu)}((l-m)\theta)}{l^{2p}m^{2p}(l-m)^{2d+2j-\nu}} - 2\sum_{l,m=1}^{\infty} \frac{(-1)^{l+m}\cos^{(\nu)}((l+m)\theta)}{l^{2p}m^{2p}(l+m)^{2d+2j-\nu}} \right\} \\ &\quad + \zeta(4p)\sum_{l=1}^{\infty} \frac{(-1)^{l}\cos(l\theta)}{l^{2p+2d}} = \sum_{\rho=0}^{d} C_{2d-2\rho}(2p)\frac{(-1)^{\rho}\theta^{2\rho}}{(2\rho)!} \ (\theta \in (-\pi,\pi)) \end{aligned} \tag{4.1}$$

and

$$\binom{2e+2p-1}{2p-1}\Big\{\zeta_{MT,2}(2e+2p,2p,2p)-\zeta_{MT,2}(2p,2p,2e+2p)\Big\} \\ =\sum_{\mu=0}^{e}C_{2e-2\mu}(2p)\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu+1)!} \tag{4.2}$$

for $p\in\mathbb{N}$ and $d,e\in\mathbb{N}_0$, where $\{C_{2\nu}(2p)\,|\,\nu\in\mathbb{N}_0\}$ are constants which can be determined inductively. In fact, applying Lemma 4.1 to (4.2), we have

$$C_{2e}(2p)=-2\sum_{\rho=0}^{e}\ \phi(2e-2\rho)\binom{2\rho+2p-1}{2p-1} \\ \times\{\zeta_{MT,2}(2\rho+2p,2p,2p)-\zeta_{MT,2}(2p,2p,2\rho+2p)\}. \tag{4.3}$$

On the other hand, letting $\theta=0$ in (4.1), we have

(4.4)

$$C_{2d}(2p)=\zeta_{MT,3}(2p,2p,2p,2d;1,1,1,1/2)+2\sum_{j=0}^{p}\ \phi(2p-2j)\ \binom{2d-1+2j}{2j} \\ \times\{\zeta_{MT,2}(2j+2d,2p,2p;1/2,1,1)-\zeta_{MT,2}(2p,2p,2j+2d;1,1,1/2)\} \\ +\zeta(4p)\phi(2p+2d)$$

for $d\in\mathbb{N}_0$. Hence, combining (4.3) and (4.4) in the case $d=e=p$, we obtain the following result which can be regarded as an alternating analogue of [5, Theorem 3.1], and a triple analogue of the previous results about double series [13, 14].

Theorem 4.2. *For $p\in\mathbb{N}$,*

$$\zeta_{MT,3}(2p,2p,2p,2p;1,1,1,1/2)=2\sum_{\nu=0}^{p}\binom{2\nu+2p-1}{2p-1}\left(2^{1-2p+2\nu}-1\right) \\ \times\zeta(2p-2\nu)\Big\{\zeta_{MT,2}(2p,2p,2p+2\nu)-\zeta_{MT,2}(2p+2\nu,2p,2p) \\ +\zeta_{MT,2}(2p,2p,2p+2\nu;1,1,1/2)-\zeta_{MT,2}(2p+2\nu,2p,2p;1/2,1,1)\Big\} \\ +\left(1-2^{1-4p}\right)\zeta(4p)^2. \tag{4.5}$$

Next we discuss special values of triple L-functions. As well as [5, (3.5)], we can prove that

$$\sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m+n}\sin((l+m+n)\theta)}{lmn(l+m+n)^2}+\theta\sum_{l,m=1}^{\infty}\frac{(-1)^{l+m}\cos((l+m)\theta)}{lm(l+m)^2} \\ -2\sum_{l,m=1}^{\infty}\frac{(-1)^{l+m}\sin((l+m)\theta)}{lm(l+m)^3}+\theta\sum_{l,m=1}^{\infty}\frac{(-1)^{m}\cos(m\theta)}{lm^2(l+m)} \\ -2\sum_{l,m=1}^{\infty}\frac{(-1)^{m}\sin(m\theta)}{lm^3(l+m)}=\frac{\zeta(2)^2}{5}\theta+\frac{\zeta(2)}{12}\theta^3 \tag{4.6}$$

for $\theta\in(-\pi,\pi)$. For example, we consider the case of $\theta=\pi/2$. For the odd Dirichlet character χ_4 of conductor 4 we know that $\chi_4(k)=\sin(k\pi/2)=-(-1)^k\sin(k\pi/2)$

$(k \in \mathbb{Z})$. Hence, by using $(-1)^k \cos(k\pi/2) = \left(i^k + i^{-k}\right)/2$ $(k \in \mathbb{Z})$, we can write (4.6) in the case of $\theta = \pi/2$ as

$$\begin{aligned}
&- L_{MT,3}(1,1,1,2;\mathbf{1},\mathbf{1},\mathbf{1},\chi_4) + \frac{\pi}{4}\Big(\zeta_{MT,2}(1,1,2;1,1,1/4) + \zeta_{MT,2}(1,1,2;1,1,-1/4)\Big) \\
&\quad + 2L_{MT,2}(1,1,3;\mathbf{1},\mathbf{1},\chi_4) + \frac{\pi}{4}\Big(\zeta_{MT,2}(1,2,1;1,1/4,1) + \zeta_{MT,2}(1,2,1;1,-1/4,1)\Big) \\
&\quad + 2L_{MT,2}(1,3,1;\mathbf{1},\chi_4,\mathbf{1}) = \frac{\pi}{10}\zeta(2)^2 + \frac{1}{96}\zeta(2)\pi^3 = \frac{13}{2880}\pi^5.
\end{aligned}$$

Thus we obtain a new evaluation formula

(4.7)

$$\begin{aligned}
&L_{MT,3}(1,1,1,2;\mathbf{1},\mathbf{1},\mathbf{1},\chi_4) = 2L_{MT,2}(1,1,3;\mathbf{1},\mathbf{1},\chi_4) + 2L_{MT,2}(1,3,1;\mathbf{1},\chi_4,\mathbf{1}) \\
&\quad + L(1,\chi_4)\Big\{\zeta_{MT,2}(1,1,2;1,1,1/4) + \zeta_{MT,2}(1,1,2;1,1,-1/4) \\
&\qquad + \zeta_{MT,2}(1,2,1;1,1/4,1) + \zeta_{MT,2}(1,2,1;1,-1/4,1)\Big\} - \frac{104}{75}L(5,\chi_4).
\end{aligned}$$

By using the partial fraction decomposition, we are able to see that $L_{MT,3}(1,1,1,2;\mathbf{1},\mathbf{1},\mathbf{1},\chi_4)$ coincides with $6\sum_{1\le l<m<n}\chi_4(n)/lmn^3$, which is an ordinary triple L-value (see [2]). This formula (4.7) can be regarded as a χ-analogue of that of $\zeta_{MT,3}$ (see, for example, [5, Example 3.2]) and as a triple analogue of that of double L-values (see [9, 10, 15]). Furthermore, by combining (3.12) and (4.7), we can also give an evaluation formula for $L_{MT,3}(1,1,2,1\,;\mathbf{1},\mathbf{1},\chi_4,\mathbf{1})$ in terms of Dirichlet L-values, double zeta and L-values.

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