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ITERATING THE CESÀRO OPERATORS

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ABSTRACT. The discrete Cesàro operator C associates to a given complex sequence $s = \{s_n\}$ the sequence $Cs \equiv \{b_n\}$, where $b_n = \frac{s_0+\cdots+s_n}{n+1}, n = 0, 1, \ldots$. When s is a convergent sequence we show that $\{C^n s\}$ converges under the sup-norm if, and only if, $s_0 = \lim_{n\to\infty} s_n$. For its adjoint operator C^*, we establish that $\{(C^*)^n s\}$ converges for any $s \in \ell^1$.

The continuous Cesàro operator, $Cf(x) \equiv \frac{1}{x}\int_0^x f(s)ds$, has two versions: the finite range case is defined for $f \in L^\infty(0,1)$ and the infinite range case for $f \in L^\infty(0,\infty)$. In the first situation, when $f : [0,1] \to \mathbb{C}$ is continuous we prove that $\{C^n f\}$ converges under the sup-norm to the constant function $f(0)$. In the second situation, when $f : [0,\infty) \to \mathbb{C}$ is a continuous function having a limit at infinity, we prove that $\{C^n f\}$ converges under the sup-norm if, and only if, $f(0) = \lim_{x\to\infty} f(x)$.

1. Introduction

We will denote by $\mathcal{S}$ the vector space consisting of all complex sequences. If $s \in \mathcal{S}$, we will write $s = \{s_n : n \in \mathbb{N}_0\}$ or $s = \{s(n) : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$. Given $s \in \mathcal{S}$, let b be the sequence given by

$$b_n \equiv \frac{s_0 + \ldots + s_n}{n+1}, \; n \in \mathbb{N}_0. \tag{1}$$

Then $C : \mathcal{S} \to \mathcal{S}$ defined by $Cs \equiv b$ is the (discrete) *Cesàro operator*.

As usual, let c be the Banach space consisting of all convergent sequences together with the sup-norm $\|\cdot\|_\infty$, and c_0 be its (closed) subspace formed by those sequences converging to 0. We will denote by e_k the sequence satisfying $e_k(m) = \delta_{k,m}, k, m \in \mathbb{N}_0$. The following two linear functionals defined on c will play a key role:

$$Ls \equiv \lim_{n\to\infty} s_n, \qquad \pi(s) \equiv s(0).$$

Clearly each of them is bounded. It is well known that $C(c) \subset c,\ C(c_0) \subset c_0$ and $LCs = Ls,\ \forall\, s \in c$. We also have $\pi Cs = \pi s,\ \forall\, s \in c$. Moreover, for $X = c,\ c_0$, the operator $C : X \to X$ is bounded and $\|C\| = 1$.

It is well known that Cs may converge, although the bounded sequence s does not converge. So in the sense of convergence, we may think of this fact as C making "better" sequences. Thus the question arises as to how does the sequence of iterates

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$\{C^n s\}$ behave? For $s \in c$, in Theorem 1 we prove that $\{C^n s\}$ converges if, and only if, $s(0) = Ls$. In this case we have that $\{C^n s\}$ converges to the constant sequence $s(0)$.

Theorem 2 deals with the iterates of C^*, the adjoint operator for $C : c_0 \to c_0$. We show that for any $y \in \ell^1$ the sequence $\{(C^*)^n y\}$ converges to $\varphi(y)e_0$ where $\varphi(y) = \sum_{j=0}^{\infty} y(j)$.

We also analyze the finite range and the infinite range cases for the continuous Cesàro operator. (One can find in [4] a very interesting exposition of the main properties of the Cesàro operators.)

In the finite range case we consider $f \in L^\infty(0,1)$ and define

$$Cf(x) \equiv \frac{1}{x}\int_0^x f(s)ds, \ \forall\, x \in (0,1). \tag{2}$$

Then $Cf \in L^\infty(0,1)$ and we obtain a linear operator $C : L^\infty(0,1) \to L^\infty(0,1)$ with $\|C\| = 1$. For $f \in C[0,1]$ we extend the above definition by taking

$$Cf(0) \equiv f(0), \quad Cf(1) = \int_0^1 f(s)ds.$$

In this situation we also have that $C : C[0,1] \to C[0,1]$ is a bounded linear operator and $\|C\| = 1$. For $f \in C[0,1]$, we show in Theorem 3 that $\{C^n f\}$ always converges to the constant function $f(0)$.

In the infinite range case we consider $f \in L^\infty(0,\infty)$ and define

$$Cf(x) \equiv \frac{1}{x}\int_0^x f(s)ds, \ \forall\, x \in (0,\infty). \tag{3}$$

Then $Cf \in L^\infty(0,\infty)$ and we obtain a linear operator $C : L^\infty(0,\infty) \to L^\infty(0,\infty)$ with $\|C\| = 1$. If $f \in C[0,\infty)$ is bounded take $Cf(0) \equiv f(0)$ and let us denote by $C[0,\infty]$ the closed subspace of $L^\infty(0,\infty)$ consisting of all continuous functions $f : [0,\infty) \to \mathbb{C}$ such that $\lim_{x\to\infty} f(x)$ exists. In this situation we also have that C takes $C[0,\infty]$ into itself and $C : C[0,\infty] \to C[0,\infty]$ is a bounded linear operator with $\|C\| = 1$. For $f \in C[0,\infty]$, we prove in Theorem 4 that $\{C^n f\}$ converges if, and only if, $f(0) = \lim_{x\to\infty} f(x)$. In this case we have that $\{C^n f\}$ converges to the constant function $f(0)$, a result corresponding to that of the discrete case.

2. Discrete case

We will start by discussing a finite dimensional case for the Cesàro operator that throws light on the general situation. So let $m = 0, 1, \ldots$ and consider $\mathbb{C}^{m+1}$ with the sup-norm $\|(b_0,\ldots,b_m)\| \equiv \max\{|b_0|,\ldots,|b_m|\}$. The Cesàro operator now takes the form

$$C(s_0, s_1, \ldots, s_m) \equiv \left(s_0, \frac{s_0+s_1}{2}, \ldots, \frac{s_0+s_1+\cdots+s_m}{m+1}\right).$$

For $s = (s_0,\ldots,s_m) \in \mathbb{C}^{m+1}$, let $M \equiv \{(s_0, x_1, \ldots, x_m) : x_1,\ldots,x_m \in \mathbb{K}\}$. Notice $s \in M$ and $C(M) \subset M$. Take $x = (s_0, x_1, \ldots, x_m)$ and $y = (s_0, y_1, \ldots, y_m) \in M$. Then,

$$\|\frac{s_0+x_1+\cdots+x_j}{j+1} - \frac{s_0+y_1+\cdots+y_j}{j+1}\| \le \frac{j}{j+1}\|x-y\|, \ 2 \le j \le m.$$

It follows that

$$\|Cx - Cy\| \le K\|x-y\|, \ \forall\, y \in M,$$

with $K \equiv (1 - \frac{1}{m+1}) < 1$. This shows that C is a contraction on M and so it has a unique fixed point, which is easily seen to be the constant vector $(s_0, \dots, s_0)$. Thus we have proved the following.

Proposition 1. *If* $s = (s_0, s_1, \dots, s_m) \in \mathbb{C}^{m+1}$, *then* $C^n s \to (s_0, \dots, s_0)$.

We now consider the infinite dimensional case.

Theorem 1. *Let* $s \in c$. *Then* $\{C^n s\}$ *converges if, and only if,* $s_0 = L(s)$. *In this case,* $\{C^n s\}$ *converges to (the constant sequence)* s_0.

Proof. Assume $C^n s \to y$. This implies $s(0) = \pi(C^n(s)) \to \pi(y) = y(0)$, when $n \to \infty$. Thus $s(0) = y(0)$. We also have $Ls = L(C^n s) \to Ly$, when $n \to \infty$. It follows that $Ly = Ls$. From $C^n s \to y$ we have that y is constant and so $Ly = y(0)$. Hence $s(0) = Ls$.

To establish the other implication, we will first prove

$$C^n(e_k) \to 0 \text{ when } n \to \infty, \ \forall\, k = 1, 2, \dots. \tag{4}$$

Let us fix $n \in \mathbb{N}$. According to G. H. Hardy [2, Sect. 11.12], C^n is the moment difference operator corresponding to the measure on the interval $[0, 1]$ given by $d\mu \equiv f_n(t)dt$, where

$$f_n(t) \equiv \frac{1}{(n-1)!} \log^{n-1} \frac{1}{t}, \ 0 < t \le 1. \tag{5}$$

(A brief discussion of this result can be found in [3, p. 125].) This means that for any $s \in c$ we have

$$C^n s(m) \equiv \sum_{j=0}^{m} \binom{m}{j} s_j \int_0^1 (1-t)^{m-j} t^j f_n(t)dt, \ \forall\, m \in \mathbb{N}_0.$$

Now take $k \in \mathbb{N}$. From above we have $C^n e_k(m) = 0, \ m < k$, and

$$C^n e_k(m) = \binom{m}{k} \int_0^1 (1-t)^{m-k} t^k f_n(t)dt, \ k \le m. \tag{6}$$

Let us define $g_n(0) = 0, \ g_n(t) = t f_n(t), \ 0 < t \le 1$ and

$$a_n \equiv \sup\{g_n(t) : 0 \le t \le 1\}. \tag{7}$$

Since $\int_0^1 (1-t)^{m-k} t^{k-1} dt = \frac{(m-k)!(k-1)!}{m!}$ [5, Thm. 7.69], from (6) we obtain $|C^n e_k(m)| \le \frac{a_n}{k}$. Thus

$$\|C^n e_k\|_\infty \ \le \ a_n, \ \forall\, k \in \mathbb{N}. \tag{8}$$

Assume in what follows that

$$a_n \to 0. \tag{9}$$

Then (4) is obtained from (8) and (9).

Take $s \in c_0$ such that $s(0) = 0$ and let $\sigma_N \equiv \sum_{k=0}^N s_k e_k \equiv \sum_{k=1}^N s_k e_k$. Given $\epsilon > 0$, we have $\|s - \sigma_{N_1}\|_\infty \le \frac{\epsilon}{2}$ for some $N_1 \in \mathbb{N}$. From (4), we can find some $N > N_1$ such that $\|C^n \sigma_N\|_\infty \le \frac{\epsilon}{2} \ \forall\, n \ge N$. Hence

$$\|C^n s\|_\infty \ \le \ \|C^n \sigma_N\|_\infty + \|C^n(s - \sigma_N)\|_\infty \ \le \ \frac{\epsilon}{2} + \|s - \sigma_N\|_\infty \le \epsilon, \ \forall\, n \ge N.$$

Finally, let $s \in c$ be such that $s(0) = Ls$. Then, $s = (s - s(0)) + s(0)$. Since $s - s_0 \in c_0$ and $\pi(s - s(0)) = 0$, we have

$$C^n s = C^n(s - s_0) + C^n s_0 = C^n(s - s_0) + s_0 \to s_0.$$

All that is now left to prove is (9), and we do this in the following lemma. □

Lemma 1. *$a_n \to 0$ when $n \to \infty$.*

Proof. Fix $n \in \mathbb{N}, n \geq 2$, and notice that $g_n : [0,1] \to \mathbb{R}$ is continuous. Its derivative is

$$g_n'(t) = \frac{\ln^{n-2}\frac{1}{t}}{(n-2)!}\left(\frac{\ln\frac{1}{t}}{n-1} - 1\right),\ 0 < t \leq 1.$$

After simple calculations it follows that g_n has $t_0 \equiv e^{-n+1}$ as its unique critical point and that $g_n(t_0)$ is its maximum value. Thus

$$a_n = g_n(t_0) = \frac{e^{-n+1}(n-1)^{n-1}}{(n-1)!}. \tag{10}$$

Stirling's formula states that $\lim_{m\to\infty}\left[\frac{m!}{e^{-m}\,m^m\sqrt{2\pi m)}}\right] = 1$ [5, Thm. 5.44]. From this and (10) the conclusion follows. □

3. Iterates of the adjoint of the Cesàro operator

The next result extends Proposition 1 to ℓ^∞ and complements Theorem 1. Since $\ell^\infty = (\ell^1)^*$, notice that ℓ^∞ can be given the weak-$*$ topology.

Corollary 1. *$\{C^n s\}$ converges weak-$*$ to (the constant sequence) s_0, for any $s \in \ell^\infty$.*

Proof. Consider $s \in \ell^\infty,\ s \neq 0$. Take $y \in \ell^1$ and let $\epsilon > 0$ be given. First we fix $N \in \mathbb{N}$ to satisfy $\|y - y_N\| \leq \frac{\epsilon}{4\|s\|_\infty}$, where $y_N \equiv \sum_{j=0}^N y(j)e_j$. Since $\|C^n\| \leq 1$, this implies

$$\begin{aligned} |\langle C^n s - s_0, y\rangle| &= |\langle C^n s - s_0, y_N\rangle| + |\langle C^n s - s_0, y - y_N\rangle| \\ &\leq \sum_{j=0}^N (C^n s\,(j) - s_0)y(j) + \frac{\epsilon}{2}. \end{aligned} \tag{11}$$

From Proposition 1 follows that, for each $j \in \mathbb{N}_0$, $C^n s\,(j) \to s_0$ when $n \to \infty$. Using this in (11), we conclude that $\langle C^n s, y\rangle \to \langle s_0, y\rangle$. □

We now consider $C : c_0 \to c_0$. After some simple calculations we find that its adjoint $C^* : \ell^1 \to \ell^1$ is given by

$$C^* y\,(m) = \sum_{j=m}^\infty \frac{y(j)}{j+1},\ m \in \mathbb{N}_0. \tag{12}$$

Theorem 2. *$(C^*)^n y \to \left(\sum_{j=0}^\infty y(j)\right) e_0,\ \forall\, y \in \ell^1$.*

Proof. Let $y \in \ell^1$. Since weakly convergent sequences in ℓ^1 are norm convergent, to obtain the conclusion we only have to show that $\{(C^*)^n y\}$ converges weakly to $\left(\sum_{j=0}^{\infty} y(j)\right) e_0$. Take $s \in \ell^\infty = {\ell^1}^*$. From Corollary 1 we now obtain

$$\begin{aligned}\langle (C^*)^n y, s\rangle &= \langle y, C^n s\rangle \to \langle y, s_0\rangle = \sum_{j=0}^{\infty} y(j) s_0 \\ &= \langle (\sum_{j=0}^{\infty} y(j)) e_0, s\rangle. \end{aligned} \qquad \square$$

4. The finite range case

Next we will see that the behavior of the Cesàro operator on the space of continuous complex functions $C[0,1]$ is the same as that of the Cesàro operator defined on $\mathbb{C}^n$.

Theorem 3. $C^n f \to f(0),\ \forall\, f \in C[0,1]$.

Proof. By a direct calculation we obtain

$$Cx^k = \frac{1}{k+1} x^k,\ k \in \mathbb{N}_0.$$

Thus $C^n 1 = 1$ and $C^n x^k \to 0,\ \forall\, k \in \mathbb{N}$. Let $P(x) \equiv c_0 + c_1 x + \cdots + c_m x^m$ be a polynomial. Hence

$$C^n P \equiv c_0 + \frac{1}{2^n} c_1 x + \cdots + \frac{1}{(m+1)^n} c_m x^m.$$

It follows that $C^n P \to c_0 = P(0)$.

We now consider an arbitrary function $f \in C[0,1]$ and let a positive real number ϵ be given. Applying Weierstrass' Theorem we find a polynomial P such that $\|f - P\| \le \frac{\epsilon}{3}$. By the case discussed above, there is some $N \in \mathbb{N}$ such that $\|C^n P - P(0)\| \le \frac{\epsilon}{3},\ \forall\, n \ge N$. Let $n \ge N$. Since $\|C^n\| \le 1$, it follows that

$$\begin{aligned}\|C^n f - f(0)\| &= \|C^n f - C^n f(0)\| \\ &\le \|(C^n f - C^n P)\| + \|(C^n (P - C^n f(0)))\| \\ &= \|(f - P)\| + \|C^n P - P(0)\| + \|P(0) - f(0)\| \le \epsilon. \end{aligned} \qquad \square$$

5. The infinite range case

Recall that $C[0,\infty]$ consists of all continuous functions $f : [0,\infty) \to \mathbb{C}$ such that $\lim_{x\to\infty} f(x)$ exists. To analyze $C : [0,\infty] \to [0,\infty]$ we will proceed in a way similar to that of the discrete Cesàro operator. We define

$$Lf \equiv \lim_{x\to\infty} f(x), \qquad \pi(f) \equiv f(0),\ \forall\, f \in C[0,\infty].$$

Clearly both L and π are bounded linear functionals. Moreover, they satisfy

$$LCf = Lf, \quad \pi C f = \pi f,\ \forall\, f \in C[0,\infty].$$

Using the change of variables $s = xt$, (3) can be written as

$$Cf(x) = \int_0^1 f(xt)dt,\ \forall\, x \in (0,\infty).$$

More generally, D. W. Boyd proved that

$$C^n f(x) = \int_0^1 f(xt) f_n(t) dt, \ \forall\, x \in (0, \infty) \tag{13}$$

where f_n is given by (5) [1, Lemma 2].

Theorem 4. *Let $f \in C[0, \infty]$. Then $\{C^n f\}$ converges if, and only if, $f(0) = Lf$. In this case, $\{C^n f\}$ converges to (the constant function) $f(0)$.*

Proof. The necessity of the condition is established as in Theorem 1.

To establish sufficiency, we will first assume $f(0) = 0 = Lf$. Let $\epsilon > 0$ be given. We can now choose δ such that $0 < \delta < 1$ and $N \in \mathbb{N}$ to satisfy

$$|f(u)| \leq \frac{\epsilon}{3} \text{ if } \ 0 \leq u \leq \delta \text{ or } \ u \geq N.$$

Let $x > N$. To estimate $C^n f(x)$ using (13), we divide the integration interval $[0, 1]$ in three parts. Since $\int_0^1 f_n(t) dt = 1$, we have

$$\int_0^{\frac{\delta}{x}} |f(xt)| \, f_n(t) dt \leq \frac{\epsilon}{3} \int_0^1 f_n(t) dt = \frac{\epsilon}{3}. \tag{14}$$

Similarly, we obtain

$$\int_{\frac{N}{x}}^1 |f(xt)| \, f_n(t) dt \leq \frac{\epsilon}{3} \int_0^1 f_n(t) dt = \frac{\epsilon}{3}. \tag{15}$$

Next, using that f_n is a decreasing function and (7), we find

$$\int_{\frac{\delta}{x}}^{\frac{N}{x}} |f(xt)| \, f_n(t) dt \leq \|f\|_\infty \frac{(N-\delta)}{x} f_n(\frac{\delta}{x}) \leq \|f\|_\infty \frac{(N-\delta)}{\delta} a_n.$$

Applying Lemma 1, this implies

$$\int_{\frac{\delta}{x}}^{\frac{N}{x}} |f(xt)| \, f_n(t) dt \leq \frac{\epsilon}{3}, \ \forall\, n \geq N_1, \tag{16}$$

for some $N_1 \in \mathbb{N}$.

Finally, by (13), (14), (15) and (16) we conclude that

$$|C^n f(x)| \leq \epsilon, \ \forall\, x > N, \forall\, n \geq N_1.$$

Now, from the finite range case (with the interval $[0, N]$ instead of $[0, 1]$) we find $N_2 \in \mathbb{N}$ such that

$$|C^n f(x)| \leq \epsilon, \ \forall\, x \in [0, N], \ \forall\, n \geq N_2.$$

This proves the theorem when $f \in C[0, \infty]$ satisfies $f(0) = Lf = 0$. If $f(0) = Lf$, then we proceed as in the discrete case. □

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