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# ADJOINTS AND FORMAL ADJOINTS OF MATRICES OF UNBOUNDED OPERATORS

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ABSTRACT. In this paper we *discuss* diverse aspects of the mutual relationship between adjoints and formal adjoints of unbounded operators bearing a matrix structure. We emphasize the behaviour of row and column operators as they turn out to be the germs of an arbitrary matrix operator, providing most of the information about the latter as it is the troublemaker.

## 1. INTRODUCTION

In recent years, $2 \times 2$ matrices of unbounded operators have attracted considerable attention, roughly divided into two groups of problems: they occur as generators for semigroups, see [2], [13] and [4], and as tools in problems from Mathematical Physics, see [8], [5], [12]. The latter has attracted much interest in spectral properties of such matrices, in particular in its essential spectrum; see [1], [7], [10], [11] and the references therein.

Adjoints of operators bearing a matrix structure have been investigated case by case; it might be difficult to find any general approach to the problem when there are rows or columns with more than one unbounded entry. So as to mention some partial results, let us refer to [6] and [14] where the only nonzero entries are those off the diagonal. On the other hand, in [3] and [13] examples are given showing that the adjoint of a matrix operator $A$ and its formal adjoint $A^\times$, that is, the matrix of adjoints of all the particular entries, may be quite different. This supports the idea of the present paper to build a common framework for all the cases. Positive results in this matter are intertwined with counterexamples; the latter indicate that $A^\times$ may not contain enough information on $A'$ itself. Thus, what is essentially trivial for bounded operators appears to become erratic for unbounded operators.

It turns out that the study of row and column operators separately is pretty much helpful as a matrix can be decomposed in a sense by means of these two. Indeed, column and row operators have a more predictable behaviour, which helps to understand the 2 by 2 matrix case; this case is interesting as well as difficult enough to deserve a careful treatment. The pith of the problem can be simply described by saying that

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> if $A$ is either a row or a column operator (even in a Hilbert space) with both entries being unbounded and closed, then it is rather unlikely that $A$ will coincide with its second adjoint.

On the other hand, it is important to stress that if a column has only one entry which is unbounded, then the problem does not appear at all; this is the most frequent case which occurs in the literature.

## 2. Preliminaries

Reasonable assumptions on $2 \times 2$ operator matrices are that their entries are closed densely defined operators and that the operators they determine are densely defined. However, when one divides bigger operator matrices into $2 \times 2$ blocks, the blocks in the latter structure will not be closed, in general. Matrices of arbitrary size occur in an attempt to determine normal extensions of unbounded operators, see [15], and our intention is to elaborate somewhere else on these kinds of matrices from the point of view of the present paper. Indeed, also for genuine $2 \times 2$ operator matrices, the blocks are often not assumed to be closed as the closedness of at least some of the blocks would be pointless anyway.

Henceforth, for $i = 1, 2$, let $E_i$ and $F_i$ be locally convex Hausdorff spaces and for $i, j = 1, 2$, let $A_{ij} \in \mathfrak{L}(E_j, F_i)$, where $\mathfrak{L}(E_j, F_i)$ denotes the set of all densely defined operators from $E_j$ to $F_i$. The closed densely defined and continuous everywhere defined operators in these spaces are denoted by $\mathfrak{C}(E_j, F_i)$ and $\mathfrak{B}(E_j, F_i)$, respectively. Finite direct sums will be denoted with the symbol $\oplus$, meaning that in the case of Hilbert spaces, the direct sum will always be identified with the orthogonal one. Define

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

from $E$ to $F$, where $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$, and the domain of $A$ is given by

$$\mathcal{D}(A) = (\mathcal{D}(A_{11}) \cap \mathcal{D}(A_{21})) \oplus (\mathcal{D}(A_{12}) \cap \mathcal{D}(A_{22})) . \tag{2.1}$$

The range of $A$ will be denoted by $\mathcal{R}(A)$.

Here we will only consider the case that $A$ is densely defined, so we require that

$$\mathcal{D}(A_{1j}) \cap \mathcal{D}(A_{2j}) \text{ is dense in } E_j \text{ for } j = 1, 2. \tag{2.2}$$

In particular, since all $A_{ij}$ are densely defined, their adjoints $A'_{ij}$ must necessarily be closed operators from $F'_i$ to $E'_j$, and we can define the operator

$$A^{\times} = \begin{pmatrix} A'_{11} & A'_{21} \\ A'_{12} & A'_{22} \end{pmatrix}$$

from $F'$ to $E'$, where, according to (2.1),

$$\mathcal{D}(A^{\times}) = (\mathcal{D}(A'_{11}) \cap \mathcal{D}(A'_{12})) \oplus (\mathcal{D}(A'_{21}) \cap \mathcal{D}(A'_{22})) .$$

Row and column operators of arbitrary size are defined as follows. A row operator $R \overset{\text{def}}{=} R_{R_1,\dots,R_n}$ is a linear mapping defined in $\bigoplus_{j=1}^n E_j$, taking values in $E_0$ and acting according to

$$R\Big(\bigoplus_{j=1}^n f_j\Big) = \sum_{j=1}^n R_j f_j, \quad \bigoplus_{j=1}^n f_j \in \mathcal{D}(R) \overset{\text{def}}{=} \bigoplus_{j=1}^n \mathcal{D}(R_j),$$

where $E_0, E_1, \dots, E_n$ are locally convex Hausdorff spaces, and the $R_j$ are linear operators from $E_j$ to $E_0$. Clearly, $R$ is densely defined if and only if all $R_j$ are densely defined.

A column operator $C \stackrel{\text{def}}{=} C_{C_1,\dots,C_n}$ is a linear mapping defined in $F_0$, taking values in $\bigoplus_{j=1}^n F_j$ and acting as

$$\mathcal{D}(C) \stackrel{\text{def}}{=} \bigcap_{j=1}^{n} D(C_j), \quad Cf \stackrel{\text{def}}{=} \bigoplus_{j=1}^{n} C_j f, \quad f \in \mathcal{D}(C),$$

the spaces $F_0, F_1, \dots, F_n$ are locally convex Hausdorff spaces, and $C_j \in \mathfrak{L}(F_0, F_j)$ for $j = 1, \dots, n$. To such a column operator we associate the row operator

$$C^\times \stackrel{\text{def}}{=} R_{C_1',\dots,C_n'}$$

which may not be densely defined even if $C$ is densely defined.

The important notice we can make at this stage more precise is that row operators behave differently from column operators. In particular, supposing all the entries of $R$ are closed,

$$R'' \text{ may not be equal to } R \text{ though always } R'^\times = R.$$

Needless to say a similar behaviour concerns column operators. This outlines once more the flavour of our paper.

In Hilbert spaces, we will take the usual Hilbert space adjoints of operators, i. e., with respect to the sesquilinear scalar product rather than the adjoints with respect to bilinear forms for dual pairs. Since the results of this paper are independent of whether the duality is realized by bilinear forms or sesquilinear forms, in Hilbert space we will always use Hilbert space adjoints. With some abuse of notation, we shall write e. g. $C^\times$ in both cases, where its meaning will be clear from the context. This is applicable in particular to Kreĭn space adjoints if each of the component spaces is a Kreĭn space, because in this case the fundamental symmetry on the direct sum is the direct sum of the fundamental symmetries on the components.

## 3. Row operators

In this section let $R \stackrel{\text{def}}{=} R_{R_1,\dots,R_n}$ be a row operator as defined above.

**Proposition 3.1.**[1] *The adjoint $R'$ of a densely defined row operator $R$ is the column operator formed by the adjoints of the $R_j$, that is,*

$$R' = C_{R_1',\dots,R_n'}.$$

*Proof.* To show $R' \subset C_{R_1',\dots,R_n'}$ let all $f_j \in \mathcal{D}(R_j)$ and $g \in \mathcal{D}(R')$. Then

$$\langle \bigoplus_j f_j, R'g \rangle = \Big\langle \sum_j R_j f_j, g \Big\rangle = \sum_j \langle R_j f_j, g \rangle.$$

Putting $f_j = 0$ for $j \neq k$, it follows that $g \in \mathcal{D}(R_k')$ for each $k = 1, \dots, n$ and $\langle R_k f_k, g \rangle = \langle f_k, R_k' g \rangle$. Thus

$$\langle \bigoplus_j f_j, R'g \rangle = \langle \bigoplus_j f_j, \bigoplus_j R_j' g \rangle,$$

which proves the inclusion "$\subset$".

[1] Propositions 1, 4.1 and 4.5 have been proved for linear relations in Hilbert spaces in [9], Proposition 2.1.

Conversely, let $g \in \mathcal{D}\left(C_{R'_1,\dots,R'_n}\right) = \bigcap_{j=1}^n \mathcal{D}(R'_j)$. Then, for all $f_j \in \mathcal{D}(R_j)$, $\langle f_j, R'_j g\rangle = \langle R_j f_j, g\rangle$, and thus

$$\langle \bigoplus_j f_j, \bigoplus_j R'_j g\rangle = \big\langle \sum_j R_j f_j, g\big\rangle = \langle R\big(\bigoplus_j f_j\big), g\rangle,$$

which shows $g \in \mathcal{D}(R')$. □

*Remark* 3.2. The simplest example showing that a row operator with closed entries may not be even closable is to consider $R_1$ and $R_2$ selfadjoint with $\mathcal{D}(R_1)\cap\mathcal{D}(R_2) = \varnothing$. Then, according to Proposition 1, $\mathcal{D}(R') = \mathcal{D}(R'_1)\cap\mathcal{D}(R'_2) = \varnothing$; hence $R$ is not closable. This is a rather extreme example in a sense; a richer one concerning the same question will be given below.

**Example 3.3.** Define $R_1$ and $R_2$ in $\ell^2$ as follows:

$$R_1 e_k \stackrel{\mathsf{def}}{=} k^2 e_k, \quad R_2 e_k \stackrel{\mathsf{def}}{=} k e_0 + k^2 e_k, \quad k = 0, 1, 2, \dots,$$

where $(e_n)_{n=0}^\infty$ is the orthodox zero-one orthonormal basis of $\ell^2$. To properly establish $R_1$ and $R_2$, we define their domains by

$$\mathcal{D}(R_1) = \left\{ f = \sum_{k=0}^\infty \gamma_k e_k : \sum_{k=0}^\infty |\gamma_k k^2|^2 < \infty \right\},$$
$$\mathcal{D}(R_2) = \left\{ f = \sum_{k=0}^\infty \gamma_k e_k \in \mathcal{D}(R_1) : \sum_{k=0}^\infty \gamma_k k \text{ converges} \right\}.$$

Then, as usual, the operators $R_i$ are defined as

$$R_i f = \sum_{k=0}^\infty \gamma_k R_i e_k, \quad f = \sum_{k=0}^\infty \gamma_k e_k \in \mathcal{D}(R_i).$$

Because $R_1$ is a diagonal operator on its maximal domain, it is closed. In order to prove the closedness of $R_2$ let

$$f_n = \sum_{k=0}^\infty \gamma_{nk} e_k \in \mathcal{D}(R_2), \quad f_n \to f = \sum_{k=0}^\infty \gamma_k e_k, \text{ and } \quad R_2 f_n \to g = \sum_{k=0}^\infty \delta_k e_k.$$

If $P$ denotes the orthogonal projection of the Hilbert space onto the closed linear span of $\{e_k\}_{k=1}^\infty$, then $P R_2 f_n = R_1 f_n$, and so $R_1 f_n \to Pg$. Since $R_1$ is closed, it follows that $f \in \mathcal{D}(R_1)$ and $Pg = R_1 f$.

Note that

$$\sum_{k=1}^\infty |\gamma_{nk} k^2 - \gamma_k k^2|^2 = \|R_1 f_n - R_1 f\|^2 \to 0 \quad \text{as} \quad n \to \infty, \tag{3.1}$$

$$(I-P) R_2 f_n = \sum_{k=1}^\infty \gamma_{nk} k e_0$$

and that $R_2 f_n \to g$ implies

$$\sum_{k=1}^\infty \gamma_{nk} k \to \delta_1 \quad \text{as} \quad n \to \infty. \tag{3.2}$$

Since for $m \geq 1$ and $n = 0, 1, \ldots$,

$$\Big|\sum_{k=0}^{m} \gamma_k k - \delta_1\Big| \leq \Big|\sum_{k=0}^{m} \gamma_{nk} k - \delta_1\Big| + \sum_{k=0}^{m} |\gamma_{nk} k - \gamma_k k|$$
$$\leq \Big|\sum_{k=0}^{m} \gamma_{nk} k - \delta_1\Big| + \Big(\sum_{k=1}^{\infty} |\gamma_{nk} k^2 - \gamma_k k^2|^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{\infty} |k|^{-2}\Big)^{\frac{1}{2}},$$

from (3.1) and (3.2), we get that for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that

$$\Big|\sum_{k=0}^{m} \gamma_k k - \delta_1\Big| \leq \Big|\sum_{k=m+1}^{\infty} \gamma_{nk} k\Big| + \varepsilon \tag{3.3}$$

for $m \geq 1$. From (3.3) and the fact that $\sum_{k=1}^{\infty} \gamma_{nk} k$ converges, we deduce that

$$\Big|\sum_{k=1}^{m} \gamma_k k - \delta_1\Big| \leq 2\varepsilon$$

for sufficiently large $m$, which proves that $\sum_{k=1}^{\infty} \gamma_k k$ converges to $\delta_1$. This shows $f \in \mathcal{D}(R_2)$ and $g = R_2 f$.

Finally, letting for $n \geq 1$, $f_n = n^{-1} e_n$, we have $f_n \in \mathcal{D}(R_2)$, $f_n \to 0$ as $n \to \infty$, and

$$R_1(-f_n) + R_2 f_n = n n^{-1} e_0 = e_0.$$

So $R$ is not closable. However the linear span of $(e_n)_{n=0}^{\infty}$ is included in $\mathcal{D}(R_1) \cap \mathcal{D}(R_2)$ as well as in $\mathcal{D}(R_1') \cap \mathcal{D}(R_2')$, and this is what makes this example more interesting than that argued for in Remark 3.2.

**Corollary 3.4.** *For $R$ to be closable it is necessary but not sufficient that $R_1, \ldots, R_n$ are closable.*

*Proof.* Use all of the above and the fact, which is implicit in Proposition 1, that $\mathcal{D}(R') = \bigcap_{j=1}^{n} \mathcal{D}(R_j')$. □

Note that in general $R$ is not closed even if all its entries are closed. However, we trivially have

**Proposition 3.5.** *Assume that at most one of the entries of $R$ is not bounded. Then $R$ is closed (closable) if and only if all $R_j$ are closed (closable). In this case $R'$ is densely defined.*

In the following we have a particular result when the closure of $R$ can be determined explicitly.

**Proposition 3.6.** *Let $n = 2$, assume that $R_1$ is injective, that $\mathcal{R}(R_2) \subset \mathcal{R}(R_1)$ and that $R_1^{-1} R_2$ is an operator with a bounded extension $K \in \mathfrak{B}(E_2, E_1)$. Then*

$$\overline{R} = (\,\overline{R_1}, 0) \begin{pmatrix} I & K \\ 0 & I \end{pmatrix},$$

*and $\mathcal{D}(\,\overline{R}\,) = \{f \oplus g \in E_1 \oplus E_2 : f + Kg \in \mathcal{D}(\,\overline{R_1}\,)\}$.*

*Proof.* The operator $\hat{R} = (\,\overline{R_1}, 0) \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}$ is closed since the $2 \times 2$ matrix operator on the right is invertible. Also, for $f \in \mathcal{D}(R_1)$, $g \in \mathcal{D}(R_2)$ we have $Kg = R_1^{-1} R_2 g \in \mathcal{D}(R_1)$ and thus $f + Kg \in \mathcal{D}(R_1)$, and

$$\hat{R}(f \oplus g) = (\,\overline{R_1}, 0)(f + R^{-1} R_2 g) \oplus g = R_1 f + R_2 g = R(f \oplus g).$$

This shows that $R \subset \hat{R}$ and thus $\overline{R} \subset \hat{R}$ as $\hat{R}$ is closed. We calculate

$$\hat{R}' = \begin{pmatrix} I & 0 \\ K' & I \end{pmatrix} \begin{pmatrix} R_1' \\ 0 \end{pmatrix} = \begin{pmatrix} R_1' \\ K'R_1' \end{pmatrix}. \tag{3.4}$$

From $R_2 \subset R_1 K$ it follows that $K'R_1' \subset (R_1K)' \subset R_2'$. In particular, $\mathcal{D}(R_1') \subset \mathcal{D}(R_2')$ and hence $K'R_1' = R_2'$ on $\mathcal{D}(R_1')$. Thus $\hat{R}' = R'$ by (3.4) and Proposition 1, which proves $\hat{R} = \overline{R}$ since $\hat{R}$ is closed. The representation of $\mathcal{D}(\overline{R})$ is now obvious from the representation of $\overline{R}$. □

## 4. Column operators

In this section, let $C \overset{\text{def}}{=} C_{C_1,\dots,C_n}$ be a densely defined column operator.

**Proposition 4.1.** *$C^\times \subset C'$.*

*Proof.* Since $C$ is densely defined, $C'$ is an operator. Let $g_j \in \mathcal{D}(C_j')$, $f \in \mathcal{D}(C)$. Then

$$\langle f, C^\times(\bigoplus_j g_j)\rangle = \langle f, \sum_j C_j' g_j\rangle = \sum_j \langle f, C_j' g_j\rangle = \sum_j \langle C_j f, g_j\rangle = \langle Cf, \bigoplus_j g_j\rangle.$$

This proves $\bigoplus_j g_j \in \mathcal{D}(C')$ and $C'(\bigoplus_j g_j) = C^\times(\bigoplus_j g_j)$. □

Referring back to the preceding section, let us recall that $R^\times = C_{R_1',\dots,R_n'}$. Then the conclusion of Proposition 1 can be restated as

$$R' = R^\times.$$

Now double applications of this and Proposition 4.1 lead to

**Corollary 4.2.** *For a row operator $R$ and a column operator $C$ we have*

$$R'' = R^{\times\prime} \supset R^{\times\times} = R'^{\times}, \quad C'' \subset C^{\times\prime} = C^{\times\times}.$$

**Proposition 4.3.** 1° *$C$ is closed if $C_1, \dots, C_n$ are closed.* 2° *$C$ is closable if $C_1, \dots, C_n$ are closable.* 3° *$\overline{C^\times} = C'$ if and only if $\overline{C} = C_{\overline{C_1},\dots,\overline{C_n}}$.*

*Proof.* 1° is straightforward, and 2° is an immediate consequence of 1°. From Corollary 4.2 we get

$$\overline{C^\times} = C' \iff C^{\times\prime} = C'' \iff C^{\times\times} = C''.$$

Then the conclusion follows by observing that $C'' = \overline{C}$ and

$$C^{\times\times} = R^\times_{C_1',\dots,C_n'} = C_{\overline{C_1},\dots,\overline{C_n}}.$$ □

*Remark* 4.4. The sufficient condition for $C$ to be closable, which is in Proposition 4.3, 2°, turns out to be not necessary. For this let $H_1$ and $H_2$ be Hilbert spaces and $C \in \mathfrak{C}(H_1, H_2) \setminus \mathfrak{B}(H_1, H_2)$, that is, $\mathcal{D}(C^*) \neq H_2$. Let $x \in H_2 \setminus \mathcal{D}(C^*)$, $P$ be the orthogonal projection onto the span of $x$, $C_1 = PC$, $C_2 = (I - P)C$. Then $C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ with $C_1$ not closable. Here, with some abuse of notation, the range spaces of $C_1$ and $C_2$ are $\mathcal{R}(P)$ and $\mathcal{R}(I-P)$, respectively.

Indeed, assume that $x \in \mathcal{D}(C_1^*)$. Then we have for $f \in \mathcal{D}(C)$ that

$$\langle Cf, x\rangle = \langle Cf, Px\rangle = \langle PCf, x\rangle = \langle f, C_1^* x\rangle,$$

and the contradiction $x \in \mathcal{D}(C^*)$ would follow. Since $\mathcal{D}(C_1^*) \subset \mathcal{R}(P)$ and $\mathcal{R}(P)$ is one-dimensional, $\mathcal{D}(C_1^*) = \{0\}$ follows. Thus $C_1$ is not closable.

For other examples with nonclosable $C_1$ but closable $C_2$ and $C$ we refer to [3, Sections 2 and 3] and [13, Section 1].

**Proposition 4.5.** *Assume that at most one of the entries of* $C$ *does not satisfy* $C_j \in \mathfrak{B}(F_0, F_j)$. *Then* $C^\times = C'$.

*Proof.* Obviously, $\overline{C} = C_{\overline{C_1},\dots,\overline{C_2}}$, and hence $\overline{C^\times} = C'$ by Proposition 4.3 3°. Clearly, $C^\times$ satisfies the assumptions of Proposition 3.5, and thus $C^\times$ is closed. □

In the following result, $C^\times$ and $C'$ are given explicitly; in particular, the structure of their domains is transparent.

**Proposition 4.6.** *Let* $C_1$, $C_2$ *be such that* $\mathcal{D}(C_1) = \mathcal{D}(C_2)$ *and assume that* $C_1$ *is injective with* $C_2C_1^{-1}$ *having an extension* $K \in \mathfrak{B}(F_1, F_2)$. *Then*

$$C^\times = (C_1', C_1'K') \quad \textit{and} \quad C' = (C_1', 0)\begin{pmatrix} I & K' \\ 0 & I \end{pmatrix}.$$

*In particular,* $\mathcal{D}(C^\times) = \{f \oplus g \in F_1' \oplus F_2' : f \in \mathcal{D}(C_1'),\ K'g \in \mathcal{D}(C_1')\}$ *and* $\mathcal{D}(C') = \{f \oplus g \in F_1' \oplus F_2' : f + K'g \in \mathcal{D}(C_1')\}$.

*Proof.* Since $C_2 = C_2C_1^{-1}C_1 = KC_1$, $C_2' = C_1'K'$, and the statements about $C^\times$ follow.

For $C$ we have the representation

$$C = \begin{pmatrix} C_1 \\ KC_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ K & I \end{pmatrix}\begin{pmatrix} C_1 \\ 0 \end{pmatrix},$$

which gives

$$C' = (C_1', 0)\begin{pmatrix} I & K' \\ 0 & I \end{pmatrix}.$$

The statement about the domain of $C'$ immediately follows from this. □

## 5. More about column operators in Hilbert spaces

Here we assume that $F_0$, $F_1$, $F_2$ are Hilbert spaces and that $C = C_{C_1,C_2}$ is a densely defined closable column operator. Let $D_0$ be a subspace of $F_0$ satisfying

$$D_0 \subset \mathcal{D}(\overline{C}C^\times). \tag{5.1}$$

Notice that, if $\overline{C^\times} = C^*$, then, by part 3° of Proposition 4.3,

$$\overline{C}C^\times = \begin{pmatrix} \overline{C_1}C_1^* & \overline{C_1}C_2^* \\ \overline{C_2}C_1^* & \overline{C_2}C_2^* \end{pmatrix}.$$

We want to investigate the following questions:[2]

- when is $D_0$ a core for $C^*$?
- when is $D_0$ a core for $C^\times$?

**Proposition 5.1.** *Let* $D_1$ *be a subspace of* $F_0$ *such that* $D_0 \subset D_1 \subset \mathcal{D}(C^*)$. *Then* $D_0$ *is a core for* $C^*|_{D_1}$ *if and only if*

$$((I + \overline{C}C^\times)(D_0))^\perp \cap D_1 = \{0\}. \tag{5.2}$$

[2] $\mathcal{D} \subset \mathcal{D}(A)$ is a core for a closable operator $A$ if $A \subset \overline{A|_{\mathcal{D}}}$.

*Proof.* Due to (5.1), for $f \in D_1$ to belong to the left-hand side of (5.2) means precisely that

$$0 = \langle f, (I + \overline{C}C^*)g\rangle \text{ for all } g \in D_0.$$

Because $D_1 \subset \mathcal{D}(C^*)$, this is what is required for $D_1$ to be a core for $C^*$. □

**Proposition 5.2.** *Let $C_1$, $C_2$ be such that $\mathcal{D}(C_1) = \mathcal{D}(C_2)$ and assume that $C_1$ is injective with $C_2C_1^{-1}$ having an extension $K \in \mathfrak{B}(F_1, F_2)$. Let $D_0$ be a subspace of $\mathcal{D}(C^*)$. Also assume that there is at least one $v_0 \in F_2 \setminus \{0\}$ such that $(-K^*v_0, v_0) \in D_0$. Then $D_0$ is a core for $C^*$ if and only if $D_0$ is dense in $F_1 \oplus F_2$ and $(I, K^*)(D_0) \cap \mathcal{D}(C_1^*)$ is a core for $C_1^*$.*

*Proof.* Since we assume that $C$ is closable, $C^*$ is densely defined, and the condition that $D_0$ is dense in $F_1 \oplus F_2$ is necessary for a core. Thus we may assume this property for the remainder of the proof. Using the graph norm of $C^*$ as in the proof of Proposition 5.1, we have to investigate when, for $f \oplus g \in \mathcal{D}(C^*)$,

$$\langle u \oplus v, f \oplus g\rangle + \langle C^*(u \oplus v), C^*(f \oplus g)\rangle = 0 \tag{5.3}$$

for all $u \oplus v \in D_0$ implies $f \oplus g = 0$. By Proposition 4.6,

$$C^*(u \oplus v) = C_1^*(u + K^*v) \quad \text{and} \quad C^*(f \oplus g) = C_1^*(f + K^*g).$$

Introducing $w = u + K^*v$ and $h = f + K^*g$, (5.3) can be written as

$$\langle u, f\rangle + \langle v, g\rangle + \langle C_1^*w, C_1^*h\rangle = 0, \tag{5.4}$$

which is the same as

$$\langle w - K^*v, h - K^*g\rangle + \langle v, g\rangle + \langle C_1^*w, C_1^*h\rangle = 0.$$

Note that

$$\langle w - K^*v, h - K^*g\rangle + \langle v, g\rangle - \langle w, h\rangle = \langle (I + KK^*)v - Kw, g\rangle - \langle K^*v, h\rangle.$$

First we want to find a condition for $h = 0$; thus we may modify $f$ and $g$ as long as $f + K^*g = h$. If $(I + KK^*)v \neq Kw$ choose $g \in F_2$ such that

$$\langle (I + KK^*)v - Kw, g\rangle = \langle K^*v, h\rangle.$$

If $(I + KK^*)v = Kw$, we replace $u \oplus v$ with $(u - K^*v_0) \oplus (v + v_0)$. Since $(u - K^*v_0) + K^*(v + v_0) = u + K^*v = w$, this does not change $w$, but now

$$(I + KK^*)(v + v_0) = Kw + (I + KK^*)v_0 \neq Kw$$

since $v_0 \neq 0$ and $(I + KK^*)$ is injective. Thus also in this case we can find $g \in F_2$ such that

$$\langle (I + KK^*)(v + v_0) - Kw, g\rangle = \langle K^*(v + v_0), h\rangle.$$

With this $g$ we put $f = h - K^*g$ in either case and deduce

$$\langle w, h\rangle + \langle C_1^*w, C_1^*h\rangle = 0$$

for all $w \in (I, K^*)(D_0)$. Hence $h = 0$ for all $h \in \mathcal{D}(C_1^*)$ if and only if $(I, K^*)(D_0) \cap \mathcal{D}(C_1^*)$ is a core for $C_1^*$. Returning to (5.4) it follows (now for our original $f$ and $g$) that $h = 0$ implies

$$\langle u, f\rangle + \langle v, g\rangle = 0$$

for all $u \oplus v \in D_0$. Since $D_0$ is dense in $F_1 \oplus F_2$, $f = 0$ and $g = 0$ follow. □

**Corollary 5.3.** *Let $C_1$, $C_2$ be such that $\mathcal{D}(C_1) = \mathcal{D}(C_2)$ and assume that $C_1$ is injective with $C_2C_1^{-1}$ having an extension $K \in \mathfrak{B}(F_1, F_2)$. Then $\overline{C^\times} = C^*$.*

*Proof.* For every $v_0 \in \mathcal{D}(C_1^* K^*)$ we have $-K^* v_0 \in \mathcal{D}(C_1^*)$, so that $(-K^* v_0, v_0) \in \mathcal{D}(C^\times)$ holds for every (and thus some) $v_0 \in F_2 \setminus \{0\}$ since $\mathcal{D}(C^\times) = \mathcal{D}(C_1^*) \oplus \mathcal{D}(C_1^* K^*)$ by Proposition 4.6. Obviously, $(I, K^*)(\mathcal{D}(C^\times)) \supset \mathcal{D}(C_1^*)$, and an application of Proposition 5.2 completes the proof. □

**Example 5.4.** Suppose $C$ is a column operator with $C_2 = T - C_1$, where both $C_1$ and $T$ are in $\mathfrak{L}(F, F_0)$, and where $F \overset{\text{def}}{=} F_1 = F_2$. Suppose moreover

$$\mathcal{D}(C_1) \subset \mathcal{D}(T) \text{ and } \mathcal{D}(C_1^*) \subsetneq \mathcal{D}(T^*).$$

Then

$$\mathcal{D}(C^\times) \subsetneq \mathcal{D}(C^*).$$

Indeed, for $h \in \mathcal{D}(C_1)$, $f, g \in F$,

$$\langle Ch, f \oplus g \rangle = \langle C_1 h \oplus (T - C_1)h, f \oplus g \rangle = \langle C_1 h, f - g \rangle + \langle Th, g \rangle.$$

This gives us immediately that $f \oplus f$ is in $\mathcal{D}(C^*)$ if and only if $f$ is in $\mathcal{D}(T^*)$; then

$$C^*(f \oplus f) = T^* f, \quad f \in \mathcal{D}(T^*)$$

and consequently

$$\mathcal{D}(C^\times) \subset \mathcal{D}(C^\times) + \{f \oplus f\colon\ f \in \mathcal{D}(T^*)\} \subset \mathcal{D}(C^*).$$

Hence, if $f \in \mathcal{D}(T^*) \setminus \mathcal{D}(C_1^*)$, then $f \oplus f \in \mathcal{D}(C^*)$ but $f \oplus f \notin \mathcal{D}(C_1^*) \oplus \mathcal{D}(C_2^*) = \mathcal{D}(C^\times)$.

## 6. The operator matrix $A$

Let $A$ be a $2 \times 2$ matrix of the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with the denseness condition (2.2) being satisfied. Writing

$$C_{A,i} = (A_{i1}, A_{i2}),\ i = 1, 2, \tag{6.1}$$

we know from Proposition 1 that

$$C'_{A,i} = \begin{pmatrix} A'_{i1} \\ A'_{i2} \end{pmatrix},$$

and thus

$$A = \begin{pmatrix} C_{A,1} \\ C_{A,2} \end{pmatrix}, \quad A^\times = (C'_{A,1}, C'_{A,2}). \tag{6.2}$$

**Theorem 6.1.** 1° $A^\times \subset A'$. 2° $A^\times$ *is closable.* 3° *If* $A^\times$ *is densely defined, then* $A$ *is closable.*

*Proof.* 1° immediately follows from (6.2) and Proposition 4.1. Since $A$ is densely defined, $A'$ is a closed operator, and hence by 1°, $A^\times$ is closable, which yields 2°. If $A^\times$ is densely defined, so is $A'$ by 1°. Consequently, $A$ is closable and 3° follows. □

**Proposition 6.2.** *If* $\mathcal{D}(C'_{A,1}) \oplus \mathcal{D}(C'_{A,2})$ *is dense in* $F'$, *then* $A$ *is closable.*

*Proof.* If $\mathcal{D}(C'_{A,1}) \oplus \mathcal{D}(C'_{A,2})$ is dense in $F'$, then $A'$ is densely defined by part 1° of Theorem 6.1, which means that $A$ is closable. □

More can be said if at most one of the operators $A_{ij}$ is not bounded.

**Proposition 6.3.** *Assume that $A_{ij} \in \mathfrak{B}(E_j, F_i)$ with the exception of at most one pair $(i,j)$ and that this exceptional $A_{ij}$ is closable. Then $A$ is closable and $A' = A^{\times}$.*

*Proof.* By Proposition 3.5, both $C_{A,1}$ and $C_{A,2}$ are closable, and hence $A$ is closable by Proposition 4.3 2°. At most one of the operators $C_{A,i}$ in (6.1) does not satisfy $C_{A,i} \in \mathfrak{B}(E, F_i)$. Then Proposition 4.5 and (6.2) lead to $A' = A^{\times}$. □

Theorem 6.1 and Proposition 6.3 immediately raise the question if the following cases can occur:

I. $A$ is not closable,
II. $A$ is closable and $A^{\times}$ is not densely defined,
III. $A^{\times}$ is densely defined but $A' \neq \overline{A^{\times}}$,
IV. $A' = \overline{A^{\times}}$ but $A' \neq A^{\times}$.

Below we will show indeed that all these cases can occur, even under the additional requirement that all $A_{ij}$ are closed operators in Hilbert spaces.

**Example 6.4.** Here we give an example for I. Let $E_1 = E_2 = F_1 = F_2 = \ell^2$ and, with $R_1$, $R_2$ from Example 3.3, put $A_{11} = R_1$, $A_{12} = R_2$, $A_{21} = A_{22} = 0$. Then all $A_{ij}$ are closed, and since $(R_1, R_2)$ is not closable, also $A$ is not closable by Proposition 4.3 2°.

**Example 6.5.** Here we give an example for II. Let $E_1 = E_2 = F_1 = F_2 = \ell^2$ and, with $R_1$, $R_2$ from Example 3.3, put $A_{11} = R_1$, $A_{12} = R_2$, $A_{21} = 0$, $A_{22} = R_2$. Then all $A_{ij}$ are closed, $A$ is closable, while $A^{\times}$ is not densely defined.

*Proof.* To show that $A$ is closable, consider any sequences

$$f_n \in \mathcal{D}(A_{11}) \cap \mathcal{D}(A_{21}),\ g_n \in \mathcal{D}(A_{12}) \cap \mathcal{D}(A_{22})$$

satisfying $f_n \to 0$, $g_n \to 0$, $A_{11}f_n + A_{12}g_n \to h_1$ and $A_{21}f_n + A_{22}g_n \to h_2$ for some $h_1$, $h_2$ in $\ell^2$. Then

$$h_2 = \lim_{n\to\infty}(A_{21}f_n + A_{22}g_n) = \lim_{n\to\infty} R_2 g_n,$$

$g_n \to 0$, and the closedness of $R_2$ imply that $h_2 = 0$. Consequently,

$$h_1 = \lim_{n\to\infty}(A_{11}f_n + A_{12}g_n) = \lim_{n\to\infty}(R_1 f_n + R_2 g_n) = \lim_{n\to\infty} R_1 f_n,$$

so that $f_n \to 0$ and the closedness of $R_1$ imply $h_1 = 0$. This completes the proof of the closability of $A$. By Example 3.3, $(A_{11}, A_{12})$ is not closable, whence, in view of Proposition 1, $\begin{pmatrix} A'_{11} \\ A'_{12} \end{pmatrix}$ is not densely defined. Hence also $A^{\times}$ is not densely defined. □

**Example 6.6.** Here we give an example for III. Let $E_1 = E_2 = F_1 = F_2 = L_2(0,1)$ and let the operators $A_{ij}$, $i,j = 1,2$ be defined by

$$\begin{aligned} &\mathcal{D}(A_{11}) = \{f \in W_2^1(0,1) : f(0) = 0\},\ A_{11}f = f', \\ &\mathcal{D}(A_{12}) = L_2(0,1),\ A_{12} = 0, \\ &\mathcal{D}(A_{21}) = \{f \in W_2^1(0,1) : f(1) = 0\},\ A_{21}f = f', \\ &\quad A_{22} = -A_{21}, \end{aligned}$$

where $W_2^1(0,1)$ denotes the usual Sobolev space of order 1. Then $A$ is densely defined, all $A_{ij}$, $i,j = 1,2$, are closed, $A^{\times}$ is densely defined, and $\overline{A^{\times}} \neq A'$.

*Proof.* The denseness of the domain of $A$ as well as the closedness of the $A_{ij}$ is well known and obvious. Since $A_{11}^* = -A_{21}$, it is also clear that $A^\times$ is densely defined. Putting $C_1 = (A_{11}, A_{12}),\ C_2 = (A_{21}, A_{22})$, we have

$$A = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

We also let $C_{12} = C_1|_{\mathcal{D}(C_1)\cap\mathcal{D}(C_2)}$ and $C_{21} = C_2|_{\mathcal{D}(C_1)\cap\mathcal{D}(C_2)}$. Clearly,

$$A = \begin{pmatrix} C_{12} \\ C_{21} \end{pmatrix} \subset \begin{pmatrix} \overline{C_{12}} \\ \overline{C_{21}} \end{pmatrix} \subset \begin{pmatrix} \overline{C_1} \\ \overline{C_2} \end{pmatrix},$$

and in view of Proposition 4.3 1° it follows that

$$\overline{A} \subset \begin{pmatrix} \overline{C_{12}} \\ \overline{C_{21}} \end{pmatrix} \subset \begin{pmatrix} \overline{C_1} \\ \overline{C_2} \end{pmatrix}. \tag{6.3}$$

Hence, by Proposition 4.3 3°, the proof will be complete if we show that the second inclusion in (6.3) is strict, i. e.,

$$\mathcal{D}(\overline{C_{12}}) \cap \mathcal{D}(\overline{C_{21}}) \neq \mathcal{D}(\overline{C_1}) \cap \mathcal{D}(\overline{C_2}). \tag{6.4}$$

Since $C_2(f \oplus f) = 0$ for $f \in \mathcal{D}(A_{21})$ we have $\{f \oplus f : f \in L_2(0,1)\} \subset \mathcal{D}(\overline{C_2})$, which immediately leads to

$$\{f \oplus f : f \in \mathcal{D}(A_{11})\} \subset \mathcal{D}(\overline{C_1}) \cap \mathcal{D}(\overline{C_2}). \tag{6.5}$$

Since $A_{11}$ and $A_{11}|_{\mathcal{D}(A_{11})\cap\mathcal{D}(A_{21})}$ are closed operators, so are $C_1$ and $C_{12}$. Hence $f \oplus g \in \mathcal{D}(\overline{C_{12}}) \cap \mathcal{D}(\overline{C_{21}})$ implies $f \in \mathcal{D}(A_{11}) \cap \mathcal{D}(A_{21})$, and thus (6.4) is proved in view of (6.5) and $\mathcal{D}(A_{11}) \not\subset \mathcal{D}(A_{21})$. □

**Example 6.7.** Here we give an example for IV. Let $E_1 = E_2 = F_1 = F_2$ be an infinite-dimensional Hilbert space and let $A_{11}$ be a closed densely defined unbounded operator in this Hilbert space. Put $A_{12} = A_{22} = 0$ and $A_{21} = -A_{11}$. Then $A$ is a densely defined closed operator, $\overline{A^\times} = A^*$, and $A^\times \neq A^*$.

Indeed, the operator $A$ is clearly densely defined and closed, and

$$A^\times = \begin{pmatrix} A_{11}^* & -A_{11}^* \\ 0 & 0 \end{pmatrix}.$$

Considering $A^\times$ as a column operator of row operators (one may look also at Theorem 10 in [15]) and applying Propositions 4.5 and 1 we come to

$$A^{\times *} = \begin{pmatrix} A_{11}^{**} & 0 \\ -A_{11}^{**} & 0 \end{pmatrix} = A$$

since $A_{11}$ is closed. Taking adjoints gives $\overline{A^\times} = A^*$.

Using Example 5.4 with $T = 0$ we get $\mathcal{D}(A^\times) \neq \mathcal{D}(A^*)$, which establishes $A^\times \neq A^*$.

## References


[1] F. V. Atkinson, H. Langer, R. Mennicken, A. A. Shkalikov, The essential spectrum of some matrix operators, Math. Nachr. 167 (1994), 5–22. MR1285306 (95f:47007)

[2] J. A. Burns, T. L. Herdman, Adjoint semigroup theory for a class of functional differential equations, SIAM J. Math. Anal. 7 (1976), 729–745. MR0417528 (54:5578)

[3] W. Desch, R. Grimmer, W. Schappacher, Wellposedness and wave propagation for a class of integrodifferential equations in Banach space, J. Diff. Equ. 74 (1988), 391–411. MR952904 (89e:45012)

[4] K.-L. Engel, Matrix representation of linear operators on product spaces, Rend. Circ. Mat. Palermo (2) Suppl. 56 (1998), 219–224. MR1710840 (2000f:47004)
[5] J. P. Friedberg, Ideal Magnetohydrodynamics, Plenum Press, New York, 1987.
[6] V. Hardt, A. Konstantinov, R. Mennicken, S. Naboko, On the spectrum of the product of closed operators, Math. Nachr 215 (2000), 91–102. MR1768195 (2001b:47008)
[7] V. Hardt, R. Mennicken, A. K. Motovilov, A factorization theorem for the transfer function associated with a $2 \times 2$ operator matrix having unbounded couplings, J. Operator Theory 48 (2002), no. 1, 187–226. MR1926050 (2003g:47026)
[8] K. Hain, R. Lüst, Zur Stabilität zylindersymmetrischer Plasmakonfigurationen mit Volumenströmen, Z. Naturforsch. A 13 (1958), 936-940. MR0106671 (21:5402)
[9] S. Hassi, A. Sandovici, H. de Snoo and H. Winkler, Extremal extensions for the sum of nonnegative selfadjoint relations, Proc. Amer. Math. Soc., 135 (2007), 3193–3204. MR2322750.
[10] A. Konstantinov, R. Mennicken, On the Friedrichs extension of some block operator matrices, Integral Equations Operator Theory 42 (2002), 472-481. MR1885445 (2003m:47039)
[11] P. Kurasov, S. Naboko, On the essential spectrum of a class of singular matrix differential operators. I: Quasiregularity conditions and essential self-adjointness, Mathematical Physics, Analysis and Geometry 5 (2002), 243–286. MR1940117 (2003m:47084)
[12] A. Lifschitz, Magnetohydrodynamics and Spectral Theory, Kluwer, Dordrecht, 1989. MR990647 (90k:76097)
[13] R. Nagel, Towards a “matrix theory” for unbounded operator matrices, Math. Z. 201 (1989), 57–68. MR990188 (90c:47004)
[14] S. Ôta, K. Schmüdgen, Some selfadjoint $2 \times 2$ operator matrices associated with closed operators, Integr. Equ. Oper. Theory 45 (2003), 475–484. MR1971749 (2004b:47041)
[15] F.H. Szafraniec, On normal extensions of unbounded operators: IV. A matrix construction, in: Operator Theory: Adv. Appl. 163 (2005), 337–350. MR2215869 (2007b:47052)

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