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RADIAL LIMITS OF INNER FUNCTIONS AND BLOCH SPACES

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ABSTRACT. Let f be an inner function in the unit ball $B_n \subset \mathbb{C}^n$, $n \geq 1$. Assume that

$$\sup_{z\in B_n} \frac{|\mathcal{R}f(z)|(1-|z|^2)^{1+\beta}}{(1-|f(z)|^2)^2} < \infty,$$

where $\beta \in (0,1)$ and $\mathcal{R}f$ is the radial derivative. Then, for all $\alpha \in \partial B_1$, the set $\{\zeta \in \partial B_n : f^*(\zeta) = \alpha\}$ has a non-zero real Hausdorff $t^{2n-1-\beta}$-content, and it has a non-zero complex Hausdorff $t^{n-\beta}$-content.

1. Introduction

Let $\mathcal{H}ol(B)$ denote the space of holomorphic functions in the unit ball $B = B_n \subset \mathbb{C}^n$, $n \geq 1$. By definition, a function $f \in \mathcal{H}ol(B)$ belongs to the Bloch space $\mathcal{B}(B)$ if

$$\|f\|_{\mathcal{B}} = \sup_{z\in B} |\mathcal{R}f(z)|(1-|z|^2) < \infty,$$

where

$$\mathcal{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

is the radial derivative (see [10], where equivalent definitions of the space $\mathcal{B}(B)$ are given). The little Bloch space $\mathcal{B}_0(B)$ consists of those $f \in \mathcal{B}(B)$ for which

$$\lim_{r\to 1-} \sup_{r<|z|<1} |\mathcal{R}f(z)|(1-|z|^2) = 0.$$

A bounded non-constant function $f \in \mathcal{H}ol(B)$ is called inner if the radial limits

$$f^*(\zeta) = \lim_{r\to 1-} f(r\zeta)$$

satisfy the identity $|f^*(\zeta)| = 1$ for almost all $\zeta \in \partial B$ (with respect to Lebesgue measure on the unit sphere $S = S_n = \partial B_n$).

If $n = 1$, then we introduce special notation $\mathbb{D} = B_1$ and $\mathbb{T} = S_1$. Rohde [7] proved that the Hausdorff dimension of the set

$$E_w(f) = \{\zeta \in \mathbb{T} : f^*(\zeta) = w\}$$

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is equal to 1 for all $w \in \mathbb{D}$ if $f \in \mathcal{B}_0(\mathbb{D})$, f is an inner function, and f is not a finite Blaschke product. Later, Donaire [3] obtained the above result for all $w \in \overline{\mathbb{D}}$.

In the present work we estimate the sizes of the sets $E_\alpha(f)$, $\alpha \in \mathbb{T}$, under different assumptions about an inner function f. To formulate the result, we recall the definitions of the real and complex Hausdorff contents. Let $\zeta \in S$, $r > 0$. Put

$$T(\zeta, r) = \{\xi \in S : |\xi - \zeta| < r\},$$
$$Q(\zeta, r) = \{\xi \in S : |1 - \langle \zeta, \xi \rangle| < r\}.$$

Fix an increasing continuous function $\phi : [0, +\infty) \to [0, +\infty)$ such that $\phi(0) = 0$. Let $E \subset S$. The quantity

$$H_\phi^{\mathbb{R}}(E) = \inf \left\{ \sum_j \phi(r_j) : \bigcup_j T(\zeta_j, r_j) \supset E \right\}$$

is called the real Hausdorff ϕ-content of the set E. The quantity

$$H_\phi^{\mathbb{C}}(E) = \inf \left\{ \sum_j \phi(r_j) : \bigcup_j Q(\zeta_j, r_j) \supset E \right\}$$

is called the complex Hausdorff ϕ-content of the set E.

If $n = 1$, then the real and complex contents coincide. In this case the Hausdorff ϕ-content is denoted by $H_\phi(E)$.

Theorem 1. *Let $\beta \in (0, 1)$ and let f be an inner function in the ball B_n. Assume that*

$$\sup_{z \in B_n} \frac{|\mathcal{R}f(z)|(1 - |z|^2)^{1+\beta}}{(1 - |f(z)|^2)^2} < \infty. \tag{1.1}$$

Then, for all $\alpha \in \mathbb{T}$, the set $E_\alpha(f) = \{\zeta \in S_n : f^*(\zeta) = \alpha\}$ has a non-zero real Hausdorff $t^{2n-1-\beta}$-content, and it has a non-zero complex Hausdorff $t^{n-\beta}$-content.

Comments and remarks. 1. Examples of inner functions with property (1.1) are constructed in [2] and [4] for $n = 1$ and $n \geq 2$, respectively.

2. Let $f \in \mathcal{H}ol(\mathbb{D})$. Assume that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Then, by the Schwarz–Pick theorem,

$$|f'(z)|(1 - |z|^2) \leq 1 - |f(z)|^2 \quad (z \in \mathbb{D}).$$

In other words, (1.1) with $1 - |f(z)|^2$ in place of $(1 - |f(z)|^2)^2$ is trivial.

3. To prove Theorem 1, we investigate the corresponding Aleksandrov–Clark measures.

4. In what follows, the symbol C denotes an absolute constant whose value may change from location to location.

2. Model example

The following assertion formally corresponds to Theorem 1 when $n = 1$ and $\beta = 0$.

Proposition 2. *Let f be an inner function in the disk $\mathbb{D}$. Assume that*

$$\sup_{z \in \mathbb{D}} \frac{|f'(z)|(1 - |z|^2)}{(1 - |f(z)|^2)^2} < \infty. \tag{2.1}$$

Then, for all $\alpha \in \mathbb{T}$, *one has*

$$H_\Phi\{\zeta \in \mathbb{T} : \ f^*(\zeta) = \alpha\} > 0,$$

where $\Phi(t) = t\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$ *for* $0 < t \le \exp(e^{-e})$ *and the function* $\Phi(t)$ *is constant when* $t \ge \exp(e^{-e})$.

Proof. 1. For $\alpha \in \mathbb{T}$, consider the function

$$H_\alpha(z) = \frac{\alpha + f(z)}{\alpha - f(z)} \quad (z \in \mathbb{D}).$$

By (2.1), straightforward calculations show that $H_\alpha \in \mathcal{B}(\mathbb{D})$. Now, note that $\operatorname{Re} H_\alpha(z)$ is a positive harmonic function in the disk. Therefore, $\operatorname{Re} H_\alpha(z)$ is the Poisson integral $P[\sigma_\alpha](z)$, where σ_α is a positive measure on the circle $\mathbb{T}$. The measure $\sigma_\alpha = \sigma_\alpha(f)$ is called an Aleksandrov–Clark measure. Next, $H_\alpha(z)$ is the Herglotz integral

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma_\alpha(z),$$

up to an additive constant. It is well known that the Herglotz integral of a positive measure μ is in $\mathcal{B}(\mathbb{D})$ if and only if the measure μ is Zygmund. By definition, this means that, for all adjacent arcs $I_-, I_+ \subset \mathbb{T}$ of equal length, we have the estimate $|\mu(I_-) - \mu(I_+)| \le C|I_+|$, where $|I_+|$ is the length of the arc I_+, and C is a positive constant. Thus, the measure σ_α is Zygmund. By Makarov's law of the iterated logarithm [6], the measure σ_α is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}_\Phi$ with $\Phi(t) = t\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$. Recall that the Hausdorff ϕ-measure of a set E is defined by the identity

$$\mathcal{H}_\phi(E) = \liminf_{\delta\to 0}\left\{\sum_j \phi(|I_j|) : \ \bigcup_j I_j \supset E,\ |I_j| \le \delta\right\}.$$

It is well known that $H_\phi(E) = 0$ if and only if $\mathcal{H}_\phi(E) = 0$.

2. The function f is inner; hence, for every $\alpha \in \mathbb{T}$, the measure σ_α is singular (with respect to Lebesgue measure on the circle). Therefore, σ_α is concentrated on the set

$$E(\sigma_\alpha) = \{\zeta \in \mathbb{T} : \ P[\sigma_\alpha]^*(\zeta) = +\infty\}.$$

In other words,

$$\sigma_\alpha(E(\sigma_\alpha)) = \sigma_\alpha(\mathbb{T}) > 0. \tag{2.2}$$

Next, observe that

$$\operatorname{Re} H_\alpha(z) = \frac{1 - |f(z)|^2}{|\alpha - f(z)|^2} \quad (z \in \mathbb{D}).$$

Thus,

$$E(\sigma_\alpha) \subset E_\alpha(f), \tag{2.3}$$

where $E_\alpha(f) = \{\zeta \in \mathbb{T} : \ f^*(\zeta) = \alpha\}$.

3. Finally, assume that $H_\Phi(E_\alpha(f)) = 0$. Then $H_\Phi(E(\sigma_\alpha)) = 0$ by (2.3). Therefore, $\mathcal{H}_\Phi(E(\sigma_\alpha)) = 0$; so, by the above-mentioned Makarov's theorem, we have $\sigma_\alpha(E(\sigma_\alpha)) = 0$. The latter identity contradicts (2.2). The proof of Proposition 2 is complete. □

3. Proof of Theorem 1

Let μ be a positive measure on the sphere S. The identity

$$\omega_{\mathbb{R}}(\mu;t) = \sup\{\mu(T(\zeta,t)) : \zeta \in S\} \quad (t>0)$$

defines the real modulus of continuity of the measure μ. The complex modulus of continuity is defined as

$$\omega_{\mathbb{C}}(\mu;t) = \sup\{\mu(Q(\zeta,t)) : \zeta \in S\} \quad (t>0).$$

We will need the following two lemmas. If $n=1$, then such results are well known (see [9]). Let

$$P(z,\zeta) = \frac{1-|z|^2}{|z-\zeta|^{2n}} \quad (z\in B,\ \zeta \in S)$$

denote the classical Poisson kernel in the ball.

Lemma 3. *Let $0<\beta<1$ and let μ be a positive measure on S_n. Assume that*

$$P[\mu](z) = \int_{S_n} P(z,\zeta)\, d\mu(\zeta) \le C(1-|z|^2)^{-\beta}.$$

Then $\omega_{\mathbb{R}}(\mu;t) \le Ct^{2n-1-\beta}$.

Proof. Let $\xi \in S$ and let $1/2 \ge t > 0$. By the definition of the set $T(\xi,t)$, we have $|\zeta-(1-t)\xi| \le Ct$ for all $\zeta \in T(\xi,t)$. Therefore,

$$\frac{\mu(T(\xi,t))}{t^{2n-1}} \le CP[\mu]((1-t)\xi) \le Ct^{-\beta}.$$

In other words, $\omega_{\mathbb{R}}(\mu;t) \le Ct^{2n-1-\beta}$. □

Let

$$\mathcal{P}(z,\zeta) = \left(\frac{1-|z|^2}{|1-\langle z,\zeta\rangle|^2}\right)^n \quad (z\in B,\ \zeta \in S)$$

denote the $\mathcal{M}$-invariant Poisson kernel in the ball.

Lemma 4. *Let $0<\beta<1$ and let μ be a positive measure on S_n. Assume that*

$$\mathcal{P}[\mu](z) = \int_{S_n} \mathcal{P}(z,\zeta)\, d\mu(\zeta) \le C(1-|z|^2)^{-\beta}.$$

Then $\omega_{\mathbb{C}}(\mu;t) \le Ct^{n-\beta}$.

Proof. Suppose that $\xi \in S$ and $1/2 \ge t > 0$. By the definition of the set $Q(\xi,t)$, the identity $|1-(1-t)\langle \xi,\zeta\rangle| \le Ct$ holds for all $\zeta \in Q(\xi,t)$. Thus,

$$\frac{\mu(Q(\xi,t))}{t^{n}} \le C\mathcal{P}[\mu]((1-t)\xi) \le Ct^{-\beta}.$$

The required inequality is proved. □

Proof of Theorem 1. 1. Fix a $\beta \in (0,1)$. Given $\zeta \in S$ and $\lambda \in \mathbb{D}$, put $f_\zeta(\lambda) = f(\lambda\zeta)$. It is well known that $\mathcal{R}f(\lambda\zeta) = \lambda f'_\zeta(\lambda)$. Therefore, (1.1) is equivalent to the following inequality:

$$\sup_{\zeta\in S,\,\lambda\in\mathbb{D}} \frac{|f'_\zeta(\lambda)|(1-|\lambda|^2)^{1+\beta}}{(1-|f_\zeta(\lambda)|^2)^2} < \infty. \tag{3.1}$$

Fix $\alpha \in \mathbb{T}$ and consider the function

$$H_\alpha(z) = \frac{\alpha + f(z)}{\alpha - f(z)} \quad (z \in B).$$

Assume that $\zeta \in S$ and $\lambda \in \mathbb{D}$. Then

$$H_{\alpha,\zeta}(\lambda) = H_\alpha(\lambda\zeta) = \frac{\alpha + f_\zeta(\lambda)}{\alpha - f_\zeta(\lambda)}.$$

Straightforward calculations and (3.1) show that

$$\sup_{\zeta\in S,\,\lambda\in\mathbb{D}} |H_{\alpha,\zeta}(\lambda)|(1-|z|^2)^{1+\beta} \le C \sup_{\zeta\in S,\,\lambda\in\mathbb{D}} \frac{|f'_\zeta(\lambda)|(1-|\lambda|^2)^{1+\beta}}{(1-|f_\zeta(\lambda)|^2)^2} < \infty.$$

Hence,

$$|H_{\alpha,\zeta}(\lambda)| \le C + C\int_0^{|\lambda|} (1-t)^{-1-\beta}\,dt \le C(\beta)(1-|\lambda|^2)^{-\beta}.$$

Therefore,

$$|H_\alpha(z)| \le C(1-|z|^2)^{-\beta}. \tag{3.2}$$

2. Since $\operatorname{Re} H_\alpha(z)$ is a positive pluriharmonic function in the ball, we have $\operatorname{Re} H_\alpha(z) = P[\sigma_\alpha](z) = \mathcal{P}[\sigma_\alpha](z)$, where σ_α is a positive measure on the sphere (see [8], Theorem 4.4.9). In other words, σ_α is a pluriharmonic Aleksandrov–Clark measure. Inequality (3.2) guarantees that $P[\sigma_\alpha](z) = \mathcal{P}[\sigma_\alpha](z) \le C(1-|z|^2)^{-\beta}$. By Lemma 3, we have

$$\omega_{\mathbb{R}}(\sigma_\alpha; t) \le Ct^{2n-1-\beta}; \tag{3.3}$$

by Lemma 4, we have

$$\omega_{\mathbb{C}}(\sigma_\alpha; t) \le Ct^{n-\beta}. \tag{3.4}$$

3. The arguments below are analogous to those given in the model case. The function f is inner, so, for every $\alpha \in \mathbb{T}$, the measure σ_α is singular (with respect to Lebesgue measure on the sphere). Thus, σ_α is concentrated on the set

$$E(\sigma_\alpha) = \{\zeta \in S: \ P[\sigma_\alpha]^*(\zeta) = +\infty\}$$

(see [8], Sections 5.2.7, 5.4.11, 5.4.12). In other words, $\sigma_\alpha(E(\sigma_\alpha)) = \sigma_\alpha(S) > 0$.

Now, we have

$$\operatorname{Re} H_\alpha(z) = \frac{1-|z|^2}{|\alpha - f(z)|^2} \quad (z \in B_n).$$

Therefore, $E(\sigma_\alpha) \subset E_\alpha(f) = \{\zeta \in S: \ f^*(\zeta) = \alpha\}$. Hence, to finish the proof, it suffices to show that

$$H^{\mathbb{R}}_{\Phi_\beta}(E(\sigma_\alpha)) > 0, \quad \text{where } \Phi_\beta(t) = t^{2n-1-\beta};$$
$$H^{\mathbb{C}}_{\Psi_\beta}(E(\sigma_\alpha)) > 0, \quad \text{where } \Psi_\beta(t) = t^{n-\beta}.$$

So, consider a covering of $E(\sigma_\alpha)$ by a countable union of sets $T(\zeta_j, r_j)$. By (3.3), we have $\sigma_\alpha(T(\zeta_j, r_j)) \le C\Phi_\beta(r_j)$. Thus,

$$0 < \sigma_\alpha(E(\sigma_\alpha)) \le \sum_j \sigma_\alpha(T(\zeta_j, r_j)) \le C \sum_j \Phi_\beta(r_j).$$

By the definition of the real Hausdorff content, the required estimate

$$H^{\mathbb{R}}_{\Phi_\beta}(E(\sigma_\alpha)) \ge \sigma_\alpha(E(\sigma_\alpha)) > 0$$

holds.

To prove the inequality $H^{\mathbb{C}}_{\Psi_\beta}(E(\sigma_\alpha)) > 0$, we argue as above and we apply (3.4) in place of (3.3). The proof of Theorem 1 is complete. □

Final remarks. For $\beta = 1$ and $n \ge 2$, the conclusion of Theorem 1 is true for all inner functions. Namely, the following assertion holds.

Proposition 5. *Let f be an inner function in the ball B_n, $n \ge 2$. Then, for all $\alpha \in \mathbb{T}$, the set $E_\alpha(f) = \{\zeta \in S_n : f^*(\zeta) = \alpha\}$ has a non-zero real Hausdorff t^{2n-2}-content, and it has a non-zero complex Hausdorff t^{n-1}-content.*

Proof. Let $\alpha \in \mathbb{T}$. Consider the Aleksandrov–Clark measure $\sigma_\alpha = \sigma_\alpha(f)$. The Poisson integral $P[\sigma_\alpha]$ is a pluriharmonic function; therefore, $\omega_{\mathbb{R}}(\sigma_\alpha; t) \le Ct^{2n-2}$ and $\omega_{\mathbb{C}}(\sigma_\alpha; t) \le Ct^{n-1}$ (see [1], Chapter 5, Section 3.3). To obtain the required conclusion, we repeat the arguments used in the proof of Theorem 1. □

Assume that $f(0) = 0$ and f is an inner function in the ball B_n, $n \ge 1$. It is proved in [5] that $c_{2n-2+\alpha}(f^{-1}(E)) \ge A(n,\alpha)c_\alpha(E)$, where $E \subset \mathbb{T}$, $A(n,\alpha)$ is a constant, and c_α is the α-dimensional capacity. In a sense, Proposition 5 is an extremal case of the latter inequality.

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