H*-ALGEBRAS AND QUANTIZATION OF PARA-HERMITIAN SPACES

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Abstract. In the present note we describe a family of H*-algebra structures on the set \( L^2(X) \) of square integrable functions on a rank-one para-Hermitian symmetric space \( X \).

INTRODUCTION

Let \( X \) be a rank-one para-Hermitian symmetric space. It is well known that \( X \) is isomorphic (up to a covering) to the quotient space \( SL(n, \mathbb{R})/GL(n-1, \mathbb{R}) \); see \([KK]\) for more details. We shall thus assume throughout this note that \( X = G/H \), where \( G = SL(n, \mathbb{R}) \) and \( H = GL(n-1, \mathbb{R}) \).

The space \( X \) allows the definition of a covariant symbolic calculus that generalizes the so-called convolution-first calculus on \( \mathbb{R}^2 \); see \([DM], [PU], [UU] \) for instance. Such a calculus, or quantization map \( \text{Op}_\sigma \), maps a suitable set of functions (or symbols) on \( X \) to linear operators acting on the representation space of the maximal degenerate series \( \pi_{-\frac{1}{2} + i\sigma} \) of the group \( G \). When applied to \( L^2(X) \), this induces a noncommutative algebra structure. On the other hand, the taking of the adjoint of an operator in such a calculus defines an involution on symbols. It turns out that these two data give rise to an H*-algebra structure on \( L^2(X) \).

According to the general theory, \([A], [L], [N] \), every H*-algebra is the direct orthogonal sum of its closed minimal two-sided ideals which are simple H*-algebras. The main result of this note is the explicit description of such a decomposition for the Hilbert algebra \( L^2(X) \) and its commutative subalgebra of \( SO(n, \mathbb{R}) \)-invariants. In \([DP] \) we have shown that the algebra structure of \( L^2(X) \) might be very useful and appropriate for a new interpretation of Rankin-Cohen brackets. Therefore a careful study of this structure is a next step in the process of its understanding.

1. DEFINITIONS AND BASIC FACTS

1.1. H*-algebras.

Definition 1.1. A set \( R \) is called an H*-algebra (or Hilbert algebra) if

1. \( R \) is a Banach algebra with involution;
2. \( R \) is a Hilbert space;
3. the norm on the algebra \( R \) coincides with the norm on the Hilbert space \( R \);

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(4) for all \( x, y, z \in R \) one has \( (xy, z) = (y, x*z) \);
(5) for all \( x \in R \) one has \( \|x^*\| = \|x\| \);
(6) \( xx^* \neq 0 \) for \( x \neq 0 \).

An example of a Hilbert algebra is the set of Hilbert-Schmidt operators \( HS(I) \) that one can identify with the set of all matrices \( (a_{\alpha\beta}) \), where \( \alpha, \beta \) belong to a fixed set of indices \( I \), satisfying the condition \( \sum_I |a_{\alpha\beta}|^2 < \infty \).

**Theorem 1.2** ([N] p. 331). Every Hilbert algebra is the direct orthogonal sum of its closed minimal two-sided ideals, which are simple Hilbert algebras.

Every simple Hilbert algebra is isomorphic to some algebra \( HS(I) \) of Hilbert-Schmidt operators.

**Definition 1.3** ([L] p. 101). An idempotent \( e \in R \) is said to be irreducible if it cannot be expressed as a sum \( e = e_1 + e_2 \) with \( e_1, e_2 \) idempotents that annihilate each other: \( e_1e_2 = e_2e_1 = 0 \).

**Lemma 1.4** ([L] p. 102). A subset \( I \) of a Hilbert algebra \( R \) is a minimal left (right) ideal if and only if it is of the form \( I = R \cdot e \) \((I = e \cdot R)\), where \( e \) is an irreducible selfadjoint idempotent. Moreover \( e \cdot R \cdot e \) is isomorphic to the set of complex numbers and \( R \) is spanned by its minimal left ideals.

Observe that any minimal left ideal is closed, since it is of the form \( R \cdot e \).

**Corollary 1.5.** If \( R \) is a commutative Hilbert algebra, then any minimal left (or right) ideal is one-dimensional.

1.2. **An algebra structure on** \( L^2(X) \). Let \( G = SL(n, \mathbb{R}) \), \( H = GL(n-1, \mathbb{R}) \), \( K = SO(n) \) and \( M = SO(n-1) \). We consider \( H \) as a subgroup of \( G \), consisting of the matrices of the form \( \begin{pmatrix} (\det h)^{-1} & 0 \\ 0 & h \end{pmatrix} \) with \( h \in GL(n-1, \mathbb{R}) \).

Let \( P^- \) be the parabolic subgroup of \( G \) consisting of \( 1 \times (n-1) \) lower block matrices \( P = \begin{pmatrix} a \\ c \end{pmatrix} \), \( a \in \mathbb{R}^* \), \( c \in \mathbb{R}^{n-1} \) and \( A \in GL(n-1, \mathbb{R}) \) such that \( a \cdot \det A = 1 \).

Similarly, let \( P^+ \) be the group of upper block matrices \( P = \begin{pmatrix} a \\ b \end{pmatrix} \), \( a \in \mathbb{R}^* \), \( b \in \mathbb{R}^{n-1} \) and \( A \in GL(n-1, \mathbb{R}) \) such that \( a \cdot \det A = 1 \).

The group \( G \) acts on the sphere \( S = \{ s \in \mathbb{R}^n, \|s\|^2 = 1 \} \) and acts transitively on the set \( \tilde{S} = S/\sim \), where \( s \sim s' \) if and only if \( s = \pm s' \), by \( g.s = \frac{g(s)}{\|g(s)\|} \), where \( g(s) \) denotes the linear action of \( G \) on \( \mathbb{R}^n \). Clearly the stabilizer of the equivalence class of the first basis vector \( e_1 \) is the group \( P^+ \); thus \( \tilde{S} \simeq G/P^+ \). If \( ds \) is the usual normalized surface measure on \( S \), then \( d(g.s) = \|g(s)\|^{-n} ds \).

For \( \mu \in \mathbb{C} \), define the character \( \omega_\mu \) of \( P^\pm \) by \( \omega_\mu(P) = |\mu|^\mu \). Consider the induced representations \( \pi^+_\mu = \text{Ind}_{P^-}^{G} \omega_{\pi^-} \mu \).

Both \( \pi^+_\mu \) and \( \pi^-_\mu \) can be realized on \( C^\infty(\tilde{S}) \), the space of even smooth functions \( \phi \) on \( S \). This action is given by
\[
\pi^+_\mu(g)\phi(s) = \phi(g^{-1}.s) \cdot \|g^{-1}(s)\|^\mu.
\]
Let \( \theta \) be the Cartan involution of \( G \) given by \( \theta(g) = {}^tg^{-1} \). Then
\[
\pi^-_\mu(g)\phi(s) = \phi(\theta(g^{-1}).s) \cdot \|\theta(g^{-1})(s)\|^\mu.
\]
Let $(\ ,\ )$ denote the usual inner product on $L^2(S): (\phi, \psi) = \int_S \phi(s)\overline{\psi}(s)ds$. Then this sesquilinear form is invariant with respect to the pairs of representations $(\pi^+, \pi^-_{-\mu})$ and $(\pi^-, \pi^+_{-\mu})$. Therefore the representations $\pi^\pm$ are unitary for $\text{Re} \mu = -\frac{n}{2}$. Notice that according to [DM2] these representations are irreducible for all nonintegral $\mu$.

The group $G$ acts also on $\bar{S} \times \bar{S}$ by

$$g(u, v) = (g(u), \theta(g)v).$$

This action is not transitive: the orbit $(\bar{S} \times \bar{S})^o = G.(\bar{c_1}, \bar{c_1}) = \{(u, v) : \langle u, v \rangle \neq 0\}/\sim$ is dense (here $\langle \ , \ \rangle$ denotes the canonical inner product on $\mathbb{R}^n$) and is of co-measure zero. Moreover, one has a $G$-equivariant diffeomorphism $(\bar{S} \times \bar{S})^o \simeq X$.

The map $f \mapsto f(u, v)|\langle u, v \rangle|^{-\frac{n}{2} + i\sigma}$, with $\sigma \in \mathbb{R}$ is a unitary $G$-isomorphism between $L^2(X)$ and $\pi^+_\frac{n}{2} + i\sigma \bar{\otimes}_2 \pi^-_{\frac{n}{2} + i\sigma}$ acting on $L^2(\bar{S} \times \bar{S})$. The latter space is provided with the usual inner product.

Define the operator $A_\mu$ on $C^\infty(\bar{S})$ by the formula

$$A_\mu \phi(s) = \int_S |\langle s, t \rangle|^{-\mu-n} \phi(t)dt.$$ 

This integral converges absolutely for $\text{Re} \mu < -1$ and can be analytically extended to the whole complex plane as a meromorphic function of $\mu$. It is easily checked that $A_\mu$ is an intertwining operator, that is, $A_\mu \pi^\pm(g) = \pi^\pm_{-\mu-n}(g)A_\mu$.

The operator $A_{-\mu-n} \circ A_\mu$ intertwines the representation $\pi^\pm_{-\mu}$ with itself and is therefore a scalar $c(\mu)$Id depending only on $\mu$. It can be computed using $K$-types.

Let $e(\mu) = \int_S |\langle s, t \rangle|^{-\mu-n}dt$. Then $c(\mu) = e(\mu)e(-\mu - n)$. But on the other hand, $c(\mu) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi} \frac{\Gamma\left(\frac{-n/2+i\sigma+1}{2}\right)}{\Gamma\left(\frac{n/2+i\sigma}{2}\right)} \frac{\Gamma\left(\frac{-n/2-i\sigma+1}{2}\right)}{\Gamma\left(\frac{-n/2-i\sigma}{2}\right)}$,

and moreover $A_{-\frac{n}{2}+i\sigma} \circ A^*_{-\frac{n}{2}+i\sigma} = c(\sigma)$Id, so that the operator $d(\sigma)A_{-\frac{n}{2}+i\sigma}$, where

$$d(\sigma) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{n/2+i\sigma}{2}\right)} \Gamma\left(\frac{-n/2-i\sigma+1}{2}\right)$$

is a unitary intertwiner between $\pi^-_{\frac{n}{2} + i\sigma}$ and $\pi^+_{\frac{n}{2} - i\sigma}$.

We thus get a $\pi^+_{\frac{n}{2} + i\sigma} \bar{\otimes}_2 \pi^-_{\frac{n}{2} + i\sigma}$ invariant map from $L^2(X)$ onto $L^2(\bar{S} \times \bar{S})$ given by

$$f \mapsto d(\sigma) \int_S f(u, v)|\langle u, v \rangle|^{-\frac{n}{2} + i\sigma} |\langle v, w \rangle|^{-\frac{n}{2} - i\sigma} dw =: (T_\sigma f)(u, v), \forall \sigma \neq 0.$$ 

This integral does not converge absolutely; it must be considered as obtained by analytic continuation.

**Definition 1.6.** A symbolic calculus on $X$ is a linear map $Op_\sigma : L^2(X) \to \mathcal{L}(L^2(\bar{S}))$ such that for every $f \in L^2(X)$ the function $(T_\sigma f)(u, v)$ is the kernel of the Hilbert-Schmidt operator $Op_\sigma(f)$ acting on $L^2(\bar{S})$.

**Definition 1.7.** The product $\#_\sigma$ on $L^2(X)$ is defined by

$$Op_\sigma(f \#_\sigma g) = Op_\sigma(f) \circ Op_\sigma(g), \forall f, g \in L^2(X).$$
We thus have

- The product $\ast_\sigma$ is associative.
- $\|f \ast_\sigma g\|_2 \leq \|f\|_2 \cdot \|g\|_2$, for all $f, g \in L^2(X)$.
- $Op_\sigma(L_x f) = \pi^\pm_{\frac{\sigma}{2} + i\sigma}(x) Op_\sigma(f) \pi^\pm_{\frac{\sigma}{2} + i\sigma}(x^{-1})$, so $L_x(f \ast_\sigma g) = (L_x f) \ast_\sigma (L_x g)$, for all $x \in G$, where $L_x$ denotes the left translation by $x \in G$ on $L^2(X)$.

This noncommutative product can be described explicitly:

\begin{equation}
(f \ast_\sigma g)(u, v) = d(\sigma) \int_S \int_S f(u, x) g(y, y) \|u, y, x, v\|^{-\frac{1}{2} + i\sigma} d\mu(x, y),
\end{equation}

where $d\mu(x, y) = |\langle x, y \rangle|^{-n} dx dy$ is a $G$-invariant measure on $\tilde{S} \times \tilde{S}$ for the $G$-action $[\Pi]$, and $[u, y, x, v] = \langle \langle u, x \rangle \langle y, v \rangle \rangle$.

On the space $L^2(X)$ there exists an (family of) involution $f \mapsto f^*$ given by: $Op_\sigma(f^*) =: Op_\sigma(f)^*$. Notice that the correspondance $f \mapsto Op_\sigma(f)^*$ is what one calls in pseudo-differential analysis “anti-standard symbolic calculus”. The link between symbols of standard and anti-standard calculus in the setting of the para-Hermitian symmetric space $X$ has been made explicit in [PU, Corollary 1.4]; see also Section 3.

Obviously we have $(f \ast_\sigma g)^* = g^* \ast_\sigma f^*$ and with the above product and involution, the Hilbert space $L^2(X)$ becomes a Hilbert algebra.

2. **The structure of the subalgebra of $K$-invariant functions in $L^2(X)$**

Let $\mathcal{A}$ be the subspace of all $K$-invariant functions in $L^2(X)$.

**Theorem 2.1.** The subset $\mathcal{A}$ is a closed subalgebra of $L^2(X)$ with respect to the product $\ast_\sigma$.

This statement clearly follows from the covariance of the symbolic calculus $Op_\sigma$, namely: $L_x(f \ast_\sigma g) = (L_x f) \ast_\sigma (L_x g)$, for all $x \in G, f, g \in L^2(X)$.

**Theorem 2.2.** Let $n > 2$. Then the subalgebra $\mathcal{A}$ is commutative.

**Proof.** For a function $f \in L^2(X)$ we set $\tilde{f}(u, v) = f(v, u)$. The map $f \mapsto \tilde{f}$ is a linear involution. Indeed,

\[(f \ast_\sigma g)(u, v) = d(\sigma) \int_S \int_S \tilde{f}(x, u) \tilde{g}(y, y) \|u, y, x, v\|^{-\frac{1}{2} + i\sigma} d\mu(x, y).
\]

Permuting $x$ and $y$ and $u$ and $v$ respectively, we get

\[(f \ast_\sigma g)(v, u) = d(\sigma) \int_S \int_S \tilde{g}(u, x) \tilde{f}(y, y) \|v, y, x, u\|^{-\frac{1}{2} + i\sigma} d\mu(x, y).
\]

But $\|v, x, y, u\| = \|u, y, x, v\|$; therefore $(f \ast_\sigma g) = \tilde{g} \ast_\sigma \tilde{f}$.

On the other hand, given a couple $(u, v) \in \tilde{S} \times \tilde{S}$ there exists an element $k \in K$ such that $k.(u, v) = (v, u)$. Geometrically $k$ can be seen as a rotation of angle $\pi [2\pi]$ around the axis defined by the bisectrix of vectors $u$ and $v$ in the plane they generate. Of course, such a $k$ exists for an arbitrary couple $(u, v)$ only if $n > 2$.

Hence for every $f \in \mathcal{A}$ we have $f = \tilde{f}$ and therefore $f \ast_\sigma g = g \ast_\sigma f$, for $f, g \in \mathcal{A}$. \qed
3. Irreducible Selfadjoint Idempotents of $\mathcal{A}$

We begin with a reduction theorem for the multiplication and involution in $L^2(X)$.

As usual, we shall identify $L^2(X)$ with $L^2(\widetilde{S} \times \widetilde{S}; |\langle x, y \rangle|^{-n} dxdy)$. If $\phi \in L^2(X)$ we shall write $\phi(u, v) = |\langle u, v \rangle|^{\nu/2-i\sigma} \phi_o(u, v)$. Then $\phi_o \in L^2(\widetilde{S} \times \widetilde{S}; dsdt) = L^2(\widetilde{S} \times \widetilde{S})$, and therefore the map $\phi \rightarrow \phi_o$ is an isomorphism.

**Theorem 3.1.** Under the isomorphism $\phi \rightarrow \phi_o$ the product $\#_\sigma$ translates into

$$\phi_o \#_\sigma \psi_o(u, v) = d(\sigma) \int_S \int_S \phi_o(u, x) \psi_o(y, v) |\langle x, y \rangle|^{-\nu/2-i\sigma} dxdy$$

and the involution becomes:

$$\tilde{\phi}_o(u, v) = d(\sigma)^2 \int_S \int_S \tilde{\phi}_o(x, y) (|\langle x, v \rangle|/|\langle u, y \rangle|)^{-\nu/2+i\sigma} dxdy.$$ 

The proof is straightforward. So we have translated the algebra structure of $L^2(X)$ to $L^2(\widetilde{S} \times \widetilde{S})$.

Let $\phi$ be an irreducible selfadjoint idempotent in $\mathcal{A}$. We shall give an explicit formula for the $\phi_o$-component of $\phi$.

Consider the decomposition of the space $L^2(\widetilde{S}) = \bigoplus_{\ell \in 2\mathbb{N}} V_\ell$, where $V_\ell$ is the space of harmonic polynomials on $\mathbb{R}^n$, homogeneous of even degree $\ell$.

Then the space $L^2(\widetilde{S} \times \widetilde{S})$ decomposes into a direct sum of tensor products $\bigoplus_{\ell, m \in 2\mathbb{N}} V_\ell \otimes V_m$ and consequently

$$L^2_K(\widetilde{S} \times \widetilde{S}) = \bigoplus_{\ell \in 2\mathbb{N}} (V_\ell \otimes \tilde{V}_\ell)^K,$$

where the sub(superscript) $K$ means: “the $K$-invariants in”.

Let $\dim V_\ell = d$ and $f_1, \ldots, f_d$ be an orthonormal basis of $V_\ell$. Then the function $\theta_\ell(u, v) = \sum_{i,j} f_i(u) f_j(v)$, that is, the reproducing kernel of $V_\ell$, is, up to a constant, the $K$-invariant element of $V_\ell \otimes \tilde{V}_\ell$.

**Theorem 3.2.** Let $\phi(u, v) = |\langle u, v \rangle|^{\nu/2-i\sigma} \phi_o(u, v)$ be an irreducible selfadjoint idempotent in $\mathcal{A}$. Then there exist complex numbers $c(\sigma, \ell)$ such that for any $\ell \in 2\mathbb{N}$ one has

$$\phi_o(u, v) = c(\sigma, \ell) \theta_\ell(u, v).$$

For different $\ell$ and $\ell'$ the idempotents annihilate each other. Moreover they span $\mathcal{A}$.

**Proof.** Firstly we shall show that $\theta_\ell$ satisfies the condition

$$\theta_\ell \#_\sigma \theta_\ell = a(\sigma, \ell) \theta_\ell$$

for some constant $a(\sigma, \ell)$. Indeed,

$$d(\sigma) \int_S \int_S \theta_\ell(u, x) \theta_\ell(y, v) |\langle x, y \rangle|^{-\frac{\nu}{2}+i\sigma} dxdy$$

$$= d(\sigma) e(\sigma) \int_S \theta_\ell(u, y) \theta_\ell(y, v) dy = d(\sigma) e(\sigma) \theta_\ell(u, v)$$

by the intertwining relation (apply $A_{-\frac{\nu}{2}+i\sigma}$ to $\theta_\ell(., x)$):

$$\int_S \theta_\ell(u, x) |\langle x, y \rangle|^{-\frac{\nu}{2}+i\sigma} dx = e(\sigma) \theta_\ell(u, y)$$
where
\[
e_\ell(\sigma) = \int_S \frac{\theta_\ell(e_1, x)}{\theta_\ell(e_1, e_1)} |x_1|^{-\frac{n}{2} - i\sigma} \, dx.
\]

Observe that \(\frac{\theta_\ell(e_1, x)}{\theta_\ell(e_1, e_1)}\) is a spherical function on \(\mathcal{S}\) with respect to \(M\) of the form \(a_\ell C_{\ell}^{n-2}(|x_1|)\) where \(C_{\ell}^{n-2}(u)\) is a Gegenbauer polynomial and
\[
a_\ell^{-1} = C_{\ell}^{n-2}(1) = 2\ell \Gamma\left(\frac{n-2}{2} + \ell\right) \Gamma\left(\frac{n-2}{2}\right)\ell!.
\]
See for instance [VI Chapter IX, §3]. Notice that \(\theta_\ell(e_1, e_1) = \dim V_\ell = \frac{(n+\ell-1)!}{(n-1)!\ell!} \neq 0\).

The integral defining \(e_\ell(\sigma)\) does not converge absolutely, but has to be considered as the meromorphic extension of an analytic function. Poles only occur in half-integer points on the real axis. So we have to restrict (and we do) to \(\sigma \neq 0\).

So we have \(\theta_\ell\#_\sigma \theta_\ell = d(\sigma) e_\ell(\sigma) \theta_\ell\) and hence \(\varphi_\ell = [d(\sigma) e_\ell(\sigma)]^{-1} \theta_\ell\) is the \(\sigma\text{-}
\integrated\) component of an idempotent in \(\mathcal{A}\). Furthermore \(\theta_\ell\#_\sigma \theta_{\ell'} = 0\) if \(\ell \neq \ell'\). Clearly \(\varphi_\ell\) is selfadjoint, since \(|d(\sigma)|^{-2} = |e_\ell(\sigma)|^2\), being equal to the constant \(c(\sigma)\) from Section 1.

So the \(\varphi_\ell\) are mutually orthogonal idempotents in the algebra \(L^2_\mathcal{F}(\mathcal{S} \times \mathcal{S}; dsdt)\) and span this space. The theorem now follows easily.

**Remark.** The constant \(e_\ell(\sigma)\) can of course be computed. Applying e.g. [G] Section 7.31, we get, by meromorphic continuation:

\[
e_\ell(\sigma) = a_\ell \int_S C_{\ell}^{n-2}(|x_1|) |x_1|^{-\frac{n}{2} - i\sigma} \, dx
\]

\[
= 2 a_\ell \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \int_0^1 u^{-\frac{n}{2} - i\sigma} (1 - u^2)^{\frac{n-2}{2}} C_{\ell}^{n-2}(u) \, du
\]

\[
= 2^{-2\ell} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}} \frac{\Gamma(n - 2 + \ell)}{\Gamma(n - 2)} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-2}{2} + \ell\right)} \frac{\Gamma\left(\frac{n}{2} - i\sigma + 1\right)}{\Gamma\left(\frac{n}{2} - i\sigma + \ell + 1\right)} \frac{\Gamma\left(\frac{n}{2} - i\sigma + \frac{\ell+1}{2}\right)}{\Gamma\left(\frac{n}{2} - i\sigma + \frac{\ell+1}{2}\right)}.
\]

4. **The structure of the Hilbert algebra \(L^2(\mathcal{X})\)**

We now turn to the full algebra \(L^2(\mathcal{X})\). We again reduce the computations to \(L^2(\mathcal{S} \times \mathcal{S})\). In a similar way as for \(\mathcal{A}\) we get:

**Lemma 4.1.** If \(\phi_o \in V_\ell \otimes V_{m'}, \psi_o \in V_{\ell'} \otimes V_{m'\ell'},\) then

\[
\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } m \neq \ell', \\ \text{in } V_\ell \otimes V_{m'} & \text{if } m = \ell'.
\end{cases}
\]

More precisely we have the following result. Let \((f_j), (g_j), (h_i)\) be orthonormal bases of \(V_\ell, V_m\) and \(V_{m'\ell'}\) respectively, and \(\phi_o(u, v) = f_j(u)\overline{g_j}(v), \psi_o(u, v) = g_j'(u)\overline{h_i}(v)\). Then

\[
\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } j \neq j', \\ d(\sigma) e_m(\sigma) f_j(u)\overline{k_i}(v) & \text{if } j = j'.
\end{cases}
\]
The proof is again straightforward and uses the intertwining relation:
\[ \int_{S} |\langle x, y \rangle|^{-n/2-\sigma} g_{j}(y)dy = e_{m}(\sigma) g_{j}(x). \]

**Theorem 4.2.** The irreducible selfadjoint idempotents of \( L^{2}(\tilde{S} \times \tilde{S}) \) are given by
\[ e_{f}^{\ell}(u, v) = \left\{ d(\sigma) e_{\ell}(\sigma) \right\}^{-1} \cdot f(u) \overline{f}(v) \]
with \( f \in V_{\ell}, \|f\|_{L^{2}(\tilde{S})} = 1 \) and \( \ell \) even. The left ideal generated by \( e_{f}^{\ell} \) is equal to
\( L^{2}(\tilde{S}) \otimes \overline{f} \).

The proof reduces to the application of Lemma (4.1).

**Remarks.**
1. The minimal right ideals are obtained in a similar way.
2. The minimal two-sided ideal generated by \( L^{2}(\tilde{S} \times \tilde{S}) \cdot e_{f}^{\ell} \) is the full algebra \( L^{2}(\tilde{S} \times \tilde{S}) \).
3. The closure of \( \bigoplus_{\ell \in 2 \mathbb{N}} V_{\ell} \otimes \overline{V}_{\ell} \) is an \( H^{*} \)-subalgebra of \( L^{2}(\tilde{S} \times \tilde{S}) \). The minimal left ideals are here \( V_{\ell} \otimes \overline{f} \) \((f \in V_{\ell}, \|f\|_{L^{2}(\tilde{S})} = 1)\); they are generated by the \( e_{f}^{\ell} \) as above. The minimal two-sided ideal generated by \( V_{\ell} \otimes \overline{f} \) is equal to \( V_{\ell} \otimes \overline{V}_{\ell} \).

5. **The case of a general para-Hermitian space**

It is not necessary to assume rank \( X = 1 \) in order to show that \( \mathcal{A} \) is commutative. Theorem 3.2 is also valid mutatis mutandis in the general case since \( (K, K \cap H) \) is a Gelfand pair, and it clearly implies the commutativity of \( \mathcal{A} \).

We shall return to the general construction of the product and the involution in another paper, but we should already mention that the case of a para-Hermitian symmetric space of Hermitian type was studied in [DP]. Results obtained in this direction gave a new interpretation of higher order Rankin-Cohen brackets in terms of branching laws for tensor products of holomorphic discrete series representations.

**References**


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