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H^*-ALGEBRAS AND QUANTIZATION OF PARA-HERMITIAN SPACES

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ABSTRACT. In the present note we describe a family of H^*-algebra structures on the set $L^2(X)$ of square integrable functions on a rank-one para-Hermitian symmetric space X.

INTRODUCTION

Let X be a rank-one para-Hermitian symmetric space. It is well known that X is isomorphic (up to a covering) to the quotient space $SL(n,\mathbb{R})/GL(n-1,\mathbb{R})$; see [KK] for more details. We shall thus assume throughout this note that $X = G/H$, where $G = SL(n,\mathbb{R})$ and $H = GL(n-1,\mathbb{R})$.

The space X allows the definition of a covariant symbolic calculus that generalizes the so-called convolution-first calculus on $\mathbb{R}^2$; see ([DM], [PU], [UU]) for instance. Such a calculus, or quantization map Op_σ, maps a suitable set of functions (or symbols) on X to linear operators acting on the representation space of the maximal degenerate series $\pi_{-\frac{n}{2}+i\sigma}$ of the group G. When applied to $L^2(X)$, this induces a noncommutative algebra structure. On the other hand, the taking of the adjoint of an operator in such a calculus defines an involution on symbols. It turns out that these two data give rise to an H^*-algebra structure on $L^2(X)$.

According to the general theory, ([A], [L], [N]), every H^*-algebra is the direct orthogonal sum of its closed minimal two-sided ideals which are simple H^*-algebras. The main result of this note is the explicit description of such a decomposition for the Hilbert algebra $L^2(X)$ and its commutative subalgebra of $SO(n,\mathbb{R})$-invariants.

In [DP] we have shown that the algebra structure of $L^2(X)$ might be very useful and appropriate for a new interpretation of Rankin-Cohen brackets. Therefore a careful study of this structure is a next step in the process of its understanding.

1. DEFINITIONS AND BASIC FACTS

1.1. H^*-algebras.

Definition 1.1. A set R is called an H^*-algebra (or Hilbert algebra) if

(1) R is a Banach algebra with involution;
(2) R is a Hilbert space;
(3) the norm on the algebra R coincides with the norm on the Hilbert space R;

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(4) for all $x, y, z \in R$ one has $(xy, z) = (y, x^*z)$;
(5) for all $x \in R$ one has $\|x^*\| = \|x\|$;
(6) $xx^* \neq 0$ for $x \neq 0$.

An example of a Hilbert algebra is the set of Hilbert-Schmidt operators $HS(I)$ that one can identify with the set of all matrices $(a_{\alpha\beta})$, where α, β belong to a fixed set of indices I, satisfying the condition $\sum_I |a_{\alpha\beta}|^2 < \infty$.

Theorem 1.2 ([N, p. 331])**.** *Every Hilbert algebra is the direct orthogonal sum of its closed minimal two-sided ideals, which are simple Hilbert algebras.*

Every simple Hilbert algebra is isomorphic to some algebra $HS(I)$ of Hilbert-Schmidt operators.

Definition 1.3 ([L, p. 101])**.** An idempotent $e \in R$ is said to be irreducible if it cannot be expressed as a sum $e = e_1 + e_2$ with e_1, e_2 idempotents that annihilate each other: $e_1e_2 = e_2e_1 = 0$.

Lemma 1.4 ([L, p. 102])**.** *A subset I of a Hilbert algebra R is a minimal left (right) ideal if and only if it is of the form $I = R \cdot e$ ($I = e \cdot R$), where e is an irreducible selfadjoint idempotent. Moreover $e \cdot R \cdot e$ is isomorphic to the set of complex numbers and R is spanned by its minimal left ideals.*

Observe that any minimal left ideal is closed, since it is of the form $R \cdot e$.

Corollary 1.5. *If R is a commutative Hilbert algebra, then any minimal left (or right) ideal is one-dimensional.*

1.2. **An algebra structure on $L^2(X)$.** Let $G = SL(n,\mathbb{R})$, $H = GL(n-1,\mathbb{R})$, $K = SO(n)$ and $M = SO(n-1)$. We consider H as a subgroup of G, consisting of the matrices of the form $\begin{pmatrix} (\det h)^{-1} & 0 \\ 0 & h \end{pmatrix}$ with $h \in GL(n-1,\mathbb{R})$.

Let P^- be the parabolic subgroup of G consisting of $1\times(n-1)$ lower block matrices $P = \begin{pmatrix} a & 0 \\ c & A \end{pmatrix}$, $a \in \mathbb{R}^*$, $c \in \mathbb{R}^{n-1}$ and $A \in GL(n-1,\mathbb{R})$ such that $a \cdot \det A = 1$. Similarly, let P^+ be the group of upper block matrices $P = \begin{pmatrix} a & b \\ 0 & A \end{pmatrix}$, $a \in \mathbb{R}^*$, $b \in \mathbb{R}^{n-1}$ and $A \in GL(n-1,\mathbb{R})$ such that $a \cdot \det A = 1$.

The group G acts on the sphere $S = \{s \in \mathbb{R}^n,\ \|s\|^2 = 1\}$ and acts transitively on the set $\widetilde{S} = S/\sim$, where $s \sim s'$ if and only if $s = \pm s'$, by $g.s = \dfrac{g(s)}{\|g(s)\|}$, where $g(s)$ denotes the linear action of G on $\mathbb{R}^n$. Clearly the stabilizer of the equivalence class of the first basis vector $\widetilde{e_1}$ is the group P^+; thus $\widetilde{S} \simeq G/P^+$. If ds is the usual normalized surface measure on S, then $d(g.s) = \|g(s)\|^{-n} ds$.

For $\mu \in \mathbb{C}$, define the character ω_μ of $P^\pm$ by $\omega_\mu(P) = |a|^\mu$. Consider the induced representations $\pi^\pm_\mu = \mathrm{Ind}^G_{P^\pm}\omega_{\mp\mu}$.

Both π^+_μ and π^-_μ can be realized on $C^\infty(\widetilde{S})$, the space of even smooth functions ϕ on S. This action is given by

$$\pi^+_\mu(g)\phi(s) = \phi(g^{-1}.s) \cdot \|g^{-1}(s)\|^\mu.$$

Let θ be the Cartan involution of G given by $\theta(g) = {}^tg^{-1}$. Then

$$\pi^-_\mu(g)\phi(s) = \phi(\theta(g^{-1}).s) \cdot \|\theta(g^{-1})(s)\|^\mu.$$

Let $(\ ,\ )$ denote the usual inner product on $L^2(S)$: $(\phi,\psi) = \int_S \phi(s)\bar{\psi}(s)ds$. Then this sesquilinear form is invariant with respect to the pairs of representations $(\pi^+_\mu, \pi^+_{-\mu-n})$ and $(\pi^-_\mu, \pi^-_{-\mu-n})$. Therefore the representations $\pi^\pm_\mu$ are unitary for $\operatorname{Re}\mu = -\frac{n}{2}$. Notice that according to [DM2] these representations are irreducible for all nonintegral μ.

The group G acts also on $\widetilde{S}\times\widetilde{S}$ by

$$g(u,v) = (g.u, \theta(g)v). \tag{1}$$

This action is not transitive: the orbit $(\widetilde{S}\times\widetilde{S})^o = G.(\widetilde{e_1},\widetilde{e_1}) = \{(u,v) : \langle u,v\rangle \neq 0\}/\sim$ is dense (here $\langle\ ,\ \rangle$ denotes the canonical inner product on $\mathbb{R}^n$) and is of co-measure zero. Moreover, one has a G-equivariant diffeomorphism $(\widetilde{S}\times\widetilde{S})^o \simeq X$.

The map $f \mapsto f(u,v)|\langle u,v\rangle|^{-\frac{n}{2}+i\sigma}$, with $\sigma\in\mathbb{R}$ is a unitary G-isomorphism between $L^2(X)$ and $\pi^+_{-\frac{n}{2}+i\sigma}\,\widehat{\oplus}_2\,\pi^-_{-\frac{n}{2}+i\sigma}$ acting on $L^2(\widetilde{S}\times\widetilde{S})$. The latter space is provided with the usual inner product.

Define the operator A_μ on $C^\infty(\widetilde{S})$ by the formula

$$A_\mu\phi(s) = \int_S |\langle s,t\rangle|^{-\mu-n}\phi(t)dt.$$

This integral converges absolutely for $\operatorname{Re}\mu < -1$ and can be analytically extended to the whole complex plane as a meromorphic function of μ. It is easily checked that A_μ is an intertwining operator, that is, $A_\mu\pi^\pm_\mu(g) = \pi^\mp_{-\mu-n}(g)A_\mu$.

The operator $A_{-\mu-n}\circ A_\mu$ intertwines the representation $\pi^\pm_\mu$ with itself and is therefore a scalar $c(\mu)\mathrm{Id}$ depending only on μ. It can be computed using K-types.

Let $e(\mu) = \int_S |\langle s,t\rangle|^{-\mu-n}dt$. Then $c(\mu) = e(\mu)e(-\mu-n)$. But on the other hand, $e(\mu) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}}\frac{\Gamma(\frac{-\mu-n+1}{2})}{\Gamma(-\frac{\mu}{2})}$.

One also shows that $A^*_\mu = A_{\bar{\mu}}$, so that, for $\mu = -\frac{n}{2}+i\sigma$ we get (by abuse of notation):

$$c(\sigma) = \frac{\Gamma\left(\frac{n}{2}\right)^2}{\pi}\cdot\frac{\Gamma\left(\frac{-n/2-i\sigma+1}{2}\right)\Gamma\left(\frac{-n/2+i\sigma+1}{2}\right)}{\Gamma\left(\frac{n/2+i\sigma}{2}\right)\Gamma\left(\frac{-n/2-i\sigma}{2}\right)},$$

and moreover $A_{-\frac{n}{2}+i\sigma}\circ A^*_{-\frac{n}{2}+i\sigma} = c(\sigma)\mathrm{Id}$, so that the operator $d(\sigma)A_{-\frac{n}{2}+i\sigma}$, where $d(\sigma) = \frac{\sqrt{\pi}}{\Gamma(\frac{n}{2})}\frac{\Gamma(\frac{n/2+i\sigma}{2})}{\Gamma(\frac{-n/2+i\sigma+1}{2})}$, is a unitary intertwiner between $\pi^-_{-\frac{n}{2}+i\sigma}$ and $\pi^+_{-\frac{n}{2}-i\sigma}$.

We thus get a $\pi^+_{-\frac{n}{2}+i\sigma}\,\widehat{\oplus}_2\,\bar{\pi}^+_{-\frac{n}{2}+i\sigma}$ invariant map from $L^2(X)$ onto $L^2(\widetilde{S}\times\widetilde{S})$ given by

$$f\mapsto d(\sigma)\int_S f(u,w)|\langle u,w\rangle|^{-\frac{n}{2}+i\sigma}|\langle v,w\rangle|^{-\frac{n}{2}-i\sigma}dw =: (T_\sigma f)(u,v),\ \forall\sigma\neq 0.$$

This integral does not converge absolutely; it must be considered as obtained by analytic continuation.

Definition 1.6. A symbolic calculus on X is a linear map $Op_\sigma : L^2(X)\to\mathcal{L}(L^2(\widetilde{S}))$ such that for every $f\in L^2(X)$ the function $(T_\sigma f)(u,v)$ is the kernel of the Hilbert-Schmidt operator $Op_\sigma(f)$ acting on $L^2(\widetilde{S})$.

Definition 1.7. The product $\#_\sigma$ on $L^2(X)$ is defined by

$$Op_\sigma(f\natural_\sigma g) = Op_\sigma(f)\circ Op_\sigma(g),\ \forall f,g\in L^2(X).$$

We thus have

- The product $\sharp_\sigma$ is associative.
- $\|f\sharp_\sigma g\|_2 \leq \|f\|_2 \cdot \|g\|_2$, for all $f, g \in L^2(X)$.
- $Op_\sigma(L_x f) = \pi^+_{-\frac{n}{2}+i\sigma}(x)\, Op_\sigma(f)\, \pi^+_{-\frac{n}{2}+i\sigma}(x^{-1})$, so $L_x(f\sharp_\sigma g) = (L_x f)\sharp_\sigma (L_x g)$, for all $x \in G$, where L_x denotes the left translation by $x \in G$ on $L^2(X)$.

This noncommutative product can be described explicitly:

$$(f\sharp_\sigma g)(u,v) = d(\sigma)\int_S\int_S f(u,x)g(y,v)|[u,y,x,v]|^{-\frac{n}{2}+i\sigma}d\mu(x,y), \tag{2}$$

where $d\mu(x,y) = |\langle x,y\rangle|^{-n}dxdy$ is a G-invariant measure on $\widetilde{S}\times\widetilde{S}$ for the G-action (1), and $[u,y,x,v] = \dfrac{\langle u,x\rangle\langle y,v\rangle}{\langle u,v\rangle\langle x,y\rangle}$.

On the space $L^2(X)$ there exists an (family of) involution $f \to f^*$ given by: $Op_\sigma(f^*) =: Op_\sigma(f)^*$. Notice that the correspondance $f \to Op_\sigma(\bar{f}^*)$ is what one calls in pseudo-differential analysis "anti-standard symbolic calculus". The link between symbols of standard and anti-standard calculus in the setting of the para-Hermitian symmetric space X has been made explicit in [PU, Corollary 1.4]; see also Section 3.

Obviously we have $(f\,\sharp_\sigma g)^* = g^*\sharp_\sigma f^*$ and with the above product and involution, the Hilbert space $L^2(X)$ becomes a Hilbert algebra.

2. The structure of the subalgebra of K-invariant functions in $L^2(X)$

Let $\mathcal{A}$ be the subspace of all K-invariant functions in $L^2(X)$.

Theorem 2.1. *The subset $\mathcal{A}$ is a closed subalgebra of $L^2(X)$ with respect to the product $\sharp_\sigma$.*

This statement clearly follows from the covariance of the symbolic calculus Op_σ, namely: $L_x(f\sharp_\sigma g) = (L_x f)\sharp_\sigma (L_x g)$, for all $x \in G, f, g \in L^2(X)$.

Theorem 2.2. *Let $n > 2$. Then the subalgebra $\mathcal{A}$ is commutative.*

Proof. For a function $f \in L^2(X)$ we set $\check{f}(u,v) = f(v,u)$. The map $f \to \check{f}$ is a linear involution. Indeed,

$$(f\,\sharp_\sigma g)(u,v) = d(\sigma)\int_S\int_S \check{f}(x,u)\check{g}(v,y)|[u,y,x,v]|^{-\frac{n}{2}+i\sigma}d\mu(x,y).$$

Permuting x and y and u and v respectively, we get

$$(f\,\sharp_\sigma g)(v,u) = d(\sigma)\int_S\int_S \check{g}(u,x)\check{f}(y,v)|[v,x,y,u]|^{-\frac{n}{2}+i\sigma}d\mu(x,y).$$

But $|[v,x,y,u]| = |[u,y,x,v]|$; therefore $(f\,\sharp_\sigma g)\check{} = \check{g}\,\sharp_\sigma \check{f}$.

On the other hand, given a couple $(u,v) \in \widetilde{S}\times\widetilde{S}$ there exists an element $k \in K$ such that $k.(u,v) = (v,u)$. Geometrically k can be seen as a rotation of angle $\pi[2\pi]$ around the axis defined by the bisectrix of vectors u and v in the plane they generate. Of course, such a k exists for an arbitrary couple (u,v) only if $n > 2$.

Hence for every $f \in \mathcal{A}$ we have $f = \check{f}$ and therefore $f\,\sharp_\sigma g = g\,\sharp_\sigma f$, for $f, g \in \mathcal{A}$. ∎

3. Irreducible selfadjoint idempotents of $\mathcal{A}$

We begin with a *reduction theorem* for the multiplication and involution in $L^2(X)$.

As usual, we shall identify $L^2(X)$ with $L^2(\widetilde{S}\times\widetilde{S}; |\langle x,y\rangle|^{-n}dxdy)$. If $\phi \in L^2(X)$ we shall write $\phi(u,v) = |\langle u,v\rangle|^{n/2-i\sigma}\,\phi_o(u,v)$. Then $\phi_o \in L^2(\widetilde{S}\times\widetilde{S}; dsdt) = L^2(\widetilde{S}\times\widetilde{S})$, and therefore the map $\phi \to \phi_o$ is an isomorphism.

Theorem 3.1. *Under the isomorphism $\phi \to \phi_o$ the product $\#_\sigma$ translates into*

$$\phi_o \#_\sigma \psi_o\,(u,v) = d(\sigma)\int_S\int_S \phi_o(u,x)\,\psi_o(y,v)\,|\langle x,y\rangle|^{-n/2-i\sigma}dxdy$$

and the involution becomes:

$$\phi_0^*(u,v) = \overline{d(\sigma)}^2\int_S\int_S \bar{\phi}_0(x,y)\,(|\langle x,v\rangle||\langle u,y\rangle|)^{-n/2+i\sigma}\,dxdy.$$

The proof is straightforward. So we have translated the algebra structure of $L^2(X)$ to $L^2(\widetilde{S}\times\widetilde{S})$.

Let ϕ be an irreducible selfadjoint idempotent in $\mathcal{A}$. We shall give an explicit formula for the ϕ_o-component of ϕ.

Consider the decomposition of the space $L^2(\widetilde{S}) = \bigoplus_{\ell\in 2\mathbb{N}} V_\ell$, where V_ℓ is the space of harmonic polynomials on $\mathbb{R}^n$, homogeneous of even degree ℓ.

Then the space $L^2(\widetilde{S}\times\widetilde{S})$ decomposes into a direct sum of tensor products $\bigoplus_{\ell,m\in 2\mathbb{N}} V_\ell \otimes \bar{V}_m$ and consequently

$$L^2_K(\widetilde{S}\times\widetilde{S}) = \bigoplus_{\ell\in 2\mathbb{N}} (V_\ell\otimes\bar{V}_\ell)^K,$$

where the sub(super-)script K means: "the K-invariants in".

Let $\dim V_\ell = d$ and $f_1,\ldots,f_d$ be an orthonormal basis of V_ℓ. Then the function $\theta_\ell(u,v) = \sum_{i=1}^d f_i(u)\bar{f}_i(v)$, that is, the reproducing kernel of V_ℓ, is, up to a constant, the K-invariant element of $V_\ell\otimes\bar{V}_\ell$.

Theorem 3.2. *Let $\phi(u,\,v) = |\langle u,\,v\rangle|^{n/2-i\sigma}\phi_o(u,\,v)$ be an irreducible selfadjoint idempotent in $\mathcal{A}$. Then there exist complex numbers $c(\sigma,\,\ell)$ such that for any $\ell\in 2\mathbb{N}$ one has*

$$\phi_o(u,\,v) = c(\sigma,\,\ell)\,\theta_\ell(u,\,v).$$

For different ℓ and ℓ' the idempotents annihilate each other. Moreover they span $\mathcal{A}$.

Proof. Firstly we shall show that θ_ℓ satisfies the condition

$$\theta_\ell \#_\sigma \theta_\ell = a(\sigma,\,\ell)\,\theta_\ell$$

for some constant $a(\sigma,\,\ell)$. Indeed,

$$\begin{aligned} & d(\sigma)\int_S\int_S \theta_\ell(u,x)\,\theta_\ell(y,v)\,|\langle x,\,y\rangle|^{-\frac{n}{2}-i\sigma}dxdy \\ = \quad & d(\sigma)\,e_\ell(\sigma)\int_S \theta_\ell(u,y)\,\theta_\ell(y,v)\,dy = d(\sigma)\,e_\ell(\sigma)\,\theta_\ell(u,v) \end{aligned}$$

by the intertwining relation (apply $A_{-\frac{n}{2}+i\sigma}$ to $\theta_\ell(.,x)$):

$$\int_S \theta_\ell(u,x)\,|\langle x,\,y\rangle|^{-\frac{n}{2}-i\sigma}dx = e_\ell(\sigma)\,\theta_\ell(u,y)$$

where

$$e_\ell(\sigma) = \int_S \frac{\theta_\ell(e_1,\, x)}{\theta_\ell(e_1,\, e_1)}\, |x_1|^{-\frac{n}{2}-i\sigma}\, dx.$$

Observe that $\frac{\theta_\ell(e_1,\, x)}{\theta_\ell(e_1,\, e_1)}$ is a spherical function on $\widetilde{S}$ with respect to M of the form $a_\ell\, C_\ell^{\frac{n-2}{2}}(|x_1|)$ where $C_\ell^{\frac{n-2}{2}}(u)$ is a Gegenbauer polynomial and

$$a_\ell^{-1} = C_\ell^{\frac{n-2}{2}}(1) = 2^\ell \frac{\Gamma(\frac{n-2}{2}+\ell)}{\Gamma(\frac{n-2}{2})\ell!}.$$

See for instance [V, Chapter IX, §3] . Notice that $\theta_\ell(e_1,\, e_1) = \dim V_\ell = \frac{(n+\ell-1)!}{(n-1)!\ell!}$ $\neq 0$.

The integral defining $e_\ell(\sigma)$ does not converge absolutely, but has to be considered as the meromorphic extension of an analytic function. Poles only occur in half-integer points on the real axis. So we have to restrict (and we do) to $\sigma \neq 0$.

So we have $\theta_\ell \#_\sigma \theta_\ell = d(\sigma)\, e_\ell(\sigma)\, \theta_\ell$ and hence $\varphi_\ell = [d(\sigma)\, e_\ell(\sigma)]^{-1}\, \theta_\ell$ is the ϕ_o-component of an idempotent in $\mathcal{A}$. Furthermore $\theta_\ell \#_\sigma \theta_{\ell'} = 0$ if $\ell \neq \ell'$. Clearly φ_ℓ is selfadjoint, since $|d(\sigma)|^{-2} = |e_\ell(\sigma)|^2$, being equal to the constant $c(\sigma)$ from Section 1.

So the φ_ℓ are mutually orthogonal idempotents in the algebra $L^2_K((\widetilde{S}\times\widetilde{S}); dsdt)$ and span this space. The theorem now follows easily. $\square$

Remark. The constant $e_\ell(\sigma)$ can of course be computed. Applying e.g. [G, Section 7.31], we get, by meromorphic continuation:

$$\begin{aligned} e_\ell(\sigma) &= a_\ell \int_S C_\ell^{\frac{n-2}{2}}(|x_1|)\, |x_1|^{-\frac{n}{2}-i\sigma}\, dx \\ &= 2\, a_\ell \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{\pi}} \int_0^1 u^{-\frac{n}{2}-i\sigma}\, (1-u^2)^{\frac{n-2}{2}}\, C_\ell^{\frac{n-2}{2}}(u)\, du \\ &= 2^{-2\ell} \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}} \cdot \frac{\Gamma(n-2+\ell)}{\Gamma(n-2)} \cdot \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-2}{2}+\ell)} \cdot \frac{\Gamma(-\frac{n}{2}-i\sigma+1)\Gamma(\frac{-\frac{n}{2}-i\sigma-\ell+1}{2})}{\Gamma(-\frac{n}{2}-i\sigma-\ell+1)\Gamma(\frac{\frac{n}{2}-i\sigma+\ell}{2})}. \end{aligned}$$

4. The strucure of the Hilbert algebra $L^2(X)$

We now turn to the full algebra $L^2(X)$. We again reduce the computations to $L^2(\widetilde{S}\times\widetilde{S})$. In a similar way as for $\mathcal{A}$ we get:

Lemma 4.1. *If $\phi_o \in V_\ell \otimes \overline{V}_m$, $\psi_o \in V_{\ell'} \otimes \overline{V}_{m'}$, then*

$$\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } m \neq \ell', \\ \text{in} & V_\ell \otimes \overline{V}_{m'} \text{ if } m = \ell'. \end{cases}$$

More precisely we have the following result. Let (f_i), (g_j), (k_l) be orthonormal bases of V_ℓ, V_m and $V_{m'}$ respectively, and $\phi_o(u,v) = f_i(u)\overline{g}_j(v)$, $\psi_o(u,v) = g_{j'}(u)\overline{k}_l(v)$. Then

$$\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } j \neq j', \\ d(\sigma)\, e_m(\sigma)\, f_i(u)\overline{k}_l(v) & \text{if } j = j'. \end{cases}$$

The proof is again straightforward and uses the intertwining relation:

$$\int_S |\langle x, y\rangle|^{-n/2-i\sigma}\, g_{j'}(y)dy = e_m(\sigma)\, g_{j'}(x).$$

Theorem 4.2. *The irreducible selfadjoint idempotents of $L^2(\widetilde{S}\times\widetilde{S})$ are given by*

$$e_f^\ell(u,v) = \{d(\sigma)\, e_\ell(\sigma)\}^{-1}\cdot f(u)\,\overline{f}(v)$$

with $f\in V_\ell$, $\|f\|_{L^2(\widetilde{S})} = 1$ and ℓ even. The left ideal generated by e_f^ℓ is equal to $L^2(\widetilde{S})\otimes\overline{f}$.

The proof reduces to the application of Lemma (4.1).

Remarks. (1) The minimal right ideals are obtained in a similar way.

(2) The minimal two-sided ideal generated by $L^2(\widetilde{S}\times\widetilde{S})\cdot e_f^\ell$ is the full algebra $L^2(\widetilde{S}\times\widetilde{S})$.

(3) The closure of $\bigoplus_{\ell\in 2\mathbb{N}} V_\ell\otimes\overline{V}_\ell$ is an H^*-subalgebra of $L^2(\widetilde{S}\times\widetilde{S})$. The minimal left ideals are here $V_\ell\otimes\overline{f}$ ($f\in V_\ell$, $\|f\|_{L^2(\widetilde{S})}=1$); they are generated by the e_f^ℓ as above. The minimal two-sided ideal generated by $V_\ell\otimes\overline{f}$ is equal to $V_\ell\otimes\overline{V}_\ell$.

5. The case of a general para-Hermitian space

It is not necessary to assume rank $X=1$ in order to show that $\mathcal{A}$ is commutative. Theorem 3.2 is also valid mutatis mutandis in the general case since $(K, K\cap H)$ is a Gelfand pair, and it clearly implies the commutativity of $\mathcal{A}$.

We shall return to the general construction of the product and the involution in another paper, but we should already mention that the case of a para-Hermitian symmetric space of Hermitian type was studied in [DP]. Results obtained in this direction gave a new interpretation of higher order Rankin-Cohen brackets in terms of branching laws for tensor products of holomorphic discrete series representations.

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