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NON-VANISHING OF THE TWISTED COHOMOLOGY ON THE COMPLEMENT OF HYPERSURFACES

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Abstract. Generically, the cohomology with coefficients in a local system of rank one on the complement in $\mathbb{P}^n$ of the union of a finite number of hypersurfaces vanishes except in the highest dimension. We study the non-generic case, in which the cohomology in other dimensions does not vanish. When the hypersurfaces are hyperplanes, many examples of this kind are known. In this paper, we consider the case in which the hypersurfaces need not be hyperplanes. We prove that the hypersurfaces given by some particular linear systems have non-vanishing local system cohomologies.

1. Introduction

Let $V_1, \dots, V_m$ be hypersurfaces in the complex projective space $\mathbb{P}^n$ of dimension n and let $\mathcal{L}$ be a complex local system of rank one over the complement $M = \mathbb{P}^n \setminus \bigcup_{i=1}^m V_i$. Call $H^k(M, \mathcal{L})$ the kth twisted cohomology group of M. In [A, KN, Ch, Di] it is shown that if $V_1, \dots, V_m$ and $\mathcal{L}$ are generic,

$$H^k(M, \mathcal{L}) = 0 \qquad \text{for } k \neq n.$$

In particular, vanishing theorems for the case of hyperplanes were found in [Ko, Yu, CDO, Ka2] (cf. [ESV, STV]). Recently, arrangements of hyperplanes with non-vanishing twisted cohomology were studied and many examples were found (cf. [CS, Fa, LY, Ka3]). The results in [Yu2] imply that in $\mathbb{P}^2$, most examples of this kind consist of special elements of pencils. In this paper we generarize this result to hypersurfaces of arbitrary dimension. We shall show that there are hypersurfaces which are supports of divisors in some linear system with the property that their complements have non-vanishing twisted cohomologies in other than the top dimension. See [Li2] for background and a discussion of related topics.

Let Ω_M denote the sheaf of germs of holomorphic forms on M and let $\mathcal{O}_M = \Omega_M^0$. Let $D_1, \dots, D_m$ be effective divisors on $\mathbb{P}^n$ such that the support of D_i is V_i and D_i and D_j are linearly equivalent for $i \neq j$. In this case, the D_i's have the same degree (see [H]). Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a complex weight with $\sum_{i=1}^m \lambda_i = 0$. For $i \neq j$, $D_i - D_j$ is the divisor of some rational function f_{ij}. Fix j and define the

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global one-form $\omega_\lambda = \sum_{i\neq j} \lambda_i \,\mathrm{d}\log f_{ij}$. It is independent of the choice of j and we can denote

$$\omega_\lambda = \sum_{i=1}^{m} \lambda_i \,\mathrm{d}\log D_i \in \Gamma(M, \Omega_M^1), \qquad \sum_{i=1}^{m} \lambda_i = 0.$$

Define the flat connection $\nabla_\lambda = d + \omega_\lambda \bigwedge : \mathcal{O}_M \to \Omega_M^1$ and let the local system $\mathcal{L}_\lambda$ be the kernel of ∇_λ. Since M is a Stein manifold, we have

$$H^k(M, \mathcal{L}_\lambda) \simeq H^k(\Gamma(M, \Omega_M), \nabla_\lambda)$$

and $H^k(M, \mathcal{L}_\lambda) = 0$ for $k > n$ (cf. [De]). The main result of this paper is the following:

Theorem 1. *Let $1 < n < s \leq m$. Let $D_1, \ldots, D_m$ be effective divisors on $\mathbb{P}^n$ with the same degree and let M be the complement of their supports. Suppose*

(A1) *$D_1, \ldots, D_s$ are elements of some linear system Λ on $\mathbb{P}^n$ with dimension $n-1$.*
(A2) *There exists a base point P of Λ such that $D_1, \ldots, D_n$ has local normal crossings at P and none of the D_i pass through P for $i = s+1, \ldots, m$.*
(A3) *$D_1, \ldots, D_s$ are in general position as points in $\Lambda = \mathbb{P}^{n-1}$, namely, no n of them lie on a hyperplane.*

Let $\lambda \notin \mathbb{Z}^m$ be a non-trivial weight such that $\sum_{i=1}^{s} \lambda_i = 0$ and $\lambda_i = 0$ for $i = s+1, \ldots, m$. Then we have

$$\dim H^{n-1}(M, \mathcal{L}_\lambda) \geq \binom{s-2}{n-1}.$$

Moreover, if $s < m$, then we have

$$\dim H^{n}(M, \mathcal{L}_\lambda) \geq \binom{s-2}{n-1}.$$

Remark. The support of $\sum_{i=1}^{s} D_i$ is of great interest when considering the non-vanishing of twisted cohomology. Condition (A2) implies that, locally at P, the set $\{D_{1,P}, \ldots, D_{s,P}\}$ becomes a generic arrangement of s hyperplanes.

Remark. If the support of $\sum_{i=1}^{m} D_i$ contains a hyperplane H, we can consider M as the complement of affine hypersurfaces in $\mathbb{C}^n = \mathbb{P}^n \setminus H$. The case of affine hypersurfaces case was considered in [KN, Ki] (cf. Section 5).

Remark. Let $D_1, \ldots, D_m$ be divisors of degree d defined only by hyperplanes. Let $\mathcal{A}$ be the set of irreducible components of the support of $\sum_{i=1}^{m} D_i$. Then $\mathcal{A}$ is an arrangement of hyperplanes in $\mathbb{P}^n$ and $M = \mathbb{P}^n \setminus \bigcup_{H\in\mathcal{A}} H$.

If $D_1, \ldots, D_m$ are divisors of degree one, then they are defined by hyperplanes. In this case, $\{D_1, \ldots, D_s\}$ becomes a generic arrangement of hyperplanes with center $P = \bigcap_{i=1}^{s} D_i$, and $D_{s+1}, \ldots, D_m$ are hyperplanes not passing through P. Theorem 1 in this case is well-known (cf. [Fa]).

In order to prove Theorem 1, we shall construct explicit non-trivial twisted cohomology classes (Section 3) using a natural generalization of the Arnold-Orlik-Solomon relation (Section 2). Condition (A3) is not essential, and in Section 4 we generalize the theorem. In Section 6 we give examples.

2. Preliminary: Rational forms

Let V be a vector space of dimension ℓ over a field K of characteristic zero. Let $S = K[V]$ be the symmetric algebra of V^* and let $F = K(V)$ be the quotient field of S. We consider S as the polynomial algebra and F as the field of rational functions on V. Let $\Omega(V) = \bigoplus_{p=0}^{\ell} \Omega^p(V)$ be the exterior algebra of the F-vector space $F \otimes V^*$ and let d be the usual differential. When we choose a basis $x_1, \dots, x_\ell$ of V^*, we have $S = K[x_1, \dots, x_\ell]$, $F = K(x_1, \dots, x_\ell)$,

$$\mathrm{d}\, f = \sum_{i=1}^{\ell} \frac{\partial f}{\partial x_i} \otimes x_i = \sum_{i=1}^{\ell} \frac{\partial f}{\partial x_i} \,\mathrm{d}\, x_i \quad \text{for } f \in F,$$

$\Omega^0(V) = F, \Omega^1(V) = F \otimes V^* = F\,\mathrm{d}\,x_1 \oplus \cdots \oplus F\,\mathrm{d}\,x_\ell$, and, $\Omega^p(V) = \bigoplus_{i_1 < \cdots < i_p} F\,\mathrm{d}\,x_{i_1} \wedge \cdots \wedge \mathrm{d}\,x_{i_p}$. For $p \geq 2$ and $\omega_1, \dots, \omega_p \in \Omega^1(V)$, define

$$\Delta[\omega_1 : \cdots : \omega_p] := \sum_{k=1}^{p} (-1)^{k-1} \omega_1 \wedge \cdots \wedge \widehat{\omega_k} \wedge \cdots \wedge \omega_p.$$

Lemma 2. *Let $p \geq 2$ and $\omega_1, \dots, \omega_p \in \Omega^1(V)$.*

(1) *For a permutation σ of $\{1, \dots, p\}$, we have*

$$\Delta\left[\omega_{\sigma(1)} : \cdots : \omega_{\sigma(p)}\right] = \operatorname{sign}(\sigma) \Delta[\omega_1 : \cdots : \omega_p].$$

(2) *If $2 \leq j \leq p-2$, then*

$$\begin{aligned}\Delta[\omega_1 : \cdots : \omega_p] =& \Delta[\omega_1 : \cdots : \omega_j] \wedge \omega_{j+1} \wedge \cdots \wedge \omega_p \\ &+ (-1)^j \omega_1 \wedge \cdots \wedge \omega_j \wedge \Delta[\omega_{j+1} : \cdots : \omega_p].\end{aligned}$$

(3) $\Delta[\omega_1 : \cdots : \omega_p] = -(\omega_1 - \omega_2) \wedge \Delta[\omega_2 : \cdots : \omega_p]$.
(4) $\Delta[\omega_1 : \cdots : \omega_p] = (-1)^{p-1}(\omega_1 - \omega_2) \wedge (\omega_2 - \omega_3) \wedge \cdots \wedge (\omega_{p-1} - \omega_p)$.
(5) $\Delta[\omega_1 : \cdots : \omega_p] = (\omega_2 - \omega_1) \wedge (\omega_3 - \omega_1) \wedge \cdots \wedge (\omega_p - \omega_1)$.
(6) $\omega_1 \wedge \Delta[\omega_1 : \cdots : \omega_p] = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_p$.

By (6), if $\Delta[\omega_1 : \cdots : \omega_p] = 0$, then $\omega_1, \dots, \omega_p$ are F-linearly dependent. However, the converse is not true in general. The rational function $f \in F$ is said to be *homogeneous of degree d*, if $f = g_1/g_2$ with homogeneous polynomials g_1, g_2 and $d = \deg g_1 - \deg g_2$.

Lemma 3. *Assume $p \geq 2$ and $f_1, \dots, f_p \in F \setminus K$ are homogeneous of the same degree. Then $\Delta[\mathrm{d}\, f_1/f_1 : \cdots : \mathrm{d}\, f_p/f_p] = 0$, if and only if, $\mathrm{d}\, f_1/f_1 \wedge \cdots \wedge \mathrm{d}\, f_p/f_p = 0$.*

Proof. Let $\omega_1, \dots, \omega_p \in \Omega^1(V)$.

From Lemma 2 we obtain:

(1) $\Delta[\omega_1 : \cdots : \omega_p] = 0$ if and only if there exists $(g_1, \dots, g_p) \in F^p \setminus \{(0, \dots, 0)\}$ such that $g_1 + \cdots + g_p = 0$ and $g_1\omega_1 + \cdots + g_p\omega_p = 0$.
(2) Take $\omega_i = \mathrm{d}\, f_i/f_i$ and $g_i = h_i f_i$. For $p \geq 2$ and $f_1, \dots, f_p \in F \setminus K$, we have $\Delta[\mathrm{d}\, f_1/f_1 : \cdots : \mathrm{d}\, f_p/f_p] = 0$ if and only if there exists $(h_1, \dots, h_p) \in F^p \setminus \{(0, \dots, 0)\}$ such that $h_1 f_1 + \cdots + h_p f_p = 0$ and $h_1\,\mathrm{d}\, f_1 + \cdots + h_p\,\mathrm{d}\, f_p = 0$.

When the f_i's are homogeneous of the same degree, we can use the Euler derivation to conclude that $h_1\,\mathrm{d}\, f_1 + \cdots + h_p\,\mathrm{d}\, f_p = 0$ implies $h_1 f_1 + \cdots + h_p f_p = 0$. □

In particular, if $c_1 f_1 + \cdots + c_p f_p = 0$ for some $(c_1, \dots, c_p) \in K^p \setminus \{(0, \dots, 0)\}$, then $\Delta[\mathrm{d}\, f_1/f_1 : \cdots : \mathrm{d}\, f_p/f_p] = 0$. If $f_1^{n_1} \cdots f_p^{n_p} = 1$ for some non-zero integers $n_1, \dots, n_p$, then $\Delta[\mathrm{d}\, f_1/f_1 : \cdots : \mathrm{d}\, f_p/f_p] = 0$.

Remark. The derivation Δ is a natural generalization of the linear derivation on Orlik-Solomon Algebras ([OT]) in the degree one case.

3. Proofs

Proof of Theorem 1. Let $[x_0 : x_1 : \ldots : x_n]$ be homogeneous coordinates of $\mathbb{P}^n$. We can assume D_i is given by a homogeneous polynomial $F_i(x)$ of degree d. So we can write $\omega_\lambda = \sum_{i=1}^m \lambda_i \mathrm{d}\, F_i/F_i$. It is easy to check that $\mathrm{d}\, F_i/F_i - \mathrm{d}\, F_j/F_j$ and ω_λ are global forms. For $1 \le i_1, \ldots, i_p \le m$, define a holomorphic form on M by

$$\eta[i_1, \ldots, i_p] := \Delta\left[\frac{\mathrm{d}\, F_{i_1}}{F_{i_1}} : \cdots : \frac{\mathrm{d}\, F_{i_p}}{F_{i_p}}\right].$$

According to Lemma 2 (4), if it is not zero, then $\eta[i_1, \ldots, i_p]$ is a global $(p-1)$-form. By (A1) and (A3), $F_1, \ldots, F_n$ becomes a basis of the vector subspace of $\Gamma(\mathbb{P}^n, \mathcal{O}(d))$ defining the linear system Λ. So we can write $F_j = a_{1j}F_1 + \cdots + a_{nj}F_n$ for some constants a_{ij}. Define the $n \times s$-matrix $A = (a_{ij})$. By (A3), any $n \times n$-minor of A is not zero. Due to Lemma 3, we have $\eta[i_1, \ldots, i_{n+1}] = 0$ for $1 \le i_1 < \cdots < i_{n+1} \le s$. Because we can write $\omega_\lambda = \sum_{i \ne i_1} \lambda_i (\mathrm{d}\, F_i/F_i - \mathrm{d}\, F_{i_1}/F_{i_1})$, using Lemma 3 we have $\omega_\lambda \wedge \eta[i_1, \ldots, i_n] = 0$. Hence $\nabla_\lambda(\eta[i_1, \ldots, i_n]) = 0$ for $1 \le i_1 < \cdots < i_n \le s$. We note that (A2) and (A3) imply that for $1 \le i_1 < \cdots < i_n \le s$, we have $\mathrm{d}\, F_{i_1}/F_{i_1} \wedge \cdots \wedge \mathrm{d}\, F_{i_n}/F_{i_n} \ne 0$, and by Lemma 3, $\eta[i_1, \ldots, i_n] \ne 0$. Thus $\eta[i_1, \ldots, i_n]$ is a ∇_λ-closed $(n-1)$-form for $1 \le i_1 < \cdots < i_n \le s$.

By (A2), we may take a local neighborhood U and coordinates $x = (x_1, \ldots, x_n)$ at P such that D_i is defined by $x_i = 0$ for $i = 1, \ldots, n$. Let $\alpha_j(x) = a_{1j}x_1 + \cdots + a_{nj}x_n$ and $H_j = \{\alpha_j(x) = 0\}$. So we get the central arrangement $\mathcal{A} = \{H_j\}_{1 \le j \le s}$ of hyperplanes in $\mathbb{C}^n \simeq U$ ($\bigcap_{i=1}^s H_i$ is the origin). Let $M(\mathcal{A})$ denote the complement of the arrangement $\mathcal{A}$. Then we have $H^k(U \cap M, \mathcal{L}_\lambda|_{U \cap M}) = H^k(M(\mathcal{A}), \tilde{\mathcal{L}}_\lambda)$, where $\tilde{\mathcal{L}}_\lambda$ is the rank one local system on $M(\mathcal{A})$ whose monodromy around the hyperplane H_j is $\exp(-2\pi\sqrt{-1}\lambda_j)$. Since λ is non-trivial and $\sum_{i=1}^s \lambda_i = 0$, without loss of generality we may assume that λ_1 and λ_s are not integers. Now, choosing $H_1 \in \mathcal{A}$, we get the decone $\mathbf{d}\,\mathcal{A}$ (see [OT]), which is an arrangement of $s-1$ affine hyperplanes in $\mathbb{C}^{n-1} \simeq H_1$. Note that $M(\mathcal{A}) \simeq M(\mathbf{d}\,\mathcal{A}) \times \mathbb{C}^*$ by the restriction of the Hopf bundle. Since any $n \times n$-minor of A is not zero, $\mathcal{A}$ is generic and $\mathbf{d}\,\mathcal{A}$ is in general position ([OT]). On the other hand, for the integer weight $k \in \mathbb{Z}^n$ with $\sum_{i=1}^s k_i = 0$, we know that the local system $\tilde{\mathcal{L}}_\lambda$ is equivalent to the local system $\tilde{\mathcal{L}}_{\lambda+k}$ associated to the integer shift weight $\lambda + k$ (see [OT2]). By shifting a weight if necessary, we can assume that $\lambda \notin (\mathbb{Z} \setminus \{0\})^n$. Since $H^k(M(\mathcal{A}), \tilde{\mathcal{L}}_\lambda) \simeq H^k(M(\mathbf{d}\,\mathcal{A}), \tilde{\mathcal{L}}_\lambda) \oplus H^{k-1}(M(\mathbf{d}\,\mathcal{A}), \tilde{\mathcal{L}}_\lambda)$ (cf. [Fa]), in this case the following is known.

Lemma 4 (cf. [Ha, KN, Ki, Ka])**.** *Let $\mathcal{A} = \{H_j = \{\alpha_j = 0\} : 1 \le j \le s\}$ be a generic arrangement of hyperplanes in $\mathbb{C}^n$ and let $M(\mathcal{A})$ be its complement. For a complex weight $\lambda = (\lambda_1, \ldots, \lambda_s)$ such that $\lambda_1 \notin \mathbb{Z}$, $\lambda_s \notin \mathbb{Z}$ and $\sum_{i=1}^s \lambda_i = 0$, we have*

(1) *$H^k(M(\mathcal{A}), \tilde{\mathcal{L}}_\lambda) = 0$ for $k \ne n, n-1$,*
(2) *$H^n(M(\mathcal{A}), \tilde{\mathcal{L}}_\lambda) \simeq H^{n-1}(M(\mathcal{A}), \tilde{\mathcal{L}}_\lambda)$ and $\dim H^n = \dim H^{n-1} = \binom{s-2}{n-1}$,*
(3) *$\{e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_{n-1}} : 1 < i_1 < \cdots < i_{n-1} < s\}$ is a basis of H^n,*
(4) *$\{\Delta[e_1 : e_{i_1} : \cdots : e_{i_{n-1}}] : 1 < i_1 < \cdots < i_{n-1} < s\}$ is a basis of H^{n-1},*

where $e_j = \mathrm{d}\,\alpha_j/\alpha_j$ and $\tilde{\mathcal{L}}_\lambda$ is the rank one local system on $M(\mathcal{A})$ whose monodromy around the hyperplane H_j is $\exp(-2\pi\sqrt{-1}\lambda_j)$.

Note that $H^k(M(\mathcal{A}), \tilde{\mathcal{L}}_\lambda)$ is isomorphic to the twisted de Rham cohomology defined by the one form $e_\lambda = \sum_{j=1}^s \lambda_j e_j$ (see [OT2]) and that $\omega_\lambda|_U = e_\lambda$ and $\eta[i_1, \dots, i_n]|_U = \Delta[e_{i_1} : \cdots : e_{i_n}]$. Now suppose that there exists a global $(n-2)$-form α such that $\eta[i_1, \dots, i_n] = \nabla_\lambda(\alpha)$. Then restricting it to U, we have $\Delta[e_{i_1} : \cdots : e_{i_n}] = \tilde{\nabla}_\lambda(\alpha|_U)$ where $\tilde{\nabla}_\lambda = d + e_\lambda \bigwedge$. However, by the above lemma, this is a contradiction. Therefore $\eta[i_1, \dots, i_n]$ defines a non-vanishing class of degree $n-1$ for $1 \le i_1 < \cdots < i_n \le s$. In a similar fashion, we obtain $\{\eta[1, i_1, \dots, i_{n-1}] : 1 < i_1 < \cdots < i_{n-1} < s\}$ as a linearly independent set of elements of $H^{n-1}(M, \mathcal{L}_\lambda)$.

Assume $s < m$ and fix m. Take $\eta[m, i_1, \dots, i_n]$ for $1 \le i_1 < \cdots < i_n \le s$. It is easy to see that $\eta[m, i_1, \dots, i_n]|_U = e_{i_1} \wedge \cdots \wedge e_{i_n}$. Therefore, $\eta[m, i_1, \dots, i_n]$ defines a non-vanishing class of degree n. Similary, using the above lemma, we have $\{\eta[m, 1, i_1, \dots, i_{n-1}] : 1 < i_1 < \cdots < i_{n-1} < s\}$ as a linearly independent set of elements of $H^n(M, \mathcal{L}_\lambda)$. This completes the proof. □

Corollary 5. *Under the assumptions of Theorem 1, suppose $s = m$ and that $D_1 + \cdots + D_s$ has an irreducible component not passing through fixed P. Namely, at least one of $D_1, \dots, D_s$ is reduced and it has an irreducible component not passing through P. For example, $D_1, \dots, D_s$ are divisors of degree $d > 1$ defined only by hyperplanes such that the arrangement of their hyperplanes is not central. Then we have*

$$\dim H^{n-1}(M, \mathcal{L}_\lambda) \ge \binom{s-2}{n-1} \text{ and } \dim H^n(M, \mathcal{L}_\lambda) \ge \binom{s-2}{n-1}.$$

Proof. We need to prove the latter inequality. Let D' be one of irreducible components of $D_1 + \cdots + D_s$ not passing through P with degree $d' < d$. We can modify the last part of the proof of Theorem 1 as follows. Let F' be a defining homogeneous polynomial of D'. The argument used in the proof of Theorem 1 shows that

$$\Delta\left[(d/d')\frac{\mathrm{d}\,F'}{F'} : \frac{\mathrm{d}\,F_1}{F_1} : \frac{\mathrm{d}\,F_{i_1}}{F_{i_1}} : \cdots : \frac{\mathrm{d}\,F_{i_{n-1}}}{F_{i_{n-1}}}\right] \ : \ 1 < i_1 < \cdots < i_{n-1} < s$$

is independent in $H^n(M, \mathcal{L}_\lambda)$. □

If a weight λ is trivial, then $H^k(M, \mathcal{L}_\lambda)$ is isomorphic to the usual de Rham cohomology $H^k(M)$ on M.

Corollary 6. *Under the assumption of Theorem 1, we have*

$$\dim H^k(M) \ge \binom{s-1}{k} \quad \text{for } 1 \le k \le n-1.$$

Moreover, if $s < m$, then we have

$$\dim H^n(M) \ge \binom{s-1}{n-1} \text{ and } \dim H^k(M) \ge \binom{s}{k} \text{ for } 1 \le k \le n-1.$$

Proof. In the proof of Theorem 1, since $\mathcal{A}$ is the generic arrangement of s hyperplanes in $\mathbb{C}^n$ and its decone $\mathbf{d}\,\mathcal{A}$ is in general position, the following is known ([OT]):

(1) $\dim H^k(M(\mathcal{A})) = \binom{s}{k}$ for $1 \le k \le n$ and $\dim H^k(M(\mathbf{d}\,\mathcal{A})) = \binom{s-1}{k}$ for $1 \le k \le n-1$,
(2) $H^k(M(\mathcal{A})) \simeq H^k(M(\mathbf{d}\,\mathcal{A})) \oplus H^{k-1}(M(\mathbf{d}\,\mathcal{A}))$ for $1 \le k \le n-1$, and $H^n(M(\mathcal{A})) \simeq H^{n-1}(M(\mathbf{d}\,\mathcal{A}))$,
(3) $\{e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_{n-1}} : 1 < i_1 < \cdots < i_{n-1} \le s\}$ is a basis of $H^n(M(\mathcal{A}))$,

(4) $\{\Delta[e_1 : e_{i_1} : \cdots : e_{i_k}] : 1 < i_1 < \cdots < i_k \leq s\} \cup \{e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} : 1 < i_1 < \cdots < i_{k-1} \leq s\}$ is a basis of $H^k(M(\mathcal{A}))$ for $1 \leq k \leq n-1$.

Since $\eta[i_1, \ldots, i_k]$ is d-closed, the same argument proves this corollary. □

4. Generalization

Let $\mathcal{A}$ be an arrangement of (affine or projective) hyperplanes. The *intersection set* $L(\mathcal{A})$ of $\mathcal{A}$ is the set of non-empty intersections of elements of $\mathcal{A}$. For $X \in L(\mathcal{A})$, define a central arrangement $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}$. Let $\mathcal{C}$ be a central arrangement with center $\bigcap_{H \in \mathcal{C}} H \neq \emptyset$. We call $\mathcal{C}$ *decomposable* if there exist non-empty subarrangements $\mathcal{C}_1$ and $\mathcal{C}_2$ so that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a disjoint union, and after a linear coordinate change, the defining polynomials for $\mathcal{C}_1$ and $\mathcal{C}_2$ have no common variables. Define $\mathrm{D}(\mathcal{A}) = \{X \in L(\mathcal{A}) : \mathcal{A}_X \text{ is not decomposable}\}$. For a complex weight λ of $\mathcal{A}$ and $X \in L(\mathcal{A})$, denote $\lambda_X = \sum_{H \in \mathcal{A}_X} \lambda_H$. The construction of a basis for the twisted cohomology for arrangements given in [FT] (cf. [OT2]), together with the arguments used in proof of Theorem 1, shows the following.

Theorem 7. *Let $1 < n < s \leq m$. Let $D_1, \ldots, D_m$ be effective divisors on $\mathbb{P}^n$ with the same degree and let M be the complement of their supports. Suppose (A1) and (A2) hold. Let $\mathcal{A}$ be the arrangement of hyperplanes in the dual projective space $\Lambda^* = (\mathbb{P}^{n-1})^* \simeq \mathbb{P}^{n-1}$ defined by $D_1, \ldots, D_s$. Let λ be a non-trivial weight such that $\sum_{i=1}^{s} \lambda_i = 0$ and $\lambda_i = 0$ for $i = s+1, \ldots, m$. If $\lambda_X \notin \mathbb{Z}_{\geq 0}$ for every $X \in \mathrm{D}(\mathcal{A})$, then we have*

$$\dim H^{n-1}(M, \mathcal{L}_\lambda) \geq \beta,$$

and moreover, if $s < m$, then we have

$$\dim H^{n}(M, \mathcal{L}_\lambda) \geq \beta,$$

where β is the Euler characteristic $\chi(M(\mathcal{A}))$ of $M(\mathcal{A}) = \mathbb{P}^{n-1} \setminus \bigcup_{H \in \mathcal{A}} H$.

Remark. Note that β is known as the beta invariant of the underlying matroid of $\mathcal{A}$. If $\mathcal{A}$ is defined over real, then β is the number of bounded chambers in $\mathbb{R}^{n-1} = \mathbb{P}^{n-1}(\mathbb{R}) \setminus H$ for $H \in \mathcal{A}$ (see [STV, OT2]).

Remark. One has $\lambda_H \notin \mathbb{Z}_{\geq 0}$ for all hyperplanes in $\mathcal{A}$. We can generalize this theorem to the case in which there is a hyperplane $H \in \mathcal{A}$ with $\lambda_H = 0$ by using [Ka2].

5. Affine case

Let $V_1^a, \ldots, V_m^a$ be hypersurfaces in the complex affine space $\mathbb{C}^n$ with coordinates $u = (u_1, \ldots, u_n)$. Write $V^a = \bigcup_{i=1}^m V_i^a$ and $M = \mathbb{C}^n \setminus V^a$. Assume that V_j^a is defined by a polynomial $f_j(u)$ of degree d_i. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a weight and let $\omega_\lambda^a = \sum_{i=1}^m \lambda_i \mathrm{d} f_i / f_i$. Then we obtain the twisted de Rham complex $(\Omega(*V^a), \nabla_\lambda^a)$ where $\Omega(*V^a)$ is the space of rational forms with poles along V^a and $\nabla_\lambda^a = d + \omega_\lambda^a \bigwedge$. The Grothendiek-Deligne comparison theorem ([De]) asserts that

$$H^k(M, \mathcal{L}_\lambda^a) \simeq H^k(\Omega(*V^a), \nabla_\lambda^a)$$

where $\mathcal{L}_\lambda^a$ is the rank one local system defined by the flat connection ∇_λ^a (cf. [KN, Ki]).

View $\mathbb{P}^n$ as a compactification of $\mathbb{C}^n$ with hyperplane H_∞ at infinity. Let $[x_0 : \ldots : x_n]$ denote homogeneous coordinates with $H_\infty = \{x_0 = 0\}$. Define the homogeneous polynomial $F_j(x) = x_0^d f_j(x_1/x_0, \ldots, x_n/x_0)$ of degree $d = \max(d_1, \ldots, d_m)$.

Then $F_j(x)$ determines the divisor D_j of degree d. If $d = d_1 = \cdots = d_m$, then the support of $\sum_{i=1}^m D_i$ does not contain H_∞; otherwise it contains H_∞. Note that the weight of H_∞ is given by $-\sum_{i=1}^m \lambda_i d_j$. If $d = d_1 = \cdots = d_m$ and $\sum_{i=1}^m \lambda_i = 0$, then the weight of H_∞ is zero.

Corollary 8. *Under the assumptions of Theorem* 1*, if a point P in (A2) is not on H_∞, then we have*

$$\dim H^{n-1}(M^a, \mathcal{L}_\lambda^a) \geq \binom{s-2}{n-1}$$

for a non-trivial weight λ with $\sum_{i=1}^s \lambda_i = 0$ and $\lambda_i = 0$ for $i = s+1, \ldots, m$.

6. Examples

6.1. $n = 2$ **and** $s = 3$**.** Let D_1, D_2 and D_3 be irreducible prime divisors on $\mathbb{P}^2$ defined by $F_1 = x_0^2 + x_1^2 - 2x_2^2$, $F_2 = x_0^2 + 2x_1^2 - 3x_2^2$ and $F_3 = 2x_0^2 + x_1^2 - 3x_2^2$, respectively. They are three generic elements of the pencil of conic curves $F_{[a:b]} = a(x_0^2 - x_2^2) + b(x_1^2 - x_2^2)$ and they transversally intersect each other at each of the four intersection points. Let $M = \mathbb{P}^2 \setminus \bigcup_{i=1}^3 D_i$ and $M^a = \mathbb{C}^2 \setminus \bigcup_{i=1}^3 (D_i \cap \mathbb{C}^2)$ where $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{x_2 = 0\}$. We have $H^1(M, \mathcal{L}_\lambda) \neq 0$ and $H^1(M^a, \mathcal{L}_\lambda) \neq 0$ for a non-trivial weight λ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$. In the following cases, we can also arrive at this conclusion:

(1) $F_1 = x_0^2 + x_1^2 - 2x_2^2$, $F_2 = x_0^2 + 2x_1^2 - 3x_2^2$ and $F_3' = x_0^2 - x_1^2$ (two conics and a set of 2-lines).
(2) $F_1 = x_0^2 + x_1^2 - 2x_2^2$, $F_2' = x_2^2 - x_0^2$ and $F_3' = x_0^2 - x_1^2$ (one conic and two sets of 2-lines).
(3) $F_1' = x_1^2 - x_2^2$, $F_2' = x_2^2 - x_0^2$ and $F_3' = x_0^2 - x_1^2$ (three sets of 2-lines).

In the last case, one has an arrangement of 6 lines as in the Ceva Theorem. In a similar fashion, we get the following. In Pascal's theorem, define divisors D_1, D_2 and D_3 with degree three by one conic and one pascal line, three lines, and another three lines, respectively. The resulting complement has a non-vanishing twisted cohomology. Needless to say, the same is true for the complement of 9 lines in Pappus's theorem ([Fa]). In the degree four case, there are two different arrangements of 12 lines in the Kirkman Theorem and the Steiner Theorem ([Ka3]). Those arrangements are 3-nets, whose combinatorial structures are matroids associated to Latin squares ([LY, Yu2, Ka3]).

On the other hand, the B_3-arrangement is an example of the case that D_i's are not prime. Let D_1, D_2 and D_3 be divisors defined by $x_2^2(x_0^2 - x_1^2)$, $x_1^2(x_0^2 - x_2^2)$ and $x_0^2(x_1^2 - x_2^2)$, respectively. Their divisors can be written by $D_1 = 2H_1 + H_2 + H_3$, $D_2 = 2H_4 + H_5 + H_6$ and $D_3 = 2H_7 + H_8 + H_9$, where H_i's are hyperplanes. The arrangement $\mathcal{A} = \{H_1, \ldots, H_9\}$ is called the B_3-arrangement. Note that a weight λ induces the weight $(2\lambda_1, \lambda_1, \lambda_1, 2\lambda_2, \lambda_2, \lambda_2, 2\lambda_3, \lambda_3, \lambda_3)$ of $\mathcal{A}$ (cf. [Fa, Ka3]).

Consider the divisors defined by $(x_0 - x_2)(x_0 + x_2)$, $x_2(x_2 - x_1)$ and $x_0^2 - x_1x_2$. They are tangent at $[0 : 1 : 0]$. However, by looking locally at $[1 : 1 : 1]$ and $[-1 : 1 : 1]$, condition (A2) holds and thus the complement of the union of these divisors has a non-vanishing twisted cohomology.

We can make an example with a singular curve. Let $F_1 = x_2^3 - x_1^2x_0$, $F_2 = x_3^2(x_0 - x_3)$ and $F_3 = x_0(x_1 - x_2)(x_1 + x_2)$. The curve $F_1 = 0$ has a cusp singularity, and the resulting complement has a non-vanishing twisted cohomology.

6.2. $n = 2$ **and** $s \geq 3$**.** We consider the pencil of cubic curves

$$F_{[a:b]} = a(x_0^3 + x_1^3 + x_2^3) + 3bx_0x_1x_2.$$

A generic element given by $a \neq 0$ and $b^3 \neq -1$ is non-singular. For s generic elements $D_1, \ldots, D_s$, we have $\dim H^1(M, \mathcal{L}_\lambda) \geq s-2$. Define non-generic elements $F_1 = x_0x_1x_2$, $F_2 = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2$, $F_3 = x_0^3 + x_1^3 + x_2^3 - 3\xi x_0x_1x_2$ and $F_4 = x_0^3 + x_1^3 + x_2^3 - 3\xi^2 x_0x_1x_2$ where $\xi = e^{2\pi\sqrt{-1}/3}$. They are four sets of 3-lines, and the set $\mathcal{A}$ of all lines is called the Hessian configuration, which is the arrangement of 12 lines passing through the nine inflection points of a non-singular cubic. In this case, we know $\dim H^1(M, \mathcal{L}_\lambda) = 2$ for a non-trivial weight λ with $\sum_{i=1}^4 \lambda_i = 0$ ([Li]).

6.3. $n = 3$**.** Let

$$F_1 = x_0(x_1 + x_2 + x_3),$$
$$F_2 = x_1(-x_0 + x_2 - x_3),$$
$$F_3 = x_2(-x_0 - x_1 + x_3)$$

and

$$F_4 = x_3(-x_0 + x_1 - x_2).$$

One can verify that the divisors defined by them satisfy the conditions in Theorem 1. Then $H^2(M, \mathcal{L}_\lambda) \neq 0$ for a weight λ with $\sum_{i=1}^4 \lambda_i = 0$. This yields an arrangement of 8 planes whose underlying matroid is of type L_8 ([Ka3]).

6.4. **A higher dimensional case.** Let $F_i = x_{i-1}^d - x_i^d$ for $i = 1, \ldots, n$ and $F_0 = x_n^d - x_0^d$. The support of the associated divisors determines an arrangement $\mathcal{A}$ of $(n+1)d$ hyperplanes in $\mathbb{P}^n$. Note that this is a projective closure of a subarrangement of the monomial arrangement $\mathcal{A}_{d,d,n+1}$ (see [OT, CS]). Since $\sum_{k=0}^n F_k = 0$, we have $H^{n-1}(M, \mathcal{L}_\lambda) \neq 0$ for a weight λ with $\sum_{i=0}^n \lambda_i = 0$. The $n = 2$ case was found in [CS]. Note that the underlying matroid of $\mathcal{A}$ is a degeneration of the matroid associated to the Latin n-dimensional hypercube given by the addition table for $(\mathbb{Z}_d)^n$ (see [Ka3]).

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