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# POINTWISE HARDY INEQUALITIES AND UNIFORMLY FAT SETS

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Abstract. We prove that it is equivalent for domain in $\mathbb{R}^n$ to admit the pointwise $p$-Hardy inequality, have uniformly $p$-fat complement, or satisfy a uniform inner boundary density condition.

## 1. Introduction

We say that a domain $\Omega \subset \mathbb{R}^n$ admits *the pointwise $p$-Hardy inequality* for $1 < p < \infty$ if there exists $1 < q < p$ such that the inequality

$$|u(x)| \leq C d_\Omega(x) \left( \sup_{r \leq 2 d_\Omega(x)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\nabla u(y)|^q \, dy \right)^{1/q} \tag{1}$$

holds for all $u \in C_0^\infty(\Omega)$ and all $x \in \Omega$ with a constant $C = C(\Omega, n, p, q) > 0$; here $d_\Omega(x)$ denotes the distance from $x \in \Omega$ to the boundary of $\Omega$. These inequalities were introduced by Hajłasz in [2], but Kinnunen and Martio also considered similar inequalities independently in [6]. It was proved in [2] (see also [6]) that if $1 < p < \infty$ and the complement of the domain $\Omega \subset \mathbb{R}^n$ is sufficiently big, namely uniformly $p$-fat (see Section 2 for precise definitions), then $\Omega$ admits the pointwise $p$-Hardy inequality. Notice that it follows immediately from the definition that if $1 < p_0 < \infty$ and a domain $\Omega$ admits the pointwise $p_0$-Hardy inequality, there exists $1 < q < p_0$ such that $\Omega$ admits pointwise $p$-Hardy inequalities for all $p > q$.

If $u \in C_0^\infty(\Omega)$ is such that (1) holds for all $x \in \Omega$ with a constant $C_1 > 0$, it is easy to see, using the Hardy-Littlewood-Wiener maximal function theorem, that $u$ satisfies the usual $p$-Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx \tag{2}$$

with a constant $C = C(C_1, n, p) > 0$. This classical inequality was first considered in the one-dimensional case by Hardy (cf. [3] and references therein). Nečas [9] generalized $p$-Hardy inequalities to higher dimensions when he proved that, for all $1 < p < \infty$, the inequality (2) holds in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ for all $u \in C_0^\infty(\Omega)$, with a constant $C = C(\Omega, n, p) > 0$ (i.e. $\Omega$ admits the $p$-Hardy inequality). Later Ancona (the case $p = 2$) [1], Lewis [8], and Wannebo [11] proved that a domain $\Omega \subset \mathbb{R}^n$ admits the $p$-Hardy inequality under the assumption that the

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complement of $\Omega$ is uniformly $p$-fat. Recall that in [2] and [6] this same assumption was shown to be sufficient for $\Omega$ to admit even the pointwise $p$-Hardy inequality. We also remark that the complement of a proper subdomain $\Omega \subsetneq \mathbb{R}^n$ is uniformly $p$-fat for all $p > n$.

However, the pointwise $p$-Hardy inequality is not equivalent to the usual $p$-Hardy inequality, since there are domains which admit the latter for some $p$, but where the corresponding pointwise inequality fails to hold. In particular, it is not true that the $p_0$-Hardy inequality would imply $p$-Hardy inequalities for all $p > p_0$, as is the case with pointwise inequalities. This can be seen by considering e.g. the punctured unit ball $B(0,1) \setminus \{0\} \subset \mathbb{R}^n$, which admits the pointwise $p$-Hardy inequality only in the trivial case $p > n$, but where the usual $p$-Hardy inequality also holds when $1 < p < n$; yet the $n$-Hardy inequality fails in this domain. This example also shows that the uniform $p$-fatness of the complement is not necessary for a domain to admit the $p$-Hardy inequality, as the complement of $B(0,1) \setminus \{0\} \subset \mathbb{R}^n$ is not uniformly $p$-fat for any $p \leq n$. Nevertheless, as a part of our main theorem, we show that uniform $p$-fatness of $\Omega^c$ is not only sufficient, but also *necessary* for $\Omega$ to admit the *pointwise* $p$-Hardy inequality.

We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies *an inner boundary density condition* with exponent $\lambda$, if there exists a constant $C > 0$ such that

$$\mathcal{H}^\lambda_\infty\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda \quad \text{for every } x \in \Omega. \tag{3}$$

It turns out that condition (3), for some exponent $\lambda > n - p$, is also necessary and sufficient for a domain $\Omega \subset \mathbb{R}^n$ to admit the pointwise $p$-Hardy inequality, and hence equivalent to the uniform $p$-fatness of $\Omega^c$. Let us now formulate our main result.

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $1 < p < \infty$. Then the following conditions are equivalent:*
*(a) The complement $\Omega^c$ is uniformly $p$-fat.*
*(b) $\Omega$ admits the pointwise $p$-Hardy inequality.*
*(c) There exists $n - p < \lambda \leq n$ such that $\Omega$ satisfies the inner boundary density condition* (3) *with the exponent $\lambda$.*

Theorem 1 can be considered as an extension of the result, proved by Ancona [1] ($n = 2$) and Lewis [8], that a domain $\Omega \subset \mathbb{R}^n$ admits the $n$-Hardy inequality if and only if the complement of $\Omega$ is uniformly $n$-fat.

Results related to Theorem 1 were also considered in [7], where the following local dichotomy was shown: Suppose that a domain $\Omega \subset \mathbb{R}^n$ admits the $p$-Hardy inequality and let $w \in \partial\Omega$, $r > 0$. Then either the Hausdorff dimension of $B(w,r) \cap \partial\Omega$ is strictly larger that $n - p$, or the Minkowski dimension of $B(w,r) \cap \partial\Omega$ is strictly less than $n - p$. Now, if $\Omega$ admits the pointwise $p$-Hardy inequality, we obtain, by Theorem 1, that only the former of the two possibilities above may occur. Indeed, when $w \in \partial\Omega$ and $r > 0$, there exists $x \in B(w, r/3) \cap \Omega$, whence $B(x, 2d_\Omega(x)) \subset B(w,r)$, and thus

$$\dim_{\mathcal{H}}\big(B(w,r) \cap \partial\Omega\big) \geq \dim_{\mathcal{H}}\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq \lambda > n - p.$$

## 2. Preliminaries

When $A$ is a subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$, $\partial A$ denotes the boundary of $A$ and $A^c = \mathbb{R}^n \setminus A$ is the complement of $A$. The characteristic function of $A$ is $\chi_A$, and $|A|$ denotes the $n$-dimensional Lebesgue measure of $A$. The Euclidean distance between two points, or a point and a set, is denoted $d(\cdot,\cdot)$. When $\Omega$ is a domain, i.e. an open and connected set, and $x \in \Omega$, we also use notation $d_\Omega(x) = d(x, \partial\Omega)$. An open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted $B(x,r)$, and the corresponding closed ball is $\overline{B}(x,r)$. If $B = B(x,r)$ and $L > 0$, we denote $LB = B(x, Lr)$. The support of a function $u\colon \Omega \to \mathbb{R}$, $\operatorname{spt}(u)$, is the closure of the set where $u$ is non-zero. We let $C$ denote various positive constants, which may vary from expression to expression.

The restricted Hardy-Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$M_R f(x) = \sup_{0<r\le R} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

The well-known maximal function theorem of Hardy, Littlewood and Wiener (see e.g. [10]) states that if $1 < p < \infty$, we have $||M_R f||_p \le C(n,p)||f||_p$ for all $0 < R \le \infty$. When $1 < q < \infty$, we denote $M_{R,q} f = (M_R f^q)^{1/q}$. Using this notation, we may now write the pointwise $p$-Hardy inequality (1) as

$$|u(x)| \le C d_\Omega(x) M_{2d_\Omega(x),q}\big(|\nabla u|\big)(x), \tag{4}$$

where $1 < q < p$.

The $\lambda$-Hausdorff content of a set $A \subset \mathbb{R}^n$ is

$$\mathcal{H}^\lambda_\infty(A) = \inf\Big\{ \sum_{i=1}^\infty r_i^\lambda : A \subset \bigcup_{i=1}^\infty B(z_i, r_i)\Big\},$$

where $z_i \in A$ and $r_i > 0$. The Hausdorff dimension of $A \subset \mathbb{R}^n$ is then

$$\dim_{\mathcal{H}}(A) = \inf\big\{\lambda > 0 : \mathcal{H}^\lambda_\infty(A) = 0\big\}.$$

We say that the boundary of a domain $\Omega \subset \mathbb{R}^n$ is $\lambda$-*thick*, if there exists a constant $C > 0$ such that

$$\mathcal{H}^\lambda_\infty\big(B(w,r) \cap \partial\Omega\big) \ge C r^\lambda$$

for all $w \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$. It is clear that $\lambda$-thickness of $\partial\Omega$ implies that condition (3) holds in $\Omega$. The converse however is not true; see Section 4 for an example.

Let $\Omega \subset \mathbb{R}^n$ be a domain. The $p$-capacity of a compact set $E \subset \Omega$ (relative to $\Omega$) is defined as

$$\operatorname{cap}_p(E,\Omega) = \inf\left\{ \int_\Omega |\nabla u|^p\, dx : u \in C_0^\infty(\Omega),\ u \ge 1 \text{ on } E\right\}.$$

A closed set $E \subset \mathbb{R}^n$ is said to be *uniformly $p$-fat* if there exists a constant $C > 0$ such that

$$\operatorname{cap}_p\big(E \cap \overline{B}(x,r), B(x,2r)\big) \ge C \operatorname{cap}_p\big(\overline{B}(x,r), B(x,2r)\big)$$

for all $x \in E$ and $r > 0$. Note that for each ball $B(x,r) \subset \mathbb{R}^n$ we have

$$\operatorname{cap}_p(\overline{B}(x,r), B(x,2r)) = C(n,p) r^{n-p}.$$

For this and other basic properties of the $p$-capacity we refer to [4].

We record the following useful lemma between Hausdorff content and $p$-capacity; for a proof, see e.g. [5, Thm. 5.9].

**Lemma 2.** *Let $E \subset B(x,r) \subset \mathbb{R}^n$ be a compact set such that*

$$\mathcal{H}^{\lambda}_{\infty}(E) \geq C_1 r^{\lambda}$$

*for some $\lambda > n-p$ and $C_1 > 0$. Then*

$$\operatorname{cap}_p\big(E \cap B(x,r), \overline{B}(x,2r)\big) \geq C\, r^{n-p},$$

*where $C = C(C_1, n, p) > 0$.*

## 3. Proof of Theorem 1

The part *(a)*$\Longrightarrow$*(b)* of Theorem 1 is contained in [2, Thm. 2]; the proof of this part relies on the self-improving property of $p$-fatness, due to Lewis [8, Thm. 1]. Let us now prove the implications *(b)*$\Longrightarrow$*(c)* and *(c)*$\Longrightarrow$*(a)* to obtain the equivalence of the conditions in the theorem.

*Proof of (b)$\Longrightarrow$(c).* Let $\Omega \subset \mathbb{R}^n$ and $1 < p < \infty$. We assume that condition (3) fails for every $n-p < \lambda \leq n$, and show that then the pointwise $p$-Hardy inequality fails in $\Omega$ as well. To this end, let $1 < q < p$ and choose $\lambda = n - q > n - p$. It is evident that (3) is equivalent to the condition that there exists some $C_1 > 0$ such that

$$\mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(x, 3d_\Omega(x)) \cap \partial\Omega\big) \geq C_1 d_\Omega(x)^{\lambda} \quad \text{for every } x \in \Omega. \tag{5}$$

Since (5) now fails for the chosen $\lambda$, there exist, for each $k \in \mathbb{N}$, a point $x_k \in \Omega$ such that

$$\mathcal{H}^{\lambda}_{\infty}(E_k) < k^{-1} R_k^{\lambda},$$

where we denote $R_k = d_\Omega(x_k)$ and $E_k = \overline{B}(x_k, 3R_k) \cap \partial\Omega$. Using this, and the fact that $E_k$ is compact, we find, for a fixed $k \in \mathbb{N}$, a finite covering $\{B_i\}_{i=1}^N$, $B_i = B(w_i, r_i)$ with $w_i \in \partial\Omega$ and $r_i > 0$, such that $E_k \subset \bigcup_{i=1}^N B_i$ and $\sum_{i=1}^N r_i^{\lambda} < k^{-1} R_k^{\lambda}$.

Define a function $\varphi_k$ by

$$\varphi_k(x) = \min_{1 \leq i \leq N} \big\{1,\, r_i^{-1} d(x, 2B_i)\big\}$$

and let $\psi_k \in C_0^{\infty}(B(x_k, 3R_k))$ be such that $0 \leq \psi_k \leq 1$ and $\psi_k(x) = 1$ for all $x \in B(x_k, 2R_k)$. Then $u_k = \psi_k\, \varphi_k \chi_\Omega$ is a Lipschitz function with compact support in $\Omega$. Since $r_i < k^{-1/\lambda} R_k$ for all $1 \leq i \leq N$, we have that

$$d(x_k, 3B_i) > \tfrac{1}{4} R_k > r_i \tag{6}$$

for all $1 \leq i \leq N$ if $k > 4^{\lambda}$, and hence $u_k(x_k) = 1$ for these $k$.

Next, denote $A_i = 3\overline{B}_i \setminus 2B_i$. Then $\operatorname{spt}(|\nabla u_k|) \cap B(x_k, 2R_k) \subset \bigcup_{i=1}^N A_i$ and we have in fact for a.e. $y \in B(x_k, 2R_k)$ that

$$|\nabla u_k(y)|^q \leq \sum_{i=1}^N r_i^{-q} \chi_{A_i}(y). \tag{7}$$

Let us now estimate the right-hand side of the pointwise $p$-Hardy inequality (4) at $x_k$. Since $\operatorname{spt}(|\nabla u_k|) \cap B(x_k, 2R_k) \subset \bigcup_{i=1}^N 3\overline{B}_i$, it follows from (6) that we must have $r > \frac{1}{4}R_k$ in order to obtain something positive when estimating the maximal function of $|\nabla u_k|$ at $x_k$. Hence, using (7), we calculate

$$\begin{aligned} M_{2R_k}\big(|\nabla u_k|^q\big)(x_k) &\le C \sup_{\frac{1}{4}R_k \le r \le 2R_k} \left( r^{-n} \int_{B(x_k,r)} |\nabla u_k(y)|^q \, dy \right) \\ &\le C R_k^{-n} \int_{B(x_k,2R_k)} |\nabla u_k(y)|^q \, dy \ \le \ C d_\Omega(x_k)^{-n} \sum_{i=1}^{N} |A_i| \, r_i^{-q} \\ &\le C d_\Omega(x_k)^{-n} \sum_{i=1}^{N} r_i^{n-q}. \end{aligned}$$

Recall that $\lambda = n - q > n - p$ and that $\sum_{i=1}^{N} r_i^\lambda < k^{-1} d_\Omega(x_k)^\lambda$. Thus

$$\begin{aligned} d_\Omega(x_k)^q M_{2R_k}\big(|\nabla u_k|^q\big)(x_k) &\le C d_\Omega(x_k)^{q-n} \sum_{i=1}^{N} r_i^{n-q} \\ &\le C d_\Omega(x_k)^{-\lambda} k^{-1} d_\Omega(x_k)^\lambda \le \frac{C}{k}, \end{aligned}$$

and so the right-hand side of the inequality (4) for $u_k$ at $x_k$ tends to zero as $k \to \infty$. However, $u_k(x_k) = 1$ for large $k$, so the pointwise $p$-Hardy inequality fails to hold with a uniform constant for all compactly supported Lipschitz functions in $\Omega$. By a standard approximation argument it is then clear that $\Omega$ does not admit the pointwise $p$-Hardy inequality. □

*Proof of (c)$\Longrightarrow$(a).* There now exists $n - p < \lambda \le n$ so that $\Omega$ satisfies the density condition (3) with the exponent $\lambda$ and with a constant $C_1 > 0$. To prove that $\Omega^c$ is uniformly $p$-fat, it is in fact enough to show that there exists a constant $C = C(C_1, n, \lambda) > 0$ such that

$$\mathcal{H}^\lambda_\infty\big(B(w,r) \cap \Omega^c\big) \ge C r^\lambda \tag{8}$$

for all $w \in \partial\Omega$ and $r > 0$. Indeed, assume that (8) holds for all $w \in \partial\Omega$ and let $z \in \Omega^c$, $r > 0$. If $B(z, r/2) \subset \Omega^c$, then it easily follows (compare to calculations in (10) below) that (8) also holds for the ball $B(z,r)$, with a constant depending only on $n$. On the other hand, if $B(z,r/2) \cap \Omega \ne \emptyset$, there is $w \in \partial\Omega$ such that $B(w, r/2) \subset B(z,r)$, and thus (8) for $B(w,r/2)$ yields (8) for $B(z,r)$, but now with a constant depending on $C$ and $\lambda$. We conclude, by Lemma 2, that (8) for all $w \in \partial\Omega$ implies the uniform $p$-fatness of $\Omega^c$.

Then let $w \in \partial\Omega$ and $r > 0$. To prove that (8) holds, first assume that

$$|B(w,r) \cap \Omega^c| \ge \tfrac{1}{4}\, |B(w,r)|. \tag{9}$$

Let $\{B_i\}_{i=1}^\infty$, $B_i = B(z_i, r_i)$ for $z_i \in \Omega^c$ and $0 < r_i \le r$, be a covering of $B(w,r) \cap \Omega^c$. Then we have that

$$\tfrac{1}{4} \le \sum_i \left(\frac{r_i}{r}\right)^n \le \sum_i \left(\frac{r_i}{r}\right)^\lambda, \tag{10}$$

and thus, by the definition of the $\lambda$-Hausdorff content, we see that (8) holds with constant $1/4$ under assumption (9).

We may hence assume that $|B(w,r) \cap \Omega| \ge \frac{3}{4}\, |B(w,r)|$. Then let $\{B_i\}_{i=1}^\infty$, $B_i = B(w_i, r_i)$ for $w_i \in \partial\Omega$ and $0 < r_i \le r$, be a covering of $B(w,r) \cap \partial\Omega$. If

$$\textstyle\sum_i |B_i| \ge \frac{1}{4}\, 2^{-n} |B(w,r)|, \tag{11}$$

it follows as in (10) that $\sum_i r_i^\lambda \geq C(n) r^\lambda$.

If (11) does not hold, i.e. we have that

$$\textstyle\sum_i |B_i| < \frac{1}{4}\, 2^{-n} |B(w,r)|, \tag{12}$$

we proceed as follows: Let $\hat{r} = (3/4)^{1/n} r$ and denote $\alpha(n) = 1 - (3/4)^{1/n}$, so that $r - \hat{r} = \alpha(n) r$. If there exists $x \in B(w,\hat{r}) \cap \Omega$ such that $d_\Omega(x) \geq \frac{1}{2}\alpha(n) r$, then, by the continuity of the distance function, there also exists $x' \in B(w,\hat{r}) \cap \Omega$ such that $d_\Omega(x') = \frac{1}{2}\alpha(n) r$. Thus $B(x', 2d_\Omega(x')) \subset B(w,r)$, and we obtain, by condition (3), that

$$\mathcal{H}^\lambda_\infty\big(B(w,r) \cap \partial\Omega\big) \geq \mathcal{H}^\lambda_\infty\big(B(x', 2d_\Omega(x')) \cap \partial\Omega\big) \geq C_1 d_\Omega(x')^\lambda \geq C r^\lambda,$$

where $C = C(C_1, n, \lambda) > 0$, and so (8) holds. We may hence assume that

$$d_\Omega(x) < \tfrac{1}{2}\alpha(n) r \;\text{ for every } x \in B(w,\hat{r}) \cap \Omega, \tag{13}$$

so that in particular $B(x, 2d_\Omega(x)) \subset B(w,r)$ for every $x \in B(w,\hat{r}) \cap \Omega$.

Let us denote $A = (B(w,\hat{r}) \cap \Omega) \setminus \bigcup_i 2B_i$. We then have, by (12) and the choice of $\hat{r}$, that

$$\begin{aligned} |A| &\geq |B(w,\hat{r}) \cap \Omega| - \textstyle\sum_i 2^n |B_i| \\ &\geq |B(w,r) \cap \Omega| - |B(w,r) \setminus B(w,\hat{r})| - 2^n \tfrac{1}{4} 2^{-n} |B(w,r)| \\ &\geq \tfrac{3}{4}\,|B(w,r)| - \tfrac{1}{4}\,|B(w,r)| - \tfrac{1}{4}\,|B(w,r)| \geq \tfrac{1}{4}\,|B(w,r)|. \end{aligned}$$

Since $A \subset \bigcup_{x \in A} B(x, 6d_\Omega(x))$, we obtain, by a standard covering lemma (cf. [10]), a countable set of points $x_k \in A$ such that the corresponding balls $6\tilde{B}_k$, where $\tilde{B}_k = B(x_k, d_\Omega(x_k))$, are pairwise disjoint and $A \subset \bigcup_k 30\tilde{B}_k$. Hence

$$\tfrac{1}{4}|B(w,r)| \leq |A| \leq \sum_k |30\tilde{B}_k| \leq 30^n \sum_k |\tilde{B}_k|. \tag{14}$$

Since the radius of $\tilde{B}_k$ is $d_\Omega(x_k) < r$ for all $k$, and $\lambda \leq n$, it now follows from (14), similar to (10), that

$$C(n)\, r^\lambda \leq \sum_k d_\Omega(x_k)^\lambda. \tag{15}$$

When $i \in \mathbb{N}$, we let $\#_i$ denote the number of the balls $2\tilde{B}_k$ such that $2\tilde{B}_k \cap B_i \neq \emptyset$. But if $2\tilde{B}_k \cap B_i \neq \emptyset$, then $d_\Omega(x_k) > \frac{1}{2} r_i$ (since $x_k \notin 2B_j$), and thus $B_i \subset 6\tilde{B}_k$. Since the balls $6\tilde{B}_k$ are pairwise disjoint, it follows that $\#_i \leq 1$ for all $i \in \mathbb{N}$. Also, we have by (13) that $2\tilde{B}_k \subset B(w,r)$, and so

$$\mathcal{H}^\lambda_\infty\big(2\tilde{B}_k \cap \partial\Omega\big) \leq \sum_{B_i \cap 2\tilde{B}_k \neq \emptyset} r_i^\lambda \tag{16}$$

for each $k$. Combining (15), (3), (16), and the fact that $\#_i \leq 1$, we finally obtain

$$\begin{aligned} r^\lambda &\leq C \sum_k d_\Omega(x_k)^\lambda \leq C \sum_k \mathcal{H}^\lambda_\infty\big(2\tilde{B}_k \cap \partial\Omega\big) \\ &\leq C \sum_k \sum_{B_i \cap 2\tilde{B}_k \neq \emptyset} r_i^\lambda \leq C \sum_i \#_i\, r_i^\lambda \leq C \sum_i r_i^\lambda, \end{aligned}$$

where $C = C(C_1, n) > 0$. Hence, by taking the infimum of the sums $\sum_i r_i^\lambda$ over all the coverings $\{B_i\}_i$ of $B(w,r) \cap \partial\Omega$, we see that equation (8) holds in this case as well. This also finishes the proof of Theorem 1. □

*Remark.* From the proof of the part *(c)*$\Longrightarrow$*(a)* of the theorem we obtain, with some minor modifications, the following result: Assume that a domain $\Omega \subset \mathbb{R}^n$ satisfies the inner boundary density condition (3) with exponent $\lambda$ and with a constant $C_1 > 0$, and let $0 < \varepsilon < 1$. Then, for each ball $B(w,r)$, where $w \in \partial\Omega$ and $r > 0$, we have

$$|B(w,r) \cap \Omega^c| \geq \varepsilon |B(w,r)| \quad \text{or} \quad \mathcal{H}^\lambda_\infty\big(B(w,r) \cap \partial\Omega\big) \geq C r^\lambda,$$

where $C = C(C_1, n, \lambda, \varepsilon) > 0$. In particular, if there exists a constant $C_2 > 0$ such that $|B(w,r) \cap \Omega| \geq C_2 |B(w,r)|$ for all $w \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$, we conclude that $\partial\Omega$ is $\lambda$-thick, with a constant $C = C(C_1, C_2, n, \lambda) > 0$.

## 4. An example

We give a brief example in which we show that the $\lambda$-thickness of the boundary of $\Omega \subset \mathbb{R}^n$, for some $\lambda > n - p$, is not necessary for $\Omega$ to admit the pointwise $p$-Hardy inequality, or equivalently, for $\Omega$ to satisfy the inner boundary density condition (3) with the exponent $\lambda$.

Let $n, k \in \mathbb{N}$ be such that $n \geq 3$ and $1 \leq k \leq n - 2$. Also let $\tau > 1$. We consider the following domain $\Omega_k \subset \mathbb{R}^n$:

$$\Omega_k = \Big\{ (x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_1, \dots, x_k < 1, \sum_{i=k+1}^{n} {x_i}^{n-k} < {x_1}^{\tau(n-k)} \Big\}.$$

Let $0 < r < 1$ and denote $B_r = B(0,r)$, $E_{k,r} = \partial\Omega_k \cap B_r$. Then $E_{k,r}$ can be covered by approximately $r^{(1-\tau)k}$ balls of radius $r^\tau$. Now, if $\lambda > k$, we have that

$$r^{-\lambda}\,\mathcal{H}^\lambda_\infty(E_{k,r}) \leq C r^{-\lambda} r^{(1-\tau)k} r^{\tau\lambda} \leq C r^{(\tau-1)(\lambda-k)} \longrightarrow 0$$

as $r \to 0$, since $(\tau - 1)(\lambda - k) > 0$. This means that $\partial\Omega_k$ is not $\lambda$-thick for any $\lambda > k$. Nevertheless, it is obvious that the inner boundary density condition (3), with $\lambda = n - 1$, holds for all $x \in \Omega_k$, and so $\Omega_k$ admits the pointwise $p$-Hardy inequality for all $p > 1$, especially for $p = n - k$.

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