ALL DIHEDRAL DIVISION ALGEBRAS
OF DEGREE FIVE ARE CYCLIC

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Abstract. In 1982 Rowen and Saltman proved that every division algebra which is split by a dihedral extension of degree 2n of the center, n odd, is in fact cyclic. The proof requires roots of unity of order n in the center. We show that for n = 5, this assumption can be removed. It then follows that 5Br(F), the 5-torsion part of the Brauer group, is generated by cyclic algebras, generalizing a result of Merkurjev (1983) on the 2 and 3 torsion parts.

1. Mathematical background

We begin with basic notions needed for this work and refer the reader to [8] or [5] for more details. Let R be a ring and let C(R) = \{r ∈ R | rx = xr ∀x ∈ R\} denote the center of R.

Definition 1.1. A ring R will be called a simple ring if R has no non-trivial two-sided ideals. In particular R is a division ring if every non-zero element is invertible.

Remark 1.2. Notice that if R is simple, its center is naturally a field.

Definition 1.3. An F-algebra R is called an F-central simple algebra if R is simple with C(R) = F and dim_F(R) < ∞.

Remark 1.4. Every F-central simple algebra A has dim_F(A) = n^2, and we define the degree of A, denoted deg(A), to be n.

By Wedderburn’s Theorem every F-central simple algebra is of the form M_n(D), where D is a division algebra with center F.

The Brauer group of a field F, denoted Br(F), is the set of isomorphism classes of F-central simple algebras modulo the following relation: two central simple algebras A, B are equivalent if and only if there exist natural numbers n, m such that M_n(A) \cong M_m(B).

Proposition 1.5. Let D be an F-central division algebra of degree n, and K a subfield of D; then K is a maximal subfield if and only if [K : F] = n.

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Definition 1.6. A crossed product is an $F$-central simple algebra $A$ of degree $n$ containing a commutative $F$-subalgebra $C$ Galois over $F$, with $[C : F] = n$. Note that if $A$ is a division algebra, then $C$ is a maximal subfield of $A$.

Definition 1.7. Let $D$ be an $F$-central division algebra of degree $n$. We will say that $D$ is split by a group $G$ if $D$ contains a maximal subfield $K$ with Galois closure $E$ such that $\text{Gal}(E/F) = G$.

Theorem 1.8. Let $A$ be a crossed product where $K \subset A$ is a maximal subfield with Galois group $\text{Gal}(K/F) = G$. Then $A$ has the following description: $A = \bigoplus_{\sigma \in G} Kx_{\sigma}$ as a left $K$-vector space, and multiplication in $A$ is according to the rules:

$$x_{\sigma}k = \sigma(k)x_{\sigma} \quad \forall k \in K$$

and

$$x_{\sigma}x_{\tau} = c(\sigma, \tau)x_{\tau}x_{\sigma}$$

where $c \in H^2(G, K^\times)$ is a 2-cocycle. In this case $A$ is denoted $A = (K, G, c)$.

Remark 1.9. If $G = \langle \sigma \rangle$ we can give a simpler representation of $A$ as follows: $A = \bigoplus_{i=0}^{n-1} Kx^i$ as a left $K$-vector space, where $n = \deg(A) = |G|$ and the multiplication is according to the rules

$$xk = \sigma(k)x \quad \forall k \in K$$

and

$$x^i x^j = \begin{cases} x^{i+j}, & i + j < n, \\ \beta x^{i+j} - n, & i + j \geq n. \end{cases}$$

In this case, $A$ is denoted as $A = (K, \sigma, \beta)$.

Remark 1.10. If $F$ contains a primitive $n$-th root of unity $\rho$, we can give an even simpler description of $A$ (since then $K = F[x \mid x^n = \alpha \in F]$) as follows:

$$A = F[x, y \mid x^n = \alpha; y^n = \beta; xy = \rho xy] \quad \alpha, \beta \in F.$$

2. Some Preliminary Results

In this section we briefly repeat the arguments of Rowen and Saltman in [9] but we do not assume $F$ contains roots of unity.

The situation we will be handling is the following: $D/F$ is a central simple algebra of odd degree $n$ having a maximal subfield $K \subset D$ with Galois closure $E \supset K \supset F$, such that

$$\text{Gal}(E/F) = D_n = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau = \sigma^{-1} \rangle$$

and $K = E^{(\tau)}$.

Extending scalars to $E^{(\sigma)}$, we may view $E \subset D' = D \otimes E^{(\sigma)}$. Now $\text{Gal}(E/E^{(\sigma)}) = \langle \sigma \rangle$, i.e. $D'$ is cyclic, so we have an element $\beta \in D'$ such that

$$\beta^{-1}x\beta = \sigma(x) \quad \forall x \in E.$$  

(1)

In particular $\beta^n \in E^{(\sigma)}$. Notice that $\tau$ can be extended to $D' = D \otimes E^{(\sigma)}$ by its action on $E^{(\sigma)}$, that is, we write $\tau$ instead of $1 \otimes \tau$.

Lemma 2.1. We may assume that $\tau(\beta) = \beta^{-1}$. 
Proof. Applying \( \tau \) to (1) yields
\[
\tau(\beta)^{-1} \tau(x) \tau(\beta) = \sigma^{-1}(\tau(x)) \quad \forall x \in E.
\]
Now since \( \tau \) is an automorphism of \( E \), \( \tau(x) \) runs over all elements of \( E \), and thus
\[
\tau(\beta)^{-1} y \tau(\beta) = \sigma^{-1}(y) \quad \forall y \in E;
\]
that is \( \tau(\beta) \) acts on \( E \) as \( \sigma^{-1} \). Now define \( \beta' = \beta^r \tau(\beta)^{-r} \), where \( r = (n + 1)/2 \),
and compute that \( \tau(\beta') = \beta'^{-1} \), and \( \beta' \) acts on \( E \) as \( \sigma \).

Let \( P_t(X) = X^n + \sum_{i=1}^n c_i(t)X^{n-i} \) denote the characteristic polynomial of \( t \in D' \). Note that \( c_1(t) = -\text{tr}(t) \) and \( c_n(t) = (-1)^n \text{N}(t) \) where \( \text{tr}(t) \) and \( \text{N}(t) \) are the reduced trace and norm of \( t \).

Lemma 2.2. Let \( t = \beta^i e \), for \( e \in E \) and \( 0 < i < n \), \( i \neq 0 \). Then \( \text{tr}(t) = 0 \).

Proof. Let \( d = \gcd(i, n) \). Clearly we have \( t^{n/d} = \beta^{ni/d} N_{\sigma}(e) \in E^{(\sigma^i)} \) where \( N_{\sigma} \), is the norm from \( E \) to \( E^{(\sigma^i)} \). Now \( [E : E^{(\sigma^i)}] = n/d \), implying \( P(X) = X^n/d - \beta^{ni/d} N_{\sigma}(e) \) is the characteristic polynomial of \( t \), hence \( \text{tr}_{E/E^{(\sigma^i)}}(t) = 0 \) which implies \( \text{tr}_{E/F}(t) = 0 \). \( \square \)

Lemma 2.3. Let \( t = (\beta + \beta^{-1}) e \) for \( e \in E \). Then the coefficients of \( P_t(X) \) satisfy \( c_i(t) = 0 \) for every odd \( 0 < i < n \).

Proof. Notice that for \( i \) odd, \( t^i \) is a sum of elements of the form \( a \beta^s \) where \( a \in E \) and \( s \) odd, \(-n < s < n \), so by Lemma 2.2 and Newton’s identities we are done in the characteristic zero case. For the general case, we refer the reader to [9] where the main idea is that you can form a model for this situation in the form of an Azumaya algebra and then use a specialization argument. \( \square \)

Corollary 2.4. There is an element \( t \in D \) such that for every \( e \in E \) (and so also for \( k \in K \subset E \)), \( c_i = 0 \) for every odd \( 0 < i < n \) in \( P_{te}(X) \).

Proof. Since \( D = D^{(\tau)} \) we have \( t = \beta + \beta^{-1} \) is the desired element. \( \square \)

Remark 2.5. Notice that if \( n = p \) is prime \( \text{Char}(F) = p \), the element \( t = \beta + \beta^{-1} \in D \) we found satisfies \( t^p \in F \) and \( t \notin F \), and so by a theorem of Albert in the “special results” chapter of his seminal book [1], which is known as Albert’s cyclicity criterion, \( D \) is cyclic (this is not a new result, as J.P. Tignol and P. Mammone did this for any field \( F \) with \( \text{Char}(F) \mid n \) in [9] using the corestriction, but it shows that the proof of Rowen and Saltman also applies to this case).

3. The case \( n = 5 \)

Now we would like to focus on the particular case where \( n = 5 \). The main tool we will be using is the following proposition taken from [8, Proposition 2.2].

Proposition 3.1. Let \( G(x_1, ..., x_n) \) be a homogeneous form of degree 3 defined over a field \( F \). If \( G \) has a solution, \( \alpha \in K^{(n)} \), defined over a quadratic extension \( K \) of \( F \), then \( G \) has a solution defined over \( F \).

Proof. The proof in [2] uses basic intersection theory which we will not use; instead we will give an algebraic proof (which is actually a translation of the proof in [2]) which will enable us to find an explicit solution in section 3. Since \( [K : F] = 2 \) the solution \( \alpha \) has the following form: \( \alpha = (\alpha_1 + \beta_1 t, ..., \alpha_n + \beta_n t) \) where \( \alpha_i, \beta_i \in k \), and
t \in K$ such that $K = F(t)$. Now specialize $G(x_1, \ldots, x_n)$ to $G(\alpha_1 + \beta_1 Z, \ldots, \alpha_n + \beta_n Z)$, denoting it by $g(Z)$. Notice that the coefficient of $Z^3$ in $g(Z)$ is $G(\beta_1, \ldots, \beta_n)$; hence if $G(\beta_1, \ldots, \beta_n) = 0$ we have a solution defined over $F$, otherwise $g(Z)$ is a degree 3 polynomial defined over $F$. Since $g(t) = 0$ we get that $g(Z) = cm_t(Z)(Z-w)$, where $c = G(\beta_1, \ldots, \beta_n)$ and $m_t(Z)$ is the minimal polynomial of $t$ over $F$. Now $c, g(Z)$ and $m_t(Z)$ are defined over $F$; hence $w$ is in $F$ and clearly $G(\alpha_1 + \beta_1 w, \ldots, \alpha_n + \beta_n w) = g(w) = 0$, so we have found a solution $\gamma = (\alpha_1 + \beta_1 w, \ldots, \alpha_n + \beta_n w) \in F^n$. \hfill \Box

**Theorem 3.2.** Let $D$ be a division algebra of degree 5 split by the group $D_5$; then $D$ is cyclic.

**Proof.** In view of Remark 3.3 we may assume $\text{Char}(F) \neq 5$. First we remark that by Albert’s cyclicity criterion it is enough to find an element $t \in D - F$ such that $t^5 \in F$, that is, $c_i = 0$ for every $0 < i < n$. Now by Corollary 3.4 we have $t \in D$ with the property $c_i(te) = 0$ for every odd $0 < i < n$ and $\forall e \in E$. Now since $P_{t-1}(x) = -N(t)^{-1}P_t(x^{-1})x^5$ we have $c_i(et^{-1}) = 0$ for every even $0 < i < n$ and $\forall e \in E$. Hence we are left with finding a solution for $c_1(et^{-1}) = 0$ (which is linear) and $c_3(et^{-1}) = 0$ (which is cubic) in the five dimensional vector space $E_t^{-1}$. Define $V := \{et^{-1} \in E_t^{-1} \mid c_1(et^{-1}) = 0\}$, which is a four dimensional subspace of $E_t^{-1}$. We have to find a solution for $c_3(v) = 0$ in $V$. Let us add a fifth root of unity to $F$, which is either a quadratic extension or a chain of two quadratic extensions. After this extension we are in the case of Rowen and Saltman where they gave an explicit element whose fifth power is in $F$ which was $(v + v^{-1})t^{-1}$, where $v \in E$. This element is clearly in $V \otimes_F F[\rho_5]$. Now by Proposition 3.1 since $c_3(v)$ is homogeneous of degree 3, we have a solution after either one or two quadratic extensions. Thus, we have a solution before the extension, and we are done. \hfill \Box

**Remark 3.3.** If the fifth root of unity is in a quadratic extension of $F$, we know $D$ is cyclic by a theorem of Vishne [10 Theorem 13.6] and D. Haile, M. A. Knus, M. Rost, J. P. Tignol, so that what actually is new is the last case of $[F[\rho] : F] = 4$.

4. A generic example

Fixing $p$ let $K = F[\rho_p]$ and denote $\text{Gal}(K/F) = \langle \tau \rangle$. In [7] Theorem 2 Merkurjev proves that $\rho_p(F)$ is generated by $F$-central simple algebras, $A$, of degree $p$ such that $A \otimes K \simeq (\alpha, \beta)$, where $K[\sqrt[p]{\alpha}]$ is cyclic over $K$ Galois over $F$.

In [10] Vishne calls these algebras quasi-symbols and gives more details about them, including generic examples. We will show that for $p = 5$ these algebras are cyclic and conclude that $5\text{Br}(F)$ is generated by cyclic algebras.

4.1. A generic quasi-symbol of degree 5. For $p = 5$ we have two possibilities for $[K : F]$. The first is $[K : F] = 2$; in this case Vishne shows that every quasi-symbol is cyclic. The second case is $[K : F] = 4$; in this case every quasi-symbol $A$ has one of the following forms (after extending scalars to $K$):

1. $A \otimes K = (\alpha, \beta)$, where $\alpha \in F$ and $\tau(\beta) \equiv \beta^2 \pmod{K^\times^5}$.
2. $A \otimes K = (\alpha, \beta)$, where $\tau(\alpha) = \alpha^{-1}$ and $\tau(\beta) \equiv \beta^{-2} \pmod{K^\times^5}$.

The first kind is known to be cyclic by [10] Theorem 10.3]. So we are left with the second kind for which Vishne gives the following generic construction which we will show is cyclic. Thus every quasi-symbol of degree 5 is cyclic and hence, by [7] Theorem 2 we conclude that $5\text{Br}(F)$ is generated by cyclic algebras.
Let \( k_0 \) be a field of characteristic \( \neq 5 \) and \( k = k_0[\rho] \) where \( \rho \) is a fixed primitive fifth root of unity, and \( \text{Gal}(k/k_0) = \langle \tau \rangle \) where \( \tau(\rho) = \rho^2 \). Set \( K = k(a,b,\eta) \) a transcendental extension and extend \( \tau \) to \( K \) by

\[
\tau(a) = a^{-1}, \quad \tau(b) = \eta^5b^{-2}, \quad \tau(\eta) = \eta^2b^{-1}.
\]

Notice that we still have \( \tau^5 = 1 \). Define \( F = K(\tau) \) and

\[
D = (a,b)_K = K[x,y \mid x^5 = a, \quad \eta^5 = b, \quad yxy^{-1} = \rho x],
\]

and extend \( \tau \) to \( D \) by \( \tau(x) = x^{-1} \), \( \tau(y) = \eta y^{-2} \). Notice that \( \tau^2(\eta) = \eta^{-1} \) and \( \tau^2(y) = y^{-1} \). Now define \( D_0 = D(\tau); \) \( D_0/F \) is the generic quasi-symbol of degree 5 of the second type.

**Remark 4.1.** Vishne’s construction is much more general and we specialized it to the above case; for the general construction we refer the reader to \([10]\).

**Proposition 4.2.** \( D_0 \) is split by \( D_5 \).

**Proof.** Notice that \( \text{Gal}(K[y]/F) = C_5 \rtimes C_4 = \langle \sigma \rangle \rtimes \langle \tau \rangle \) and now we will see how \( \tau \) acts on \( \sigma \). Applying \( \tau \) to \( x^{-1}tx = \sigma(t) \), which holds for every \( t \in K[y] \), yields \( \tau(\sigma(t)) = \tau(x^{-1})\tau(t)\tau(x) = x\tau(t)x^{-1} = \sigma^{-1}(\tau(t)) \) and so we get \( \tau\sigma\tau^{-1} = \sigma^{-1} \).

Hence \( \tau^2 \) is a central element in \( \text{Gal}(K[y]/F) \) and it is clear that \( E = K[y]^{\langle \tau^2 \rangle} \subset K[y] \) is Galois over \( F \) with \( \text{Gal}(E/F) = D_5 = \langle \sigma \rangle \rtimes \langle \tau \rangle \), and we are done. \( \square \)

**Corollary 4.3.** \( D_0 \) is cyclic.

In \([7]\) Merkurjev proves the following theorem:

**Theorem 4.4.** Let \( F \) be a field. \( _n\text{Br}(F) \) is generated by cyclic algebras, for \( n = 2, 3 \).

Now as a result of the above we can extend Merkurjev’s theorem to \( n = 5 \) and get

**Theorem 4.5.** \( _5\text{Br}(F) \) is generated by cyclic algebras.

**Proof.** By section 8 of \([10]\) \( _5\text{Br}(F) \) is generated by quasi-symbols of degree 5, and so we are done. \( \square \)

### 4.2. Finding an explicit solution

Since the above example is a generic one, it would be nice to give an explicit element with fifth power in \( F \), which is what we do now by going over the general proof.

Let \( P_l(X) = X^n + \sum_{i=1}^n c_iX^{n-i} \) denote the characteristic polynomial of \( t \in D_0 \). \( V = \langle x+x^{-1} \rangle^{-1}K[y]^{\langle \tau \rangle} \) is a 5-dimensional \( F \)-subspace of \( D_0 \), satisfying \( c_2(\tau) = c_4(\tau) = 0 \) for all \( v \in V \). Also, we want to find a solution in \( V \) for \( tr(Z) = c_2((x+x^{-1})^{-1}Z) = 0 \) and \( G(Z) = c_2((x+x^{-1})^{-1}Z) = 0 \). Extending scalars from \( F \) to \( F[\rho+\rho^{-1}] \), we have the solutions \( Z_1 = y+y^{-1} = \alpha + \beta (\rho + \rho^{-1}) \) and \( Z_2 = \tau(Z_1) = \alpha + \beta \tau(\rho^2 + \rho^{-2}) \) where \( \alpha = (\alpha_1, ..., \alpha_5), \beta = (\beta_1, ..., \beta_5) \in \langle y \rangle \) so \( \alpha, \beta \in F \). Now define the following line: \( L = \{ \alpha + \beta t \} = \{ (\alpha_1 + \beta_1 t, ..., \alpha_5 + \beta_5 t) \} \) defined over \( F \).

**Proposition 4.6.** For every \( l \in L \) we have \( tr(l) = 0 \).

**Proof.** By standard linear algebra, \( L \cap \{ tr(Z) = 0 \} \) is either one point or the whole line \( L \). Since \( Z_1, Z_2 \in L \cap \{ tr(Z) = 0 \} \), we get \( L \cap \{ tr(Z) = 0 \} = L \) and we are done. \( \square \)
Now let us study the variety \( \{ G(Z) = 0 \} \cap L \). First we need to compute \( G(Z) \). In order to do that we use the representation of \( D \) induced by right multiplication on \( D = K[y] + K[y]x + K[y]x^2 + K[y]x^3 + K[y]x^4 \), namely

\[
x \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & a \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

\( m \in K[y] \mapsto \text{Diag}(m, \sigma(m), \sigma^2(m), \sigma^3(m), \sigma^4(m)) \).

Now the minimal polynomial of \( x + x^{-1} \) is

\[
\lambda^5 - 5\lambda^3 + 5\lambda - (a + a^{-1});
\]

hence

\[
(x + x^{-1})^{-1} = ((x + x^{-1})^4 - 5(x + x^{-1})^2 + 5)(a + a^{-1})^{-1}
\]

\[
= (a + a^{-1})^{-1}(x^4 + x^{-4} - x^2 - x^{-2} + 1)
\]

implying

\[
(x + x^{-1})^{-1} \mapsto (a + a^{-1})^{-1} \begin{pmatrix}
1 & a & -1 & -a & 1 \\
-a & 1 & a & -1 & -a \\
-a & -1 & a & -1 & a \\
1 & -a & 1 & a & -1 \\
\end{pmatrix}.
\]

Now when we compute the characteristic polynomial of \( (x + x^{-1})^{-1} m \) we get that

\[
c_3((x + x^{-1})^{-1} m) = (a + a^{-1})^{-1}(m\sigma(m)\sigma^2(m) + \sigma(m)\sigma^2(m)\sigma^3(m) + \sigma^2(m)\sigma^3(m)\sigma^4(m) + \sigma^3(m)\sigma^4(m)m + \sigma^4(m)m\sigma(m))
\]

\[
= (a + a^{-1})^{-1} \text{tr}_\sigma(m\sigma(m)\sigma^2(m)),
\]

yielding \( F(Z) = (a + a^{-1})^{-1} \text{tr}_\sigma(Z\sigma(Z)\sigma^2(Z)) \). Now clearly \( \{ F(Z) = 0 \} \cap L \) is defined over \( F \) by the polynomial

\[
f(t) = F(\alpha + \beta t) = (a + a^{-1})^{-1} \text{tr}_\sigma(\alpha + \beta t)\sigma(\alpha + \beta t)\sigma^2(\alpha + \beta t)
\]

\[
= (a + a^{-1})^{-1} \text{tr}_\sigma(\beta\sigma(\beta)\sigma^2(\beta)t^3 + ...) = F(\beta)t^3 + ...
\]

But we know two solutions for \( f(t) \), namely \( t_1 = \rho + \rho^{-1} \) and \( t_2 = \rho^2 + \rho^{-2} \), so we get \( f(t) = F(\beta)(t - t_1)(t - t_2)(t - t_3) \). Now since \( f(t) \) and \( F(\beta)(t - t_1)(t - t_2) \) are defined over \( F \), we get \( t_3 \in F \). Explicitly, \( f(0) = -t_1 t_2 t_3 F(\beta) \) implies \( t_3 = \frac{-f(0)}{t_1 t_2 F(\beta)} = \frac{f(0)}{f(\beta)} = \frac{F(\alpha)}{F(\beta)} \) is in \( F \). Hence we get:

**Theorem 4.7.** The element \( w = (x + x^{-1})^{-1}(\alpha + \beta \frac{F(\alpha)}{F(\beta)}) \in D_0 - F \) satisfies \( w^5 \in F \).

Now we are left with solving for \( \alpha, \beta \) from the two equations

\[
y + y^{-1} = \alpha + \beta (\rho + \rho^{-1}),
\]

\[
\eta y^{-2} + \eta^{-1} y^2 = \tau(y + y^{-1}) = \alpha + \beta(\rho^2 + \rho^{-2}).
\]
Hence
\[
\begin{align*}
\beta &= y + y^{-1} - \eta y^{-2} - \eta^{-1} y^2, \\
\alpha &= y + y^{-1} - \beta (\rho + \rho^{-1}).
\end{align*}
\]

4.3. The general case. We will now show that the above solution for the case of quasi-symbols, where we descend from \(F[\rho + \rho^{-1}]\) to \(F\), is valid for the general case of \(D_5 = \langle \sigma, \tau : \sigma^5 = \tau^2 = 1, \tau \sigma \tau = \sigma^{-1} \rangle\) division algebras, where we need to descend from \(F[\rho] \otimes E(\sigma)\) to \(F\). The situation is the following: we look for a solution to \(c_2(t) = c_1(t) = 0\) where \(c_i(t)\) are as in Section 3 and \(t \in (\beta + \beta^{-1})^{-1} E(\tau)\). Let \(\text{Gal}(F[\rho]/F) = \langle \pi \rangle\); hence \(\text{Gal}(E \otimes F[\rho]/F) = D_5 \times \langle \pi \rangle\) and so after extending scalars to \(F[\rho]\) we want a solution in \((\beta + \beta^{-1})^{-1}(E \otimes F[\rho])^{(\tau) \times (\pi)}\), which will then be defined over \(F\).

Proposition 4.8. We may assume \(v + v^{-1} \in (E \otimes F[\rho])^{(\tau) \times (\pi^2)}\), for \(v\) as in the proof of Theorem 3.2.

Proof. Since \(v = x^r \tau (x)^{-r}\), where \(x\) is any eigenvector of \(\sigma\) with eigenvalue \(\rho\), we may write \(x = \sum_{i=0}^{4} \rho^{-i} \sigma^i (k)\) for \(k \in E^{(\tau) \times (\pi)}\). Now \(\tau(x) = \pi^2 (x)\) and so 
\[
\tau(v) = \tau(x)^r x^{-r} = \pi^2(x)^r x^{-r} = \pi^2(x^r \pi^2(x)^{-r}) = \pi^2(x^r \tau(x)^{-r}) = \pi^2(v)
\]
implying \(\tau(v + v^{-1}) = v + v^{-1}\), hence \(v + v^{-1}\) is in \((E \otimes F[\rho])^{(\tau) \times (\pi^2)}\), as desired. \(\square\)

Now it is clear that after extending scalars to \(F[\rho + \rho^{-1}]\) we have the solution \((\beta + \beta^{-1})^{-1}(v + v^{-1})\), and so we are in the same situation as in the quasi-symbol case; hence the above solution is valid for the general case, too.

References

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