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THE COMPLEX HESSIAN EQUATION WITH INFINITE DIRICHLET BOUNDARY VALUE

NI XIANG AND XIAO-PING YANG

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Abstract. The existence and nonexistence of the Γ-subharmonic solutions for the complex Hessian equations with infinite Dirichlet boundary value are proved in the certain bounded domain in C^n. We calculate the k-Hessian of the radially symmetric function and use radial functions to construct various barrier functions in this paper. Moreover, it is shown that the growth rate conditions are nearly optimal.

1. Introduction

Let Ω be a bounded domain in C^n and let $u \in C^2(\Omega)$ be a real-valued function. Then the complex Hessian of u,

$$H[u](z) = [\frac{\partial^2 u(z)}{\partial z_i \partial \overline{z_j}}]_{n\times n}, \tag{1.1}$$

is an $n\times n$ Hermitian matrix at each point $z \in \Omega$. Let $\lambda(H[u]) = (\lambda_1(z), \dots, \lambda_n(z))$ be all eigenvalues of $H[u](z)$ as a vector in C^n. Then the kth elementary symmetric function σ_k is defined as follows:

$$\sigma_k(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k}. \tag{1.2}$$

In particular,

$$\det H[u] = \sigma_n(\lambda(H[u])), \quad \Delta u = \mathrm{tr}(H[u]) = \sigma_1(\lambda(H[u])). \tag{1.3}$$

It was proved in [8] that $(\sigma_k(\lambda))^{1/k}$ is a symmetric concave strictly increasing function on the symmetric convex cone :

$$\Gamma_k = \{\lambda \in R^n : \sigma_k(\lambda) > 0\}, \tag{1.4}$$

and

$$\Gamma_n = \{\lambda \in R^n : \lambda_i > 0, 1 \le i \le n\}, \quad \Gamma_1 = \{\lambda \in R^n : \sum_{i=1}^n \lambda_i > 0\}. \tag{1.5}$$

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Also Γ_k is symmetric in $\lambda = (\lambda_1, \ldots, \lambda_n)$, which means that if $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k$, then $\widetilde{\lambda} = (\lambda_{i_1}, \ldots, \lambda_{i_n}) \in \Gamma_k$ where $(i_1, i_2, \ldots, i_n)$ is any permutation of $1, 2, \ldots, n$. It follows from [16] that

$$\Gamma^+ = \Gamma_n \subset \cdots \subset \Gamma_{k+1} \subset \Gamma_k \subset \cdots \subset \Gamma_1. \tag{1.6}$$

We say that u is plurisubharmonic in Ω if $\lambda(H[u](z)) \in \overline{\Gamma_n}$, for all $z \in \Omega$; we say that u is subharmonic in Ω if $\lambda(H[u](z)) \in \overline{\Gamma_1}$ for all $z \in \Omega$. We shall let Γ be a convex cone which is symmetric with vertex 0 , so that $\Gamma_n \subset \Gamma \subset \Gamma_1$. Let $M(n, \Gamma)$ be the subset of all $n \times n$ Hermitian matrices H, so that $\lambda(H) \in \Gamma$, where $\lambda(H)$ is a vector in R^n being formed by all eigenvalues of H. So we say that a real-valued function u is Γ-*subharmonic* if $\lambda(H[u](z)) \in \Gamma$ for all $z \in \Omega$.

In this paper, we shall consider the complex Hessian equation

$$\sigma_k(\lambda(H[u](z))) = f(\operatorname{Re} z, \operatorname{Im} z, u, \nabla u) \quad \text{in } \Omega \tag{1.7}$$

with the singular boundary value condition

$$u(z) = +\infty \qquad \text{on } \partial\Omega, \tag{1.8}$$

where $\nabla u(z_0) = p_0$ (seen as a function of $(\operatorname{Re} z, \operatorname{Im} z)$ and $p_0 \in R^{2n}$). A natural class of functions for the solutions to (1.7)-(1.8) is Γ_k-*subharmonic* functions. We shall look for Γ_k-*subharmonic* solutions to (1.7)-(1.8).

Hessian operators have received considerable study. See, for example, [2, 5, 6, 14, 15, 16]. The existence of the classical solution for the Dirichlet problem of the eigenvalues of the real Hessian was proved by Caffarelli, Nirenberg and Spruck in [2]. The results concerning the boundary blow-up problem of Monge-Ampère in [11] were extended by Salani in [13] to some Hessian equations. Recently, Bo Guan and Huai-Yu Jian in [7] studied the real Monge-Ampère equation with infinite boundary value conditions, and obtained the existence and nonexistence theorems by barriers and the maximum principle. Then Huai-Yu Jian in [9] extended the results in [7] to real Hessian operators.

Song-Ying Li in [10] obtained the existence and regularity for the Dirichlet problem for symmetric functions of the eigenvalues of the complex Hessian. The aim of this paper is to extend the main results in [18] to the case $k \in \{1, 2, \ldots, n-1\}$. We obtain the existence and nonexistence of the Γ-subharmonic solutions for the complex Hessian equations with infinite Dirichlet boundary value by calculating the k-Hessian of the radially symmetric function and using radial functions to construct various barrier functions. Moreover, we show that the growth rate conditions are nearly optimal.

Based on the example (Lemma 3.7) constructed in [9], we have the following result, then:

Theorem 1.1. *Let Ω be a bounded domain in C^n. If*

$$0 \le f(\operatorname{Re} z, \operatorname{Im} z, \phi, P) \le M(1 + (\phi^+)^q)(1 + |P|^\gamma) \tag{1.9}$$

where $(z, \phi, P) \in \Omega \times R \times R^{2n}$, for some $q, \gamma \ge 0$, $q + \gamma \le k$, $\phi^+ = \max\{\phi, 0\}$, there is no Γ_k-subharmonic solution for (1.7) *and* (1.8).

Theorem 1.2. *If there are constants $\alpha > 1$ and $M > 0$ such that*

$$f(\operatorname{Re} z, \operatorname{Im} z, \phi, P) \ge M(1 + |P|^k)^\alpha, \tag{1.10}$$

and Ω is a domain containing some ball of radius a such that

$$a > [\frac{(n-1)!(2n-k)(k+1)}{2^{k+1}k!(n-k)!M(\alpha-1)}]^{\frac{1}{k}},$$

then there exists no Γ_k-subharmonic solution for (1.7) *and* (1.8).

We shall deal with the existence of problem (1.7)-(1.8) in the viscosity sense. For the details of viscosity solutions to Hessian equations like (1.7), as for the notion in the viscosity sense, we refer to [17].

Theorem 1.3. *Let Ω be a bounded strictly convex domain in C^n. Suppose that $f \in C^\infty(\overline{\Omega}, R, R^{2n})$ satisfies for all $(z, \phi, P) \in \Omega \times R \times R^{2n}$,*

$$f_\phi(\operatorname{Re} z, \operatorname{Im} z, \phi, P) > 0, \tag{1.11}$$

and there exist $q > k$ and $M > 0$ such that

$$\varphi(\phi)(1 + |P|^k) \geq f(z, \phi, P) \geq M(\phi^+)^q, \tag{1.12}$$

where $\varphi \in C^1(R^n)$ is a positive nondecreasing function satisfying

$$\sup_{\phi \leq 0}(e^{-\epsilon\phi}\varphi(\phi)) < +\infty \tag{1.13}$$

for some $\epsilon > 0$. Also assume

$$f^{\frac{1}{k}}(z, \phi, P) \in C^{1,1}(\Omega, R, R^{2n}) \text{ is positive and convex in } P. \tag{1.14}$$

Then there is a strictly Γ_k-subharmonic solution $u \in C^{0,1}(\Omega)$ to (1.7)-(1.8). *Moreover, there are functions $\underline{h}, \overline{h} \in C(R^+)$ with $\underline{h}(r), \overline{h}(r) \to \infty$ as $r \to 0$, such that*

$$\underline{h}(d(z)) \leq u(z) \leq \overline{h}(d(z)), \quad \forall z \in \Omega, \tag{1.15}$$

where d is the distance function to $\partial\Omega$.

2. Preparation

In this section, we will give a comparison principle, in analogy with the real case (see [7, 9]). Moreover, we will construct some radially symmetric strictly plurisubharmonic functions that will be used as barriers. The proofs in this section will be omitted because they have an analogy to the real case (see [9]).

As the first step, let us start with some notation that we shall use later. Letting $z = (z_1, \ldots, z_n) \in C^n$, we may write z in real coordinates as $z = (t_1, \ldots, t_{2n})$. For the complex variables, $\partial_{i\bar{j}}u = \partial^2 u/(\partial z_i \partial z_{\bar{j}})$, $u_{i\bar{j}} = \partial_{i\bar{j}}u$. A domain $\Omega \subset C^n$ with a smooth boundary $\partial\Omega$ is called strongly pseudoconvex if there exists a C^∞-function r defined on a neighborhood of $\partial\Omega$ such that $dr \neq 0$, and $r < 0$ in Ω; $r = 0$ on $\partial\Omega$; $r > 0$ outside of Ω. r is strictly plurisubharmonic.

Lemma 2.1. *Let $\Omega \subset C^n$ be a bounded domain. If $u, v \in C^\infty(\overline{\Omega})$ are Γ_k-subharmonic, with $u \leq v$ on $\partial\Omega$ and $f_\phi(\operatorname{Re} z, \operatorname{Im} z, \phi, P) > 0$, then $u \leq v$ in Ω.*

Theorem 2.2. *Suppose that $u = +\infty$, $v = +\infty$ on $\partial\Omega$ and u is Γ_k-subharmonic and v is strictly Γ_k-subharmonic in Ω. Let Ω be a bounded domain containing the origin in C^n and let f satisfy*

$$(t_1, \ldots, t_{2n}) \cdot \nabla f(\operatorname{Re} z, \operatorname{Im} z, \phi, \nabla\phi) \leq 0, \quad P \cdot D_P f(\operatorname{Re} z, \operatorname{Im} z, \phi, \nabla\phi) \geq 0$$

where $(z, \phi, P) \in \Omega \times R \times R^{2n}$. If either

$$f(\operatorname{Re} z, \operatorname{Im} z, \mu\phi^+, P) \geq \mu^q f(\operatorname{Re} z, \operatorname{Im} z, \phi, P) \quad \forall \mu \geq 1, \tag{2.1}$$

where $(z,\phi,P)\in\Omega\times R\times R^{2n}$, $\quad q>k$, *or there is* $\epsilon>0$ *such that*

$$f_\phi(\operatorname{Re} z,\operatorname{Im} z,\phi,\nabla\phi)\geq \epsilon f(\operatorname{Re} z,\operatorname{Im} z,\phi,\nabla\phi) \tag{2.2}$$

where $(z,\phi,P)\in\Omega\times R\times R^{2n}$, *then* $u\leq v$ *in* Ω.

Then, we shall construct the functions that will serve as the lower and upper barriers. Let $u(z)=u(|z|)$ be a radially symmetric function. A straightforward calculation shows that

$$\sigma_k(\lambda(H[u](|z|)))=C_{n-1}^{k-1}[\frac{u''}{4}(\frac{u'}{2r})^{k-1}+\frac{2n-k}{2k}(\frac{u'}{2r})^k]. \tag{2.3}$$

Lemma 2.3. *Let* $\eta\in C^1(R)$ *satisfy* $\eta(\phi)>0$, $\eta'(\phi)\geq 0$ *for all* $\phi\in R$. *Then for any* $a>0$ *there is a strictly* Γ_k*-subharmonic radially symmetric function* $v\in C^2(B_a(0))$ *satisfying*

$$\begin{cases}\sigma_k(\lambda(H[v](|z|)))\geq e^v\eta(v)(1+|\nabla v|^k) & in\quad B_a(0),\\ v(0)\leq 0;\ v=+\infty & on\quad \partial B_a(0).\end{cases} \tag{2.4}$$

Remark 2.4. Given a and η, in the sequel we will denote the function $v\in C^2(B_a(0))$ in Lemma 2.3 by $v^{a,\eta}$. We will also write $v^{a,\eta}(z)=v^{a,\eta}(|z|)$, since it is radially symmetric.

By Lemma 2.1 we have:

Lemma 2.5. *Let* $u\in C^2(\Omega)$ *be a strictly* Γ_k*-subharmonic solution of* (1.7) *and* (1.8), *where* Ω *is a domain contained in a ball* $B_a(z_0)$. *Assume that*

$$f(\operatorname{Re} z,\operatorname{Im} z,\phi,P)\leq e^\phi\eta(\phi)(1+|P|^k),\forall(z,\phi,P)\in\overline{\Omega}\times R\times R^{2n},$$

where $\eta\in C^1(R)$ *satisfies* $\eta(\phi)>0$ *and* $\eta'(\phi)\geq 0$. *Then* $u(z)\geq v^{a,\eta}(z-z_0)$ *for all* $z\in\Omega$.

A straightforward calculation shows that when $q>k$ the function

$$w(z):=(1-|z|^2)^{\frac{k+1}{k-q}}$$

is strictly Γ_k-*subharmonic* and satisfies the inequality

$$\sigma_k(\lambda(H[w](|z|)))\leq C(n,k,q)w^q \quad \text{in } B_1(0),$$

where C is a constant only depending on n,k,q. By rescaling, we have:

Lemma 2.6. *Let* $a,M>0$ *and* $q>k$ *and define* $w^{a,M}\in C^\infty(B_a(0))$, $w^{a,M}=+\infty$ *on* $\partial B_a(0)$, *by*

$$w^{a,M}(z):=\mu w(\frac{z}{a}),\quad z\in B_a(0),$$

where

$$\mu=(\frac{C(n,k,q)}{a^{2k}M})^{1/(q-k)}.$$

Then

$$\sigma_k(\lambda(H[w](|z|)))\leq M(w^{a,M})^q \quad \text{in } B_a(0).$$

Immediately from Lemmas 2.5 and 2.1 we have the following lemma:

Lemma 2.7. *Let* $u\in C^2(\Omega)$ *be a strictly* Γ_k*-subharmonic solution of* (1.7). *Suppose that* f *satisfies* (1.11) *with* $q>k$ *and that* Ω *contains a ball* $B_a(z_0)$ *and* $M>0$. *Then* $u(z)\leq w^{a,M}(z-z_0)$ *for all* $z\in B_a(z_0)$.

Note that for any domain Ω and any $z \in \Omega$, the ball $B_{d(z)} \subset \Omega$, where $d(z)$ is the distance function to $\partial\Omega$. Then Lemma 2.6 implies

Remark 2.8. Let $u \in C^2(\Omega)$ be a strictly Γ_k-*subharmonic* solution of (1.7). Suppose f satisfies (1.12) for some $q > k$ and $M > 0$. Then

$$u(z) \le \overline{h}(d(z)), \forall z \in \Omega,$$

where $\overline{h}(r) := w^{r,M}(0) \in C^\infty(R^+)$, where $r > 0$.

3. The proofs of Theorem 1.1 and Theorem 1.2

For the proof of Theorem 1.1, we need to construct subsolutions to (1.7) defined on the whole space C^n when f satisfies (1.9) with $q + \gamma \le k$.

Lemma 3.1. *Assume that $\gamma, q \ge 0$, $\gamma + q \le k$ and $M > 0$. Then there is a strictly Γ_k-subharmonic radially symmetric positive function $\tilde{u} \in C^\infty(C^n)$ satisfying*

$$\sigma_k(\lambda(H[\tilde{u}](|z|))) \ge M(1 + (\tilde{u})^q)(1 + |\nabla\tilde{u}|^\gamma), \quad \forall z \in C^n. \tag{3.1}$$

Proof. We just give the outline. First, for any $p', q' \ge 0, p' + q' \le n$, and $M' > 0$, by Lemma 3.7 in [9], one has $\tilde{u} \in C^\infty(C^n)$ satisfying

$$\sigma_n(\lambda(H[\tilde{u}](|z|))) \ge M'(1 + (\tilde{u})^{p'})(1 + |\nabla\tilde{u}|^{q'}).$$

Second, notice that in [13],

$$[\frac{1}{C_n^k}\sigma_k]^{\frac{1}{k}} \le [\frac{1}{C_n^{k-1}}\sigma_{k-1}]^{\frac{1}{k-1}} \le \cdots \le \frac{1}{n}\sigma_1. \tag{3.2}$$

Hence,

$$C_n^k \sigma_n^{\frac{k}{n}} \le \sigma_k, \quad \forall \lambda \in \Gamma_n. \tag{3.3}$$

Finally, there is a positive constant $C'(n,k) = 2^{\frac{k}{n}-1}$ such that

$$(1+t)^{\frac{k}{n}} \ge C'(1 + t^{\frac{k}{n}}), \quad \forall t \ge 0. \tag{3.4}$$

Then we obtain (3.1). □

Proof of Theorem 1.1. Let $u \in C^2(\Omega)$ be a Γ_k-*subharmonic* solution of (1.7) and (1.8), where Ω is bounded and f satisfies (1.9). Let $\tilde{u} \in C^\infty(C^n)$ satisfy (3.1) in Lemma 3.1. Note that $u - C\tilde{u} = \infty$ on $\partial\Omega$ for any $C > 0$. Since $\tilde{u} > 0$, we can choose $C > 1$ such that

$$u(z_0) - C\tilde{u}(z_0) = \min_{\Omega}(u - C\tilde{u}) < 0,$$

where $z_0 \in \Omega$. Hence $\nabla u(z_0) = C\nabla\tilde{u}(z_0)$ and $H(u) - CH(\tilde{u})(z_0)$ is a positive semi-definite matrix. By (1.9) we have

$$\begin{aligned} \sigma_k(\lambda(H[u](z_0))) &\le M(1 + (u^+(z_0))^q)(1 + |\nabla u(z_0)|^\gamma) \\ &\le M(1 + (C\tilde{u}(z_0))^q)(1 + C|\nabla\tilde{u}(z_0)|^\gamma) \\ &< C^k M(1 + (\tilde{u}(z_0))^q)(1 + |\nabla\tilde{u}(z_0)|^\gamma) \\ &\le C^k \sigma_k(\lambda(H[\tilde{u}](z_0))) \\ &= \sigma_k(C(H[\tilde{u}](z_0))), \end{aligned} \tag{3.5}$$

contradicting the fact that $((u - C\tilde{u})_{i\bar{j}}(z_0))$ is a positive semi-definite matrix. The proof is complete. □

Lemma 3.2. *For any $\alpha > 1$ and $a > 0$, there exists a strictly Γ_k-subharmonic radially symmetric function $\bar{u} \in C^2(B_a(0))$ satisfying*

$$\sigma_k(\lambda(H[\bar{u}])) \leq \frac{(n-1)!(2n-k)(k+1)}{2^{k+1}k!(n-k)!a^k(\alpha-1)}[1+|\nabla \bar{u}|^k]^\alpha, \quad in\ B_a(0) \tag{3.6}$$

and

$$\frac{\partial \bar{u}}{\partial \nu} = +\infty, \quad on\ \partial B_a(0), \tag{3.7}$$

where ν is the unit normal to $\partial B_a(0)$.

Proof. Let

$$\varphi(r) = \begin{cases} \int_0^r [\frac{(1-t^{k+1})^{-1/(\alpha-1)}-1}{t}]^{\frac{1}{k}} dt, & r \in (0,1), \\ 0, & r = 0. \end{cases} \tag{3.8}$$

It is easy to verify that

$$\varphi \in C^2[0,1],\ \varphi(0) = \varphi'(0) = 0,\ \varphi' \geq 0,\ \varphi'' > 0 \text{ in } [0,1),$$

and

$$\begin{aligned} (r(\varphi'(r))^k)' &= (\varphi'(r))^k + kr(\varphi'(r))^{k-1}\varphi''(r) \\ &= \frac{k+1}{\alpha-1} r^k (1 + r(\varphi'(r))^k)^\alpha. \end{aligned} \tag{3.9}$$

Then, let $\bar{u}(z) = a\varphi(a^{-1}|z|), \quad z \in B_a(0)$. So $\bar{u} \in C^2(B_a(0))$ and is strongly Γ_k-subharmonic. A direct calculation yields

(3.10)

$$\begin{aligned} \sigma_k(\lambda(H[\bar{u}](z))) &= \frac{C_{n-1}^{k-1}}{a^k}(\frac{2|z|}{a})^{-k}[\frac{2n-k}{2k}(\varphi'(\frac{|z|}{a}))^k + \frac{|z|}{2a}(\varphi'(\frac{|z|}{a}))^{k-1}(\varphi''(\frac{|z|}{a}))] \\ &= \frac{C_{n-1}^{k-1}}{2^{k+1}ka^k}(\frac{|z|}{a})^{-k}[(2n-k)(\varphi'(\frac{|z|}{a}))^k + k\frac{|z|}{a}(\varphi'(\frac{|z|}{a}))^{k-1}(\varphi''(\frac{|z|}{a}))] \\ &\leq \frac{C_{n-1}^{k-1}(2n-k)(k+1)}{2^{k+1}ka^k(\alpha-1)}[1 + \frac{|z|}{a}(\varphi'(\frac{|z|}{a}))^k]^\alpha \\ &\leq \frac{(n-1)!(2n-k)(k+1)}{2^{k+1}k!(n-k)!a^k(\alpha-1)}[1+|\nabla(\bar{u})(z)|^k]^\alpha \qquad \text{in } B_a(0). \end{aligned}$$

The proof is complete. ☐

Proof of Theorem 1.2. We may assume $\overline{B_a(0)} \subset \Omega$. Suppose to the contrary that there is a Γ_k-*subharmonic* solution $u \in C^2(\Omega)$ of (1.7) and (1.8). Choose a function $\bar{u}$ in Lemma 3.2. Then we have

$$\sigma_k(\lambda(H[\bar{u}](z))) < M(1 + |\nabla(\bar{u}(z))|^k)^\alpha \ \text{ in } \ B_a(0), \tag{3.11}$$

and

$$\frac{\partial(\bar{u}-u)}{\partial \nu} = +\infty \ \text{ on } \partial B_a(0). \tag{3.12}$$

For some $y \in B_a(0)$, we have

$$\bar{u}(y) - u(y) = \min_{B_a(0)} (\bar{u} - u). \tag{3.13}$$

Using the assumption (1.10) and repeating the same arguments as in the proof of Theorem 1.1, we obtain a contradiction. ☐

4. The proof of Theorem 1.3

Proof. We first assume Ω to be smooth. We shall find a solution of (1.7)-(1.8) as required as in Theorem 1.3 by the limit of solution, u_m. For each integer $m \geq 1$, we consider the Dirichlet problem

$$\begin{cases}\sigma_k(\lambda(H[u](z))) = f(\operatorname{Re} z, \operatorname{Im} z, u, \nabla u) & \text{in } \Omega,\\ u = m & \text{on } \partial\Omega.\end{cases} \tag{4.1}$$

By assumption (1.13), we may find a positive nondecreasing function $\eta \in C^\infty$ such that

$$\max_{y \leq \phi} \varphi(y) \leq e^{\epsilon\phi}\eta(\phi).$$

We choose $\epsilon = 1$. Since Ω is bounded, we may choose $r > 0$ large enough such that $\Omega \subset B_r(0)$ and $v^{r,\eta} \leq 1$ on $\partial\Omega$ (see (2.4) in Lemma 2.3 and Remark 2.4). It follows from Lemma 2.1 that for any Γ_k-*subharmonic* solution $u_m \in C^2$ of (4.1), $v^{r,\eta} \leq u \leq m$ in $\overline{\Omega}$. Let

$$C_m^0 = \max\{m, \sup|v^{r,\eta}|\}, \forall m \geq 1.$$

In order to show the existence of (4.1), we want to use results of Guan [6]. By a result of Caffarelli-Kohn-Nirenberg-Spruck in [1], for each m and any constant $C_m > 0$, there is a solution $\underline{u}_m \in C^2(\overline{\Omega})$, plurisubharmonic, which satisfies

$$\begin{cases}\det((\underline{u}_m)_{i\bar{j}}) = C_m(1 + |\nabla \underline{u}_m|^n) & \text{in } \Omega,\\ \underline{u}_m = m & \text{on } \partial\Omega.\end{cases} \tag{4.2}$$

Using this, (3.3) and (3.4), choosing C_m such that $2^{k-1}C_n^k(C_m)^{\frac{k}{n}} \geq \varphi(C_m^0)$, we see that

$$\begin{cases}\sigma_k(\lambda(H[\underline{u}_m](z))) \geq \varphi(C_m^0)(1 + |\nabla \underline{u}_m|^k) & \text{in } \Omega,\\ \underline{u}_m = m & \text{on } \partial\Omega,\end{cases} \tag{4.3}$$

which means that $\underline{u}_m$ is a subsolution of (4.1) for each m, since φ is nondecreasing and $\underline{u}_m \leq m$ in Ω. Then by the subsolution and the assumptions, Theorem 1.2 of Guan [6] implies that problem (4.1) has a unique Γ_k-*subharmonic* solution $u_m \in C^\infty$ for each m. Moreover, $u_m = m < m + 1 = u_{m+1}$ *on* $\partial\Omega$ and assuming (1.11) we have

$$u_m \leq u_{m+1} \quad \forall m \geq 1 \tag{4.4}$$

by Lemma 2.1.

Lemma 4.1. *There is $a > 0$ depending only on Ω and a decreasing sequence $a_m \to a (m \to \infty)$ such that*

$$v^{a_m,\eta}(a - d(z)) \leq u_m(z) \leq \overline{h}(d(z)) \quad \forall z \in \Omega,\ m \geq 1, \tag{4.5}$$

where $d(z)$ is the distance function to $\partial\Omega$.

Proof of Lemma 4.1. In fact, the second inequality in (4.5) follows directly from Remark 2.8. To show the first one, we use the strict convexity of Ω to find the smallest positive number a, such that for any $\bar{z} \in \partial\Omega$, there is a ball $B_a(z_0) \supset \Omega$ satisfying $\overline{\Omega} \cap \partial B_a z_0 = \bar{z}$. Choose $a_1 > a_2 > \cdots > a_m > a_{m+1} > \cdots$, $a_m \to a$ as $m \to \infty$, such that $v^{a_m,\eta}(a) = m$ for each $m \geq 1$. For any $y \in \Omega$, choose $\bar{y} \in \partial\Omega$

and a ball $B_a(z_0)$ such that $d(y) = |y - \bar{y}|$, $\Omega \subset B_a(z_0)$, and $\overline{\Omega} \cap \partial B_a(z_0) = \bar{y}$. Observing that

$$v^{a_m,\eta}(z - z_0) \le v^{a_m,\eta}(a) = m \le u_m(z), \quad \forall z \in \partial\Omega,$$

by Lemma 2.1 we obtain

$$v^{a_m,\eta}(z - z_0) \le u_m(z), \quad \forall z \in \Omega.$$

In particular,

$$v^{a_m,\eta}(a - d(y)) = v^{a_m,\eta}(y - z_0) \le u_m(y).$$

Since $y \in \Omega$ is arbitrary, the first inequality in (4.5) holds. □

Now continuing the proof of Theorem 1.3, we use (4.4) and (4.5) to see that, for each $z \in \Omega$, the limit

$$u(z) = \lim_{m\to\infty} u_m(z)$$

exists and it satisfies

$$v^{a,\eta}(a - d(z)) \le u(z) \le \overline{h}(d(z)), \quad \forall z \in \Omega. \tag{4.6}$$

Moreover, by Theorem 3.1 of [15] and the methods in [5], the convergence is uniform in every compact set $K \subset \Omega$ and $u \in C^{0,1}_{loc}(\Omega)$. By the stability theorem of viscosity solutions under uniform convergence, we see that u is a viscosity Γ_k-*subharmonic* solution of (1.7) and (1.8) which satisfies (1.15) by (4.6).

Finally, suppose that Ω is not smooth. We choose a sequence of smooth strictly pseudoconvex domains

$$\Omega_1 \subset \cdots \subset \Omega_m \subset \cdots \subset \Omega$$

such that

$$\Omega = \bigcup_{m=1}^{\infty} \Omega_m.$$

The rest of the proof is similar to the real case in [9]. We have completed the proof of Theorem 1.3. □

Remark 4.2. The nonexistence results can indicate that the conditions in Theorem 1.3 are nearly optimal.

References

1. L. Caffarelli, J.J. Kohn, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations, II:Complex Monge-Ampère and uniformly elliptic equations*, Comm. Pure. Appl. Math., 38 (1985), 209-252. MR780073 (87f:35097)
2. L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Math., 155(1985), 261-301. MR806416 (87f:35098)
3. S.Y. Cheng and S.-T. Yau, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure. Appl. Math., 33(1980), 507-544. MR575736 (82f:53074)
4. S.Y. Cheng and S.T. Yau, *The real Monge-Ampère equation and affine flat structures*, Proc. 1980 Beijing Symp. on Diff. Geom. and Diff. Equations, Vol. I (S.S. Chern and W.T. Wu, eds.), Science Press, Beijing, 1982, 339-370. MR714338 (85c:53103)
5. K.S. Chou and X. Wang, *A variational theory of the Hessian equation*, Comm. Pure. Applied Math., 54(2001), 1029-1064. MR1835381 (2002e:35072)
6. B. Guan, *The Dirichlet problem for a class of fully nonlinear elliptic equations*, Comm. Partial Differential Equations, 19(1994), 339-416. MR1265805 (95c:35100)
7. B. Guan and H.-Y. Jian, *The Monge-Ampère equation with infinite boundary value*, Pacific Jour. Math., 216(2004), 77-94. MR2094582 (2005f:35100)

8. N.M. Ivochkina, *Description of cones of stability generated by differential operators of Monge-Ampère type*, Mat. Sb., 122(1983), 265-275 (Russian); English transl. Math. USSR Sb., 50(1985), 259-268. MR717679 (85g:35043)
9. Huai-Yu Jian, *Hessian equations with infinite Dirichlet boundary value*, Indiana Univ. Math. J., 55(2006), 1045–1062. MR2244597
10. Song-Ying Li, *On the Dirichlet problem for symmetric function equations of the eigenvalues of the complex Hessian*, Asian J. Math., 8(2004), 87-106. MR2128299 (2006d:32057)
11. J. Matero, *The Bieberbach-Rademacher problem for the Monge-Ampère operator*, Manuscripta Math., 91(1996), 379-391. MR1416719 (97i:35107)
12. R. Osserman, *On the inequality* $\triangle u \geq f(u)$, Pacific J. Math., 7(1957), 1641-1647. MR0098239 (20:4701)
13. P. Salani, *Boundary blow-up problems for Hessian equations*, Manuscripta Math., 96(1998), 281-294. MR1638149 (99e:35071)
14. N.S. Trudinger, *On the Dirichlet problem for Hessian equations*, Acta Math., 175(1995), 151-164. MR1368245 (96m:35113)
15. N.S. Trudinger, *Weak solutions of Hessian equations*, Comm. Partial Diff. Eqns., 22(1997), 1251-1261. MR1466315 (99a:35077)
16. N.S. Trudinger and Xu-Jia Wang, *Hessian measure II*, Annals Math., 150(1999), 597-604. MR1726702 (2001f:35141)
17. J. Urbas, *On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations*, Indiana Univ. Math. J., 39(1990), 355-382. MR1089043 (92h:35074)
18. Ni Xiang and Xiao-Ping Yang, *The complex Monge-Ampère equation with infinite Dirichlet boundary value*, Nonlinear Analysis, to appear.

School of Science, Nanjing University of Science & Technology, Nanjing, People's Republic of China 210094

E-mail address: nixiang_810715@yahoo.com.cn

School of Science, Nanjing University of Science & Technology, Nanjing, People's Republic of China 210094

E-mail address: xpyang@mail.njust.edu.au