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HYPONORMAL TOEPLITZ OPERATORS AND ZEROS OF POLYNOMIALS

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Abstract. The problem of hyponormality for Toeplitz operators with (trigonometric) polynomial symbols is studied. We give a necessary and sufficient condition using the zeros of the analytic polynomial induced by the Fourier coefficients of the symbol.

Let L^p be the Lebesgue space on the unit circle T and let H^p be the corresponding Hardy space for $1 \leq p \leq \infty$. The Toeplitz operator T_ϕ with symbol ϕ in L^∞ is the operator on H^2 defined by $T_\phi f = P(\phi f)$ for f in H^2, where P is the orthogonal projection from L^2 onto H^2. In this paper, we are interested in when T_ϕ is hyponormal.

Two characterizations of the hyponormality of T_ϕ are known as the following :

(I) Suppose ϕ_1 and ϕ_2 are functions in H^2 with $\phi = \phi_1 + \bar{\phi}_2$ in L^∞. Then T_ϕ is hyponormal if and only if there exists a constant c and a function k in H^∞ with $\|k\|_\infty \leq 1$ such that $\phi_2 = c + T_{\bar{k}}\phi_1$.

(II) T_ϕ is hyponormal if and only if there exist two functions k and g in H^∞ such that $\phi = k\bar{\phi} + g$ and $\|k\|_\infty \leq 1$.

The characterization (I) is due to Cowen [1]. Cowen [1] and Zhu [6] used this characterization. (II) is due to Nakazi-Takahashi [4, Lemma 1]. It is easy to prove (II) if we use (I). Nakazi-Takahashi [4] and Hwang-Lee [3] used this one. Hwang-Lee [3] established an explicit and useful criterion using (II) when the symbol ϕ is a trigonometric polynomial. Their criterion involves the zeros of an analytic polynomial induced by the Fourier coefficients of ϕ. On the other hand, Zhu [6] gave a characterization which is related to the coefficients of the analytic polynomial induced by the Fourier coefficients of ϕ, using (I) and a theorem of Schur [5]. In this paper, we give a necessary and sufficient condition which is related to the zeros of an analytic polynomial induced by the Fourier coefficient of ϕ, using (II) and the Carathéodory-Schur interpolation theorem (cf. [3]).

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Theorem 1. *Suppose ϕ is a trigonometric polynomial such that*

$$\phi = \bar{z}^{\ell} \prod_{j=1}^{t} (z - \alpha_j) \prod_{j=1}^{s} (z - \beta_j),$$

where $\ell \geq 1$, $\mid \alpha_j \mid < 1$ and $\mid \beta_j \mid \geq 1$. When $t = 0$ or $s = 0$, we assume that

$$\prod_{j=1}^{t} (z - \alpha_j) = 1 \qquad \text{or} \qquad \prod_{j=1}^{s} (1 - \beta_j) = 1.$$

Let

$$f = \prod_{j=1}^{t} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \qquad \text{and} \qquad h = \prod_{j=1}^{s} \frac{1 - \bar{\beta}_j z}{z - \beta_j}.$$

Then T_ϕ is hyponormal if and only if $2\ell \leq t + s$ and there exists a solution $a_0, \ldots, a_{\ell-1}$ of the linear system of equations

$$f^{(i)}(0) = \sum_{j=0}^{i} (i-1)(i-2)\cdots(i-j+1) a_j h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1)$$

for which the associated lower triangular Toeplitz matrix

$$T(a_0, \ldots, a_{\ell-1}) = \begin{bmatrix} a_0 & & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell-1} & a_{\ell-2} & \cdots & a_0 \end{bmatrix}$$

has $\|T(a_0, \ldots, a_{\ell-1})\| \leq 1$.

Proof. By the characterization (II), T_ϕ is hyponormal if and only if there exists a function K in H^∞ with $\|K\|_\infty \leq 1$ and a function g in H^∞ such that $\phi = K\bar{\phi} + g$. Hence T_ϕ is hyponormal if and only if $2\ell \leq t + s$ by (1) of Corollary 5 in [4] and there exists a function K in H^∞ with $\|K\|_\infty \leq 1$ and a function g in H^∞ such that

$$\bar{z}^{\ell} \prod_{j=1}^{t} (z - \alpha_j) \prod_{j=1}^{s} (z - \beta_j) = K z^{\ell} \prod_{j=1}^{t} (\bar{z} - \bar{\alpha}_j) \prod_{j=1}^{s} (\bar{z} - \bar{\beta}_j) + g.$$

The above equality can be written as follows:

$$f = K z^{2\ell - (t+s)} h + z^{\ell} G,$$

where $G = g / \prod_{j=1}^{t} (1 - \bar{\alpha}_j z) \prod_{j=1}^{s} (1 - \bar{\beta}_j z)$. Since $z^{(t+s)-2\ell}(f - z^{\ell} G) = Kh$,

$$z^{(t+s)-2\ell}(f - z^{\ell} G) \prod_{j=1}^{s} (z - \beta_j) = K \prod_{j=1}^{s} (1 - \bar{\beta}_j z).$$

This implies that K is divisible in H^∞ by $z^{(t+s)-2\ell}$ because $\mid \beta_j \mid \leq 1$. Hence, if $k = z^{2\ell-(t+s)} K$, then k belongs to H^∞ and $f = kh + z^{\ell} G$. Hence T_ϕ is hyponormal if and only if $2\ell \leq t + s$ and there exists a function $k \in H^\infty$ with $\|k\|_\infty \leq 1$ such that

$$f^{(i)}(0) = \sum_{j=0}^{i} {}_iC_j k^{(j)}(0) h^{(i-j)}(0) \quad (0 \leq i \leq \ell - 1),$$

where ${}_iC_j = i!/j!(i-j)!$. Put $k = \sum_{j=0}^{\infty} a_j z^j$. Then $k^{(j)}(0) = j! a_j$ and so

$$f^{(i)}(0) = \sum_{j=0}^{i} (i-1)(i-2)\cdots(i-j+1) a_j h^{(i-j)}(0)$$

for $0 \leq i \leq \ell - 1$. Now the theorem follows from the Carathéodory-Schur interpolation theorem (cf. [2]).

In the characterization (II) of hyponormality, put $\mathcal{E}(\phi) = \{k \in H^\infty \ ; \ \phi = k\bar{\phi} + g, g \in H^\infty, \text{ and } \|k\|_\infty \leq 1\}$. $\mathcal{E}(\phi)$ has been studied and it may contain more than two elements (see [5]). Hence the k in the proof of Theorem 1 may not be unique in general, and so $(a_j)_{j=0}^{\ell}$ may not be unique.

By a result in the previous paper [5, Corollary 5], if $\{1/\bar{\beta}_j\}_{j=1}^{s} \subseteq \{\alpha_j\}_{j=1}^{t}$ (see Theorem 1), then T_ϕ is hyponormal. Here we give a necessary and sufficient condition for hyponormality of T_ϕ in terms of a relation between $\{\alpha_j\}_{j=1}^{t}$ and $\{\beta_j\}_{j=1}^{s}$ when $\ell = 1$ or 2. □

Corollary 1. *Let $\ell = 1$ in Theorem 1. Then T_ϕ is hyponormal if and only if*

$$\prod_{j=1}^{t} |\alpha_j| \times \prod_{j=1}^{s} |\beta_j| \leq 1.$$

When $t = 0$ or $s = 0$, we assume

$$\prod_{j=1}^{t} |\alpha_j| = 1 \qquad \text{or} \qquad \prod_{j=1}^{s} |\beta_j| = 1.$$

Proof. By Theorem 1, T_ϕ is hyponormal if and only if $f(0) = a_0 h(0)$ and $|a_0| \leq 1$. □

Corollary 2. *Let $\ell = 2$ in Theorem 1. Then T_ϕ is hyponormal if and only if there exist constants a_0, a_1 such that $|a_1| \leq 1 - |a_0|^2$ and*

$$\prod_{j=1}^{t} |\alpha_j| \times \prod_{j=1}^{s} |\beta_j| \leq 1$$

$$\sum_{k=1}^{t} \left\{ (1 - |\alpha_k|^2) \prod_{j\neq k}^{t} (-\alpha_j) \right\} = a_0 \sum_{k=1}^{s} \frac{(|\beta_k|^2 - 1)}{\beta_k^2} \prod_{j\neq k}^{s} \left(-\frac{1}{\beta_j}\right) + a_1 \prod_{j=1}^{s} \left(-\frac{1}{\beta_j}\right).$$

If $s = 0$, then

$$\sum_{k=1}^{t} \left\{ (1 - |\alpha_k|^2) \prod_{j\neq k}^{t} (-\alpha_j) \right\} = 1$$

and if $t = 0$, then there exist constants a_0, a_1 such that $|a_1| \leq 1 - |a_0|^2$ and

$$a_0 \sum_{k=1}^{s} (|\beta_k|^2 - 1) \prod_{j\neq k}^{s} \left(-\frac{1}{\beta_j}\right) + a_1 \prod_{j=1}^{s} \left(-\frac{1}{\beta_j}\right) = 0.$$

Proof. By Theorem 1, T_ϕ is hyponormal if and only if $f(0) = a_0 h(0)$, $f'(0) = a_0 h(0) + a_1 h'(0)$, and $|a_1| \leq 1 - |a_0|^2$. □

Corollary 3. *Let $s = 0$ in Theorem 1. Then T_ϕ is hyponormal if and only if $f^{(i)}(0) = a_i (0 \leq i \leq \ell - 1)$ and $\|T(a_0, a_1, \ldots, a_{l-1})\| \leq 1$.*

Proof. By Theorem 1, T_ϕ is hyponormal if and only if $f^{(i)}(0) = a_0$ for $i = 0, 1, 2$ and $\|T(a_0, 0, 0)\| = \mid a_0 \mid \leq 1$. □

Corollary 4. *Let $t = 0$ in Theorem 1. Then T_ϕ is hyponormal if and only if $1 = a_0 h(0), \sum_{j=0}^{i} a_j h^{(i-j)}(0) = 0$ $(1 \leq i \leq \ell - 1)$ and $\|T(a_0, a_1, \ldots, a_{\ell-1}\| \leq 1$.*

Our corollaries are new and different from Examples 6 and 7 in [6]. The author of [6] proved them under some condition $a_2 \neq 0$ in Example 6. Of course, his result is not for zeros of a polynomial.

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References

[1] C. C. Cowen, Hyponormality of Toeplitz operators, Proc. Amer. Math. Soc. 103(1988), 809-812. MR0947663 (89f:47038)

[2] J. A. Ball, I. Gohberg and L. Rodman, Interpolation of Rational Matrix Functions, Operator Theory: Advances and Applications, 45, Birkhäuser-Verlag, Basel, 1990. MR1083145 (92m:47024)

[3] I. S. Hwang and W. Y. Lee, Hyponormality of trigonometric Toeplitz operators, Trans. Amer. Math. Soc. 354(2002), 2461-2474. MR1888332 (2003a:47057)

[4] T. Nakazi and K. Takahashi, Hyponormal Toeplitz operators and extremal problems of Hardy spaces, Trans. Amer. Math. Soc. 338(1993), 753-767. MR1162103 (93j:47040)

[5] I. Schur, Über Potenzreihen die im Innern des Einheitskreises beschränkt sind, J. Reine Angew. Math. 147(1917), 205-232.

[6] K. Zhu, Hyponormal Toeplitz operators with polynomial symbols, Integr. Equat. Oper. Th. 21(1995), 376-381. MR1316550 (95m:47044)

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