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ON THE APPROXIMATION OF ISOLATED EIGENVALUES OF ORDINARY DIFFERENTIAL OPERATORS

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ABSTRACT. We extend a result of Stolz and Weidmann on the approximation of isolated eigenvalues of singular Sturm–Liouville and Dirac operators by the eigenvalues of regular operators.

1. INTRODUCTION

The approximation of isolated eigenvalues of singular ordinary differential operators by the eigenvalues of regular operators is an important and well studied topic since the latter ones can be computed numerically with arbitrary precision. See the recent monograph by Zettl [8] or in particular the recent survey [7] by Weidmann.

While the case of eigenvalues below the essential spectrum is well understood, the case of eigenvalues in essential spectral gaps was only recently solved by Stolz and Weidmann in [3] (see also [4]).

Let τ be either a Sturm–Liouville expression

$$\tau = \frac{1}{r}\Big(-\frac{d}{dx} p \frac{d}{dx} + q\Big), \tag{1}$$

or a Dirac system

$$\tau = \frac{1}{r}\Big(\mathrm{i}\sigma_2 \frac{d}{dx} + q\Big), \tag{2}$$

where $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ is the Pauli matrix, on the interval $I = (a, b)$.

As usual, we will assume the coefficients p^{-1}, q, r are real-valued locally integrable functions with $p, r > 0$ in the Sturm-Liouville case and q, r are real, symmetric 2×2 matrices with $r > 0$ in the Dirac case.

An endpoint a or b is called regular if it is finite and the coefficients are integrable near this endpoint. If both endpoints are regular, we will call τ regular.

Let $\mathfrak{D}(\tau)$ be the maximal domain of definition of τ. Then τ is self-adjoint on $\mathfrak{D}(\tau)$ if it is limit point (l.p.) near both a and b. Otherwise we will impose an additional boundary condition at every endpoint where τ is limit circle (l.c.). In this way we obtain a self-adjoint operator H associated with τ.

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A solution $\psi_a(z,x)$ ($\psi_b(z,x)$) of $\tau\psi = z\psi$ which is square integrable near a (b) and satisfies the boundary condition at a (b) (if any) is called a Weyl solution. Such a solution is unique up to a constant, and it exists at least for $z \in \mathbb{C}\backslash\sigma_{ess}(H)$.

Our aim is to approximate H by regular operators H_n obtained by restricting τ to a finite interval $(a_n, b_n) \subseteq (a,b)$ ([6, Chap. 14]). The case $a_n = a$ or $b_n = b$ is allowed.

Fix functions $u, v \in \mathfrak{D}(\tau)$. Pick $a_n \downarrow a$, $b_n \uparrow b$. Define H_n

$$H_n : \begin{array}{ccl} \mathfrak{D}(H_n) & \to & L^2((a_n,b_n); r\,dx) \\ f & \mapsto & \tau f \end{array}, \tag{3}$$

where

$$\mathfrak{D}(H_n) = \{f \in \mathfrak{D}(\tau_n) | W_{a_n}(u,f) = W_{b_n}(v,f) = 0\}. \tag{4}$$

Then Stolz and Weidmann prove the following

Theorem 1 ([3]). *Define H_n as above with $u = \psi_a(\lambda_a)$ and $v = \psi_b(\lambda_b)$ with $\lambda_a, \lambda_b \in [\lambda_0, \lambda_1]$ (in particular, assume that the corresponding Weyl solutions exist). Let $P_\Omega(H)$ be the spectral projection of H corresponding to the Borel set $\Omega \subseteq \mathbb{R}$.*

If $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H) < \infty$, then the eigenvalues of H in (λ_0,λ_1) are exactly the limits of eigenvalues of H_n which lie in (λ_0,λ_1). The corresponding (one-dimensional) eigenprojections converge in norm.

If $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H) = \infty$, then the eigenvalues of H_n accumulate in (λ_0,λ_1) as $n \to \infty$.

In fact, in [3] this result is only proven in the case $[\lambda_0,\lambda_1] \cap \sigma_{ess}(H) = \emptyset$, and λ_0, λ_1 are not eigenvalues of H. Hence we will provide a proof of this slightly generalized version below.

As pointed out in [3], this result has of course one practical drawback: The Weyl solutions used to generate the boundary conditions of H_n will not be known explicitly in general. To evade this obstacle they show that their result still holds if the Weyl solution of a nearby operator is chosen instead.

The main purpose of this note is to propose an alternate way of approximating H which only involves one Weyl solution at one endpoint. More precisely, we show that if τ is l.p. at one endpoint, the Weyl solution of the other endpoint can also be chosen instead:

Theorem 2. *Suppose τ is l.p. at b. Define H_n as above with $u = \psi_a(\lambda_a)$, $\lambda_a \in [\lambda_0,\lambda_1]$, and $v = \psi_a(\lambda_0)$ or $v = \psi_a(\lambda_1)$ (in particular, we assume that the corresponding Weyl solutions exist).*

If $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H) < \infty$, then the eigenvalues of H in (λ_0,λ_1) are exactly the limits of eigenvalues of H_n which lie in (λ_0,λ_1). The corresponding (one-dimensional) eigenprojections converge in norm.

If $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H) = \infty$, then the eigenvalues of H_n accumulate in (λ_0,λ_1) as $n \to \infty$.

In particular, if one endpoint is regular, a solution satisfying the boundary condition at this endpoint can be taken in this case. Clearly the result cannot hold if τ is l.c. at b, since our assumptions contain no information on the boundary condition of H at b in this case.

Remark 3. (i) As shown in [3], if τ is l.c. at a, then u can be choosen to be any function in $\mathfrak{D}(\tau)$ generating the boundary condition of H at a.

(ii) The same result (with the same proof) holds for Jacobi operators (see [5]).

2. Approximation by regular operators

We begin by recalling that H_n converges to H strongly ([6]). Strictly speaking this statement makes no sense since H_n and H live in different Hilbert spaces. This can be easily fixed by using $\alpha\mathbb{1} \oplus H_n \oplus \alpha\mathbb{1}$ on $L^2((a,b); r\,dx) = L^2((a,b_n); r\,dx) \oplus L^2((a_n,b_n); r\,dx) \oplus L^2((a_n,b); r\,dx)$ where α is a fixed real constant outside $[\lambda_0,\lambda_1]$. Alternatively, one can also use generalized strong convergence as introduced in [3].

Lemma 4. *Suppose that either H is limit point at a or that $u = \psi_-(\lambda_0)$ for some λ_0 and, similarly, that either H is limit point at b or $v = \psi_+(\lambda_1)$ for some λ_1. Then H_m converges to H in strong resolvent sense as $m \to \infty$.*

In addition, we need the following abstract result. In this respect we remark that for a self-adjoint projector P we have

$$\dim \operatorname{Ran}(P) = \operatorname{tr}(P) = \|P\|_1, \tag{5}$$

where $\|.\|_1$ denotes the trace class norm. If P is not finite rank, then it is of course not trace class, and all three numbers are equal ∞.

Lemma 5. *Let A_n, A be self-adjoint operators such that $A_n \to A$ in strong resolvent sense. Then*

$$\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A)) \le \liminf \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A_n)). \tag{6}$$

If in addition, $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A_n)) \le \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A))$, *then*

$$\lim_{n\to\infty} \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A_n)) = \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A)) \tag{7}$$

and if $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A)) < \infty$ *we even have*

$$\lim_{n\to\infty} \|P_{(\lambda_0,\lambda_1)}(A_n) - P_{(\lambda_0,\lambda_1)}(A)\|_1 = 0. \tag{8}$$

Proof. The first part is just Lemma 5.2 from [1]. This also implies the second if $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A)) = \infty$. Otherwise, if $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A)) < \infty$, we have

$$\limsup \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A_n)) \le \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(A))$$

and the first claim follows. The second is then a consequence of Grümm's theorem ([2, Thm. 2.19]). □

Now it remains to show that this result is applicable in our situation.

Lemma 6. *Let H_n be defined as in* (3) *with*

(i) $u = \psi_a(\lambda_a)$, $v = \psi_b(\lambda_b)$ *with* $\lambda_a, \lambda_b \in [\lambda_0, \lambda_1]$ *or*

(ii) $u = \psi_a(\lambda_a)$ *with* $\lambda_a \in [\lambda_0,\lambda_1]$ *and* $v = \psi_a(\lambda_b)$ *with* $\lambda_b \in \{\lambda_0, \lambda_1\}$.

Then,

$$\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H_n)) \le \operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H)). \tag{9}$$

Proof. Abbreviate $P = P_{(\lambda_0,\lambda_1)}(H)$, $P_n = P_{(\lambda_0,\lambda_1)}(H_n)$.

(i) Since this part is identical to the proof in [3], we just give an outline. Let $\tilde\psi_1, \dots, \tilde\psi_k \in \operatorname{Ran} P_n$ be the normalized eigenfunctions of H_n, and construct

$$\psi_j(x) = \begin{cases} \gamma_{a,j} u(x), & x < a_n, \\ \tilde\psi_j(x), & a_n \le x \le b_n, \\ \gamma_{b,j} v(x), & x > b_n, \end{cases}$$

where $\gamma_{a,j}$, $\gamma_{b,j}$ are chosen such that $\psi_j \in \mathfrak{D}(\tau)$. A computation now shows that

$$\|(H - \frac{\lambda_1 + \lambda_0}{2})\psi\| < \frac{\lambda_1 - \lambda_0}{2}\|\psi\|$$

for any ψ in the linear span of the ψ_j's, which yields the first result.

(ii) By considering two steps from (a_n, b_n) to (a, b_n) and from (a, b_n) to (a, b), we see that the first step is covered by (i), and hence it is no restriction to assume $a_n = a$. Now proceed as in the previous case but use

$$\psi_j(x) = \begin{cases} \tilde{\psi}_j(x) - \gamma_j v(x), & x \le b_n, \\ 0, & x > b_n, \end{cases}$$

where γ_j are chosen such that $\psi_j \in \mathfrak{D}(\tau)$. Now let $\psi = \sum_j c_j \psi_j$ be in the linear span of the ψ_j's. Then, since v is also an eigenvector of H_n and hence orthogonal to the ψ_j's, we have

$$\begin{aligned} \|(H - \frac{\lambda_1 + \lambda_0}{2})\psi\|^2 &= \|\sum_j c_j \Big(\frac{2\lambda_j - \lambda_1 - \lambda_0}{2}\tilde{\psi}_j(x) - \frac{2\lambda_b - \lambda_1 - \lambda_0}{2}\gamma_j v(x)\Big)\|^2 \\ &= \sum_j |c_j|^2 (\frac{2\lambda_j - \lambda_1 - \lambda_0}{2})^2 + |\gamma|^2 (\frac{\lambda_1 - \lambda_0}{2})^2 \|v\|^2_{(a,b_n)} \\ &< (\frac{\lambda_1 - \lambda_0}{2})^2 \|\psi\|^2, \end{aligned}$$

where $\gamma = \sum_j c_j \gamma_j$. Hence the second result follows. □

Theorem 1 and Theorem 2 now follow by combining the last two lemmas.

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