A FORMULA FOR THE EULER CHARACTERISTICS
OF EVEN DIMENSIONAL TRIANGULATED MANIFOLDS

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Abstract. An alternative formula for the Euler characteristics of even dimensional triangulated manifolds is deduced from the generalized Dehn-Sommerville equations.

A finite simplicial complex $K$ is called an\ Eulerian\ manifold\ (or\ a\ semi-Eulerian\ complex\ in\ the\ literature)\ if\ all\ of\ the\ maximal\ faces\ have\ the\ same\ dimension\ and, for\ every\ nonempty\ face\ $\sigma \in K$,

$$\chi(\text{Lk} \sigma) = \chi(S^{\dim K - \dim \sigma - 1})$$

holds, where $\text{Lk} \sigma$ is the link of $\sigma$ in $K$ and $S^n$ is the $n$-dimensional sphere. Note that $K$ is not necessarily connected. Any triangulation of a closed manifold is an Eulerian manifold. More generally, a triangulation of a homology manifold without boundary provides an Eulerian manifold. The purpose of this short note is to prove the following alternative formula for the Euler characteristics of even dimensional Eulerian manifolds.

Theorem 1. Let $K$ be a $2m$-dimensional Eulerian manifold. Then

$$\chi(K) = \sum_{i=0}^{2m} \left( -\frac{1}{2} \right)^i f_i(K)$$

holds, where $f_i(K)$ is the number of $i$-simplices of $K$.

A finite simplicial complex $L$ is called a flag complex if every collection of vertices of $L$ which are pairwise adjacent spans a simplex of $L$. The formula (1) was proved in [1] under the additional assumptions that $K$ is a PL-triangulation of a closed $2m$-manifold and is a flag complex. M. W. Davis pointed out that the formula (1) follows from a result in [3], provided $K$ is a flag complex (see Note added in proof in [1]). Both results follow from the considerations of the Euler characteristics of Coxeter groups. In this note, we deduce formula (1) from the generalized Dehn-Sommerville equations proved by Klee [4].

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2571
Let $K$ be a finite $(d - 1)$-dimensional simplicial complex and $f_i = f_i(K)$ the number of $i$-simplices of $K$ as before. The $d$-tuple $(f_0, f_1, \ldots, f_{d-1})$ is called the $f$-vector of $K$. The $f$-polynomial $f_K(t)$ of $K$ is defined by

$$f_K(t) = t^d + f_0t^{d-1} + \cdots + f_{d-2}t + f_{d-1}.$$  

Define the $h$-polynomial $h_K(t)$ of $K$,

$$h_K(t) = h_0t^d + h_1t^{d-1} + \cdots + h_{d-1}t + h_d,$$

by the rule $h_K(t) = f_K(t-1)$. The $(d+1)$-tuple $(h_0, h_1, \ldots, h_d)$ is called the $h$-vector of $K$. The $h$-vector of $K$ satisfies the generalized Dehn-Sommerville equations, as stated below in Theorem 2.

**Theorem 2** ([1]). Let $K$ be a $(d - 1)$-dimensional Eulerian manifold. Then

$$h_{d-i} - h_i = (-1)^i \binom{d}{i}(\chi(K) - \chi(S^{d-1}))$$

holds for all $i$ $(0 \leq i \leq d)$.

**Remark.** Klee stated the generalized Dehn-Sommerville equations in terms of the $h$-vector rather than the $f$-vector. The formulae quoted in Theorem 2 are equivalent to those in [4] and can be found in [5]. Theorem 2 was also proved in [2] by a quite different method, provided that $K$ is a triangulation of a closed manifold.

Now we prove Theorem 1. We have

$$h_K(-1) = \sum_{i=0}^{2m+1} (-1)^{2m+1-i}h_i = \sum_{i=0}^{m} (-1)^i(h_{2m+1-i} - h_i).$$

Now Theorem 2 asserts that

$$h_{2m+1-i} - h_i = (-1)^i \binom{2m+1}{i}(\chi(K) - 2).$$

Hence we obtain

(2) \hspace{1cm} h_K(-1) = (\chi(K) - 2) \sum_{i=0}^{m} \binom{2m+1}{i} = 2^{2m}(\chi(K) - 2).

On the other hand, we have

(3) \hspace{1cm} f_K(-2) = (-2)^{2m+1} + \sum_{i=0}^{2m} (-2)^{2m-i}f_i = 2^{2m} \left(-2 + \sum_{i=0}^{2m} \left(-\frac{1}{2}\right)^i f_i\right).

Since $h_K(-1) = f_K(-2)$ by the definition of the $h$-polynomial $h_K(t)$, Theorem 1 follows from (2) and (3).

**References**


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