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# CUT-AND-STACK SIMPLE WEAKLY MIXING MAP WITH COUNTABLY MANY PRIME FACTORS

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Abstract. Via the cut-and-stack construction we produce a 2-fold simple weakly mixing transformation which has countably many proper factors and all of them are 2-to-1 and prime.

## 0. Introduction

Let $T$ be an (invertible) transformation of a probability space $(X, \mathfrak{B}, \mu)$. A measure $\lambda$ on $X \times X$ is called a 2-*fold self-joining* of $T$ if it is $T \times T$-invariant and it projects onto $\mu$ on both coordinates. Denote by $J_2^e(T)$ the set of all ergodic 2-fold self-joinings of $T$. Let $C(T)$ stand for the centralizer of $T$, i.e. the group of all $\mu$-preserving transformations commuting with $T$. Given $S \in C(T)$, we let $\mu_S(A \times B) := \mu(A \cap SB)$ for all $A, B \in \mathfrak{B}$. Of course, $\mu_S \in J_2^e(T)$. If $J_2^e(T) \subset \{\mu_S \mid S \in C(T)\} \cup \{\mu \times \mu\}$, then $T$ is called 2-*fold simple* [Ve], [dJR]. By a *factor* of $T$ we mean a $T$-invariant sub-$\sigma$-algebra of $\mathfrak{B}$. If $T$ has no non-trivial proper factors, then $T$ is called *prime*. In [Ve] it was shown that if $T$ is 2-fold simple, then for each non-trivial factor $\mathfrak{F}$ of $T$ there exists a compact (in the strong operator topology) subgroup $K_{\mathfrak{F}} \subset C(T)$ such that

$$\mathfrak{F} = \{A \in \mathfrak{B} \mid \mu(kA \triangle A) = 0 \text{ for all } k \in K_{\mathfrak{F}}\}.$$

Hence $\mathfrak{F}$ (or, more precisely, the restriction of $T$ to $\mathfrak{F}$) is prime if and only if $K_{\mathfrak{F}}$ is a maximal compact subgroup of $C(T)$.

The purpose of our paper is to produce via cutting-and-stacking a 2-fold simple weakly mixing transformation which has countably many factors and all of them are prime. We also specify which of these factors are conjugate.

Note that the only known example of a 2-fold simple $T$ with non-unique prime factors was constructed by Glasner and Weiss [GlW] as an inverse limit of certain horocycle flows, i.e. in a quite different way. The subtle results of M. Ratner on joinings of horocycle flows [Ra], as well as the existence of a lattice in $SL_2(\mathbb{R})$ with rather special properties, play a crucial role in [GlW]. We also notice that $T$ has many non-prime factors as well. Note that for some time it was not obvious at all whether it was possible to construct such an example by means of the more elementary cutting-and-stacking technique (see [Th]). To achieve this purpose we apply a method of *auxiliary bigger group actions*: we construct a rank-one action

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of an auxiliary group $G = \mathbb{Z} \times (\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z})$ such that the $\mathbb{Z}$-subaction is 2-fold simple and has centralizer coinciding with the full $G$-action. It remains to list all non-trivial finite subgroups of $G$:

$$\{\{(0, b, 1), (0, 0, 0)\} \mid b \in \mathbb{Z}, b \neq 0\}$$

and note that all of them are maximal. While constructing this action we follow the $(C, F)$-formalism developed in [Da4, Section 6].

For other recent applications of the method of auxiliary bigger group actions in the theory of simple and quasi-simple actions, root problems, spectral theory, etc., we refer to [dJ], [Ma], [Da3], [Da4], [Da5].

## 1. $(C, F)$-construction

We recall here the $(C, F)$-construction of funny rank-one actions (see [Da1]–[Da4], [DaS] and [dJ] for details). Let $G$ be a countable group. Given a finite subset $F \subset G$, we denote by $\lambda_F$ the probability equidistributed on $F$. Now let $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ be two sequences of finite subsets in $G$ such that the following are satisfied:

$$F_0 = \{0\},\ \#C_n > 1, \tag{1.1}$$

$$F_n C_{n+1} \subset F_{n+1}, \tag{1.2}$$

$$F_n c \cap F_n c' = \emptyset \text{ for all } c \neq c' \in C_{n+1}, \tag{1.3}$$

$$\lim_{n\to\infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} < \infty. \tag{1.4}$$

We put $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$ and define a map $i_n : X_n \to X_{n+1}$ by setting

$$i_n(f_n, d_{n+1}, d_{n+2}, \dots) := (f_n d_{n+1}, d_{n+2}, \dots).$$

In view of (1.1), $X_n$ endowed with the infinite product topology is a compact Cantor space. It follows from (1.2) and (1.3) that $i_n$ is well defined and it is a topological embedding of $X_n$ into $X_{n+1}$. Denote by $X$ the topological inductive limit of the sequence $(X_n, i_n)_{n=1}^{\infty}$. In the sequel we will suppress the canonical embedding maps and just write $X = \bigcup_{n\geq 0} X_n$ with $X_0 \subset X_1 \subset \cdots$. Clearly, $X$ is a locally compact Polish totally disconnected space without isolated points. We define a finite measure $\mu_n$ on $X_n$ by setting

$$\mu_n := \alpha_n(\lambda_{F_n} \times \lambda_{C_{n+1}} \times \lambda_{C_{n+2}} \times \cdots),$$

where $\alpha_n$ is a positive coefficient such that $\alpha_0 := 1$ and $\alpha_{n+1} := \alpha_n \frac{\#F_{n+1}}{\#F_n \#C_{n+1}}$. The latter 'matching' condition yields that $\mu_{n+1} \restriction X_n = \mu_n$. Hence there exists a unique $\sigma$-finite measure $\mu$ on the standard Borel $\sigma$-algebra $\mathfrak{B}$ of $X$ generated by the topology such that $\mu \restriction X_n = \mu_n$. In particular, $\mu(X_n) = \alpha_n$ for all $n \geq 0$. It is easy to check that $\mu(X) < \infty$ if and only if (1.4) holds. After a normalization (i.e. by an appropriate change of $\alpha_0$) we may assume that $\mu(X) = 1$. Suppose also that the following is satisfied:

$$\text{for any } g \in G, \text{ there exists } m \geq 0 \text{ with } gF_n C_{n+1} \subset F_{n+1} \text{ for all } n \geq m. \tag{1.5}$$

For such $n$, take any $x \in X_n \subset X$ and write the expansion $x = (f_n, c_{n+1}, c_{n+2}, \dots)$ with $f_n \in F_n$ and $c_{n+i} \in C_{n+i}$, $i > 0$. Then we let

$$T_g x := (g f_n c_{n+1}, c_{n+2}, \dots) \in X_{n+1} \subset X.$$

It follows from (1.5) that $T_g$ is a well defined homeomorphism of $X$. Moreover, $T_gT_{g'} = T_{gg'}$, i.e. $T := (T_g)_{g\in G}$ is a topological action of $G$ on $X$.

**Definition 1.1.** We call $(X, \mathfrak{B}, \mu, T)$ the $(C, F)$-action of $G$ associated to the sequence $(F_n, C_{n+1})_{n=0}^{\infty}$ (cf. [dJ], [Da1], [Da4], [DaS]).

We list without proof several properties of $T$. They can be easily verified by the reader.

- $T$ is a minimal uniquely ergodic (i.e. strictly ergodic) free action of $G$.
- Two points $x = (f_n, c_{n+1}, c_{n+2}, \dots)$ and $x' = (f'_n, c'_{n+1}, c'_{n+2}, \dots) \in X_n$ are $T$-orbit equivalent if and only if $c_i = c'_i$ eventually (i.e. for all large enough $i$). Moreover, $x' = T_g x$ if and only if
$$g = \lim_{i\to\infty} f'_n c'_{n+1} \cdots c'_{n+i} c_{n+i}^{-1} \cdots c_{n+1}^{-1} f_n^{-1}.$$

For each $A \subset F_n$, we let $[A]_n := \{x = (f_n, c_{n+1}, \dots) \in X_n \mid f_n \in A\}$ and call it an *$n$-cylinder*. The following holds:
$$\begin{aligned}
&[A]_n \cap [B]_n = [A \cap B]_n, \text{ and } [A]_n \cup [B]_n = [A \cup B]_n,\\
&[A]_n = \bigsqcup_{c\in C_{n+1}} [Ac]_{n+1},\\
&T_g[A]_n = [gA]_n \text{ if } gA \subset F_n,\\
&\mu([Ac]_{n+1}) = \frac{1}{\#C_{n+1}} \mu([A]_n) \text{ for any } c \in C_{n+1},\\
&\mu([A]_n) = \mu(X_n)\lambda_{F_n}(A).
\end{aligned}$$

Moreover, for each measurable subset $B \subset X$,
$$\lim_{n\to\infty} \min_{A\subset F_n} \mu\Bigl(B \triangle \bigsqcup_{a\in A} T_a[0]_n\Bigr) = 0. \tag{1.6}$$

This means that $T$ has *funny rank one* (see [Fe] for the case of $\mathbb{Z}$-actions and [So] for the general case).

## 2. Main result

Let $G = \mathbb{Z} \times \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ with multiplication as follows:
$$(n, m, a)(n', m', a') = (n + n', m + (-1)^a m', a + a').$$

Then the center $C(G)$ of $G$ is $\mathbb{Z} \times \{0\} \times \{0\}$. Each finite subgroup of $G$ coincides with $G_b := \{(0, b, 1), (0, 0, 0)\}$ for some $b \in \mathbb{Z}$. Notice also that $G_b$ is a maximal finite subgroup of $G$ if $b \neq 0$.

Let $H := \mathbb{Z}^2$. Given $a > 0$, we denote by $I[a]$ the symmetric interval $\{m \in \mathbb{Z} \mid |m| < a\}$. We also set $I_+[a] := I[a] \cup \{a\}$. The Cartesian square of $I[a]$ and $I_+[a]$ are denoted by $I^2[a]$ and $I_+^2[a]$, respectively. Let $(r_n)_{n=0}^{\infty}$ be an increasing sequence of positive integers such that
$$\lim_{n\to\infty} n^4/r_n = 0. \tag{2.1}$$

Below—just after Lemma 2.1—one more restriction on the growth of $(r_n)_{n=0}^{\infty}$ will be imposed. We recurrently define two other sequences $(a_n)_{n=0}^{\infty}$ and $(\widetilde{a}_n)_{n=0}^{\infty}$ by setting
$$a_0 = \widetilde{a}_0 = 1,\ a_{n+1} := \widetilde{a}_n(2r_n - 1),\ \widetilde{a}_{n+1} := a_{n+1} + (2n + 1)\widetilde{a}_n.$$

For each $n \in \mathbb{N}$, we let

$$H_n := I^2[r_n],\ F_n := I_+^2[a_n] \times \mathbb{Z}/2\mathbb{Z},\ \widetilde{F}_n := I_+^2[\widetilde{a}_n] \times \mathbb{Z}/2\mathbb{Z} \text{ and}$$
$$S_n := I_+^2[(2n-1)\widetilde{a}_{n-1}] \times \mathbb{Z}/2\mathbb{Z}.$$

We also consider a homomorphism $\phi_n : H \to G$ given by

$$\phi_n(i,j) := (2i\widetilde{a}_n, 2j\widetilde{a}_n, 0).$$

We then have

$$S_n \subset F_n, \qquad F_nS_n = S_nF_n \subset \widetilde{F}_n \subset G, \tag{2.2}$$

$$F_{n+1} = \bigsqcup_{h \in H_n} \widetilde{F}_n \phi_n(h) = \bigsqcup_{h \in H_n} \phi_n(h)\widetilde{F}_n \text{ and} \tag{2.3}$$

$$S_n = \bigsqcup_{h \in I^2[n]} \widetilde{F}_{n-1}\phi_{n-1}(h). \tag{2.4}$$

Now fix a sequence $\epsilon_n \to 0$ as $n \to \infty$. For any two finite sets $A, B$ and a map $\phi : A \to B$, the probability $\frac{1}{\#A}\sum_{a\in A}\delta_{\phi(a)}$ on $B$ will be denoted by $\mathrm{dist}_{a\in A}\phi(a)$. Given two measures $\kappa, \rho$ on a finite set $B$, we let $\|\kappa - \rho\|_1 := \sum_{b\in B}|\kappa(b) - \rho(b)|$. Given $h = (h_1, h_2) \in H$, we let $h^* := (h_1, -h_2)$.

**Lemma 2.1.** *If $r_n$ is sufficiently large, then there exists a map $s_n : H_n \to S_n$ such that*

(i) *for any $D \geq n^{-2}r_n$,*

$$\|\mathrm{dist}_{t\in I[D]\times\{0\}}(s_n(h+t), s_n(h'+t)) - \lambda_{S_n} \times \lambda_{S_n}\|_1 < \epsilon_n$$

*whenever $h \neq h' \in H_n$ with $\{h, h'\} + (I[D] \times \{0\}) \subset H_n$ and*

(ii) *$s(h) = s(h^*)$ for all $h \in H_n$.*

We skip the proof of Lemma 2.1 since it is the same as that of Lemma 3.1 of [dJ].

From now on we will assume that $r_n$ is large so that the conclusion of Lemma 2.1 is satisfied. For every $n \in \mathbb{N}$, we fix a map $s_n$ whose existence is asserted in the lemma. Without loss of generality we may assume that the following *boundary condition* holds:

$$s_n(i, r_n - 1) = s_n(i, 1 - r_n) = s_n(r_n - 1, i) = s_n(1 - r_n, i) = 0 \text{ for all } i \in I[r_n]. \tag{2.5}$$

Now we define a map $c_{n+1} : H_n \to F_{n+1}$ by setting $c_{n+1}(h) := s_n(h)\phi_n(h)$. We also put $C_{n+1} := c_{n+1}(H_n)$. It is easy to derive from (2.2) and (2.3) that (1.1)–(1.3) are satisfied for the sequence $(F_n, C_{n+1})_{n=0}^{\infty}$. Moreover,

$$\frac{\#F_{n+1}}{\#F_n\#C_{n+1}} = \frac{a_{n+1}^2}{a_n^2(2r_n-1)^2} = \frac{\widetilde{a}_n^2}{a_n^2} = \left(1 + \frac{(2n-1)\widetilde{a}_{n-1}}{a_n}\right)^2 = \left(1 + \frac{2n-1}{2r_{n-1}-1}\right)^2.$$

From this and (2.1) we deduce that (1.4) holds. Moreover, (2.5) implies (1.5). Thus the conditions (1.1)–(1.5) are all satisfied for $(F_n, C_{n+1})_{n=0}^{\infty}$. Hence the associated $(C, F)$-action $T = (T_g)_{g\in G}$ of $G$ is well defined on a standard probability space $(X, \mathfrak{B}, \mu)$.

We now state the main result of the paper.

**Theorem 2.2.** *The transformation $T_{(1,0,0)}$ is weakly mixing and 2-fold simple. All non-trivial proper factors of $T_{(1,0,0)}$ are 2-to-1 and prime. They are as follows:*

$\mathfrak{F}_{G_b}$, $b \in \mathbb{Z} \setminus \{0\}$. *Two factors* $\mathfrak{F}_{G_b}$ *and* $\mathfrak{F}_{G_{b'}}$ *are isomorphic if and only if* $b$ *and* $b'$ *have the same parity.*

To prove Theorem 2.2 we will need an auxiliary lemma. Denote by $G^0$ the subgroup $\mathbb{Z}\times\mathbb{Z}\times\{0\}$ of index 2 in $G$. We now set $\widetilde{F}_n^0 := \widetilde{F}_n \cap G^0$ and $\widetilde{F}_n^1 := \widetilde{F}_n \setminus \widetilde{F}_n^0$.

**Lemma 2.3.** *Let* $f = f'\phi_{n-1}(h)$ *with* $f' \in \widetilde{F}_{n-1}$ *and* $h \in H$.

(i) *Then we have*

$$\begin{aligned} L &:= \widetilde{F}_{n-1}^{\alpha}\phi_{n-1}(h + I^2[n-1]) \sqcup \widetilde{F}_{n-1}^{\beta}\phi_{n-1}(h^* + I^2[n-1]) \\ &\subset fS_n \subset \widetilde{F}_{n-1}^{\alpha}\phi_{n-1}(h + I^2[n+1]) \sqcup \widetilde{F}_{n-1}^{\beta}\phi_{n-1}(h^* + I^2[n+1]), \end{aligned}$$

*where* $\{\alpha,\beta\} = \{0,1\}$ *and* $\alpha = 1$ *if and only if* $f \in G^0$. *Hence*

$$\frac{\#(fS_n \triangle L)}{\#S_n} \leq \frac{8}{n-2}.$$

(ii) *If, in addition,* $fS_n \subset F_n$, *then for any subset* $A \subset F_{n-1}$,

$$\frac{\#(AC_n \cap fS_n)}{\#S_n} = \lambda_{F_{n-1}}(A) \pm \frac{10}{n}.$$

*Proof.* (i) Suppose that $f \in G^0$ (the case $f \notin G^0$ is considered in a similar way). We have

$$\begin{aligned} fS_n &= f'\phi_{n-1}(h)\widetilde{F}_{n-1}\phi_{n-1}(I^2[n]) \\ &= f'\widetilde{F}_{n-1}^0\phi_{n-1}(h + I^2[n]) \sqcup f'\widetilde{F}_{n-1}^1\phi_{n-1}(h^* + I^2[n]). \end{aligned}$$

For each $u = (u_1, u_2) \in \mathbb{Z}^2$, we let $|u| := \max(|u_1|, |u_2|)$. Since $\widetilde{F}_{n-1}^0\widetilde{F}_{n-1}^{\alpha} \subset \bigsqcup_{|u|\leq 1} \widetilde{F}_{n-1}^{\alpha}\phi_{n-1}(u)$, there exists a partition of $\widetilde{F}_{n-1}^{\alpha}$ into subsets $A_u^{\alpha}$, $|u| \leq 1$, such that $f'A_u^{\alpha} \subset \widetilde{F}_{n-1}^{\alpha}\phi_{n-1}(u)$ for any $u$ and $\alpha = 0, 1$. Therefore

$$fS_n = \bigsqcup_{|u|\leq 1} (f'A_u^0\phi_{n-1}(u)^{-1}\phi_{n-1}(u+h+I^2[n]) \sqcup f'A_u^1\phi_{n-1}(u)^{-1}\phi_{n-1}(u+h^*+I^2[n])).$$

It remains to notice that $\bigsqcup_{|u|\leq 1} f'A_u^{\alpha}\phi_{n-1}(u)^{-1} = \widetilde{F}_{n-1}^{\alpha}$.

(ii) Since $fS_n \subset F_n$ and $F_n = \bigsqcup_{k\in H_{n-1}} \widetilde{F}_{n-1}\phi_{n-1}(k)$, it follows from (i) that the subsets $K := h + I^2[n-1]$ and $K^* := h^* + I^2[n-1]$ are contained in $H_{n-1}$. Therefore

$$\begin{aligned} \frac{\#(AC_n \cap fS_n)}{\#S_n} &= \frac{\sum_{k\in H_{n-1}} \#(As_{n-1}(k)\phi_{n-1}(k) \cap L) \pm \frac{8\#S_n}{n-2}}{\#S_n} \\ &= \sum_{k\in K} \frac{\#(As_{n-1}(k) \cap \widetilde{F}_{n-1}^{\alpha})}{\#S_n} + \sum_{k\in K^*} \frac{\#(As_{n-1}(k) \cap \widetilde{F}_{n-1}^{\beta})}{\#S_n} \pm \frac{8}{n-2}. \end{aligned}$$

Since $s_{n-1}(k) = s_{n-1}(k^*)$ for all $k \in K$ by Lemma 2.1(ii), we obtain that

$$\begin{aligned}\frac{\#(AC_n \cap fS_n)}{\#S_n} &= \sum_{k\in K} \frac{\#(As_{n-1}(k) \cap (\widetilde{F}^{\alpha}_{n-1} \cup \widetilde{F}^{\beta}_{n-1}))}{\#S_n} \pm \frac{8}{n-2} \\ &= \frac{\#A \cdot \#K}{\#S_n} \pm \frac{8}{n-2} \\ &= \frac{\#A}{\#F_{n-1}} \cdot \frac{\#F_{n-1}}{\#\widetilde{F}_{n-1}} \cdot \frac{\#K\#\widetilde{F}_{n-1}}{\#S_n} \pm \frac{8}{n-2} \\ &= \lambda_{F_{n-1}}(A)\left(1 \pm \frac{1}{n}\right)\frac{\#I^2[n-1]}{\#I^2[n]} \pm \frac{8}{n-2}.\end{aligned}$$

□

Now we are ready to prove the first half of the first claim of Theorem 2.2.

**Proposition 2.4.** *The transformation $T_{(1,0,0)}$ is weakly mixing.*

*Proof.* Let $h_0 := (1, 0) \in H$ and $g_n := \phi_n(h_0)$. Since $g_n = (1, 0, 0)^{2\widetilde{a}_n}$, it suffices to show that the sequence $(g_n)_{n=1}^{\infty}$ is *mixing* for $T$, i.e.

$$\lim_{n\to\infty} \mu(T_{g_n} D \cap D') = \mu(D)\mu(D')$$

for every pair of measurable subsets $D, D' \subset X$. Take any $A, B \subset F_n$. Since $g_n \in C(G)$ for all $n \in \mathbb{N}$, we have

$$g_nAc_{n+1}(h) = As_n(h)\phi_n(h_0 + h) = As_n(h)s_n(h_0 + h)^{-1}c_{n+1}(h_0 + h)$$

whenever $h, h_0h \in H_n$. We set $F'_n := F_n \cap F_nS_nS_n^{-1}$, $A' := A \cap F'_n$, $B' := B \cap F'_n$, $H'_n := H_n \cap (h_0^{-1}H_n)$. Then

$$\begin{aligned}\mu(T_{g_n}[A]_n \cap [B]_n) &= \mu(T_{g_n}[A']_n \cap [B']_n) \pm 2\mu([F_n \setminus F'_n]_n) \\ &= \sum_{h\in H_n} \mu(T_{g_n}[A'c_{n+1}(h)]_{n+1} \cap [B']_n) + \overline{o}(1) \\ &= \sum_{h\in H'_n} \mu(T_{g_n}[A'c_{n+1}(h)]_{n+1} \cap [B']_n) + \overline{o}(1) \\ &= \sum_{h\in H'_n} \mu([A's_n(h)s_n(h_0 + h)^{-1}c_{n+1}(h_0 + h)]_{n+1} \cap [B']_n) + \overline{o}(1) \\ &= \sum_{h\in H'_n} \mu([(A's_n(h)s_n(h_0 + h)^{-1} \cap B')c_{n+1}(h_0 + h)]_{n+1}) + \overline{o}(1) \\ &= \frac{1}{\#H_n} \sum_{h\in H'_n} \mu([A's_n(h)s_n(h_0 + h)^{-1} \cap B']_n) + \overline{o}(1) \\ &= \frac{1}{\#H_n} \sum_{h\in H'_n} \lambda_{F_n}(A's_n(h)s_n(h_0 + h)^{-1} \cap B')\mu(X_n) + \overline{o}(1) \\ &= \frac{1}{\#H'_n} \sum_{h\in H'_n} \lambda_{F_n}(A's_n(h) \cap B's_n(h_0 + h)) + \overline{o}(1) \\ &= \frac{1}{\#H'_n} \sum_{h\in H'_n} \lambda_{F_n}(As_n(h) \cap Bs_n(h_0 + h)) + \overline{o}(1),\end{aligned}$$

where $\overline{o}(1)$ means (here and below) a sequence that goes to 0 and that does not depend on the choice of $A, B \subset F_n$. Let $\xi_n := \mathrm{dist}_{h \in H'_n}(s_n(h), s_n(h_0 + h))$. Notice that

$$\xi_n = \frac{1}{2r_n - 1} \sum_{i \in I[r_n]} \mathrm{dist}_{-r_n < t < r_n - 1}(s_n(t, i), s_n(t + 1, i)).$$

It follows from Lemma 2.1 that $\|\xi_n - \lambda_{S_n} \times \lambda_{S_n}\|_1 < \epsilon_n$. We define a function $f : S_n \times S_n \to \mathbb{R}$ by setting $f(v, w) := \lambda_{F_n}(Av \cap Bw)$. Then

$$\frac{1}{\#H'_n} \sum_{h \in H'_n} \lambda_{F_n}(As_n(h) \cap Bs_n(h_0 + h)) = \int f \, d\xi_n = \int f \, d(\lambda_{S_n} \times \lambda_{S_n}) \pm \epsilon_n.$$

Thus we obtain

$$\mu(T_{g_n}[A]_n \cap [B]_n) = \int_{S_n \times S_n} \lambda_{F_n}(Av \cap Bw) \, d\lambda_{S_n}(v) \, d\lambda_{S_n}(w) + \overline{o}(1). \tag{2.6}$$

Now we take $A := A^* C_n$ and $B := B^* C_n$ for some subsets $A^*, B^* \subset F_{n-1}$. Then the integral in the right-hand side of (2.6) equals the sum

$$\sum_{a \in A^*} \sum_{b \in B^*} \sum_{h, h' \in H_{n-1}} \frac{\#(ac_n(h)S_n \cap bc_n(h')S_n \cap F_n)}{(\#S_n)^2 \#F_n}. \tag{2.7}$$

It follows from the definition of $c_n$ and Lemma 2.3(i) that $ac_n(h)S_n \cap bc_n(h')S_n \neq \emptyset$ only if $h' - h \in I^2[2n+1]$ or $h' - h^* \in I^2[2n+1]$. If one of the two latter conditions is satisfied we say that $h$ and $h'$ are *partners*. Denote by $P(h)$ the set of partners of $h$ that belong to $H_{n-1}$. Clearly, $\#P(h) \leq 2(4n+1)^2$. Therefore we deduce from (2.6), (2.7) and Lemma 2.3(i) that

$$\begin{aligned}
&\mu(T_{g_n}[A^*]_{n-1} \cap [B^*]_{n-1}) \\
&\quad = \sum_{a \in A^*} \sum_{b \in B^*} \sum_{h \in H_{n-1}} \sum_{h' \in P(h)} \frac{\#(c_n(h)S_n \cap c_n(h')S_n \cap F_n) \pm \frac{32\#S_n}{n-2}}{(\#S_n)^2 \#F_n} + \overline{o}(1) \\
&\quad = \frac{\#A^* \#B^*}{(\#F_{n-1})^2} \cdot \theta_n \pm \frac{(\#F_{n-1})^2 \#H_{n-1} \cdot 2(4n+1)^2 \cdot 32\#S_n}{(\#S_n)^2 \#F_n \cdot (n-2)} + \overline{o}(1) \\
&\quad = \lambda_{F_{n-1}}(A^*)\lambda_{F_{n-1}}(B^*)\theta_n \pm 2^{11} n \frac{\#F_{n-1}}{\#S_n} + \overline{o}(1),
\end{aligned}$$

where $\theta_n = \frac{(\#F_{n-1})^2}{(\#S_n)^2 \#F_n} \sum_{h \in H_{n-1}} \sum_{h' \in P(h)} \#(c_n(h)S_n \cap c_n(h')S_n \cap F_n)$. Substituting $A^* = B^* = F_{n-1}$ and passing to the limit, we obtain that $\theta_n \to 1$ as $n \to \infty$. Hence

$$\mu(T_{g_n}[A^*]_{n-1} \cap [B^*]_{n-1}) = \mu([A^*]_{n-1})\mu([B^*]_{n-1}) + \overline{o}(1). \tag{2.8}$$

Since $\overline{o}(1)$ does not depend on the choice of $A^*$ and $B^*$ inside $F_{n-1}$, it follows from (1.6) and (2.8) that $(g_n)_{n=1}^{\infty}$ is mixing for $T$. □

Notice that slightly modifying the techniques from Ornstein's work [Or] one can show that $T_{(1,0,0)}$ is mixing indeed (cf. [Ma]). Moreover, imposing an extra condition on $s_n$ we can achieve that the entire action $T$ of $G$ is mixing. However we will not need this.

Our next task is to describe all ergodic 2-fold self-joinings of $T_{(1,0,0)}$.

**Theorem 2.5.** *The transformation $T_{(1,0,0)}$ is 2-fold simple and*

$$C(T_{(1,0,0)}) = \{T_g \mid g \in G\}.$$

*Proof.* Take any joining $\nu \in J_2^e(T_{(1,0,0)})$. Let $I_n := I[n^{-2}a_n]$, $J_n := I[n^{-2}r_n]$ and $\Phi_n := I_n + 2\widetilde{a}_n J_n$. We first notice that $(\Phi_n)_{n=1}^\infty$ is a Følner sequence in $\mathbb{Z}$. Since

$$\frac{a_n}{n^2} + \frac{2\widetilde{a}_n r_n}{n^2} < \frac{\widetilde{a}_n(2r_n+1)}{n^2} < \frac{2a_{n+1}}{(n+1)^2},$$

it follows that $\Phi_n \subset I_{n+1} + I_{n+1}$ and hence $\bigcup_{m=1}^n \Phi_m \subset I_{n+1} + I_{n+1}$. This implies that

$$\#\left(\Phi_{n+1} - \bigcup_{m=1}^{n} \Phi_m\right) \le 3\#\Phi_{n+1} \quad \text{for every } n \in \mathbb{N},$$

i.e. Shulman's condition [Li] is satisfied for $(\Phi_n)_{n=1}^\infty$. By [Li], the pointwise ergodic theorem holds along $(\Phi_n)_{n=1}^\infty$ for any ergodic transformation. Since $T_{(1,0,0)}$ is ergodic by Proposition 2.4, we have

$$\frac{1}{\#\Phi_n}\sum_{i\in\Phi_n} \chi_D(T_{(i,0,0)}x)\chi_{D'}(T_{(i,0,0)}x') \to \nu(D\times D') \tag{2.9}$$

as $n \to \infty$ for $\nu$-a.a. $(x,x') \in X \times X$ and for all cylinders $D, D' \subset X$. We call such $(x,x')$ a *generic point* for $(T_{(1,0,0)} \times T_{(1,0,0)}, \nu)$. Fix one of them. Then $x, x' \in X_n$ for all sufficiently large $n$ and we have the following expansions:

$$\begin{aligned} x &= (f_n, c_{n+1}(h_n), c_{n+2}(h_{n+1}), \dots, ), \\ x' &= (f'_n, c_{n+1}(h'_n), c_{n+2}(h'_{n+1}), \dots, ) \end{aligned}$$

with $f_n, f'_n \in F_n$ and $h_i, h'_i \in H_i$, $i > n$. We let $H_n^- = I^2[(1-n^{-2})r_n] \subset H_n$. Then $\#H_n^-/\#H_n \ge 1 - 0.5n^{-2}$. Since the marginals of $\nu$ are both equal to $\mu$, by the Borel-Cantelli lemma we may assume without loss of generality that $h_n, h'_n \in H_n^-$ for all sufficiently large $n$. This implies, in turn, that

$$f_{n+1} = f_n c_{n+1}(h_n) \in \widetilde{F}_n \phi_n(H_n^-) \subset I_+^2[(2r_n(1-n^{-2})-1)\widetilde{a}_n] \times \mathbb{Z}/2\mathbb{Z},$$

and, similarly, $f'_{n+1} \in I_+^2[(2r_n(1-n^{-2})-1)\widetilde{a}_n] \times \mathbb{Z}/2\mathbb{Z}$. Notice that given $g \in \Phi_n$, we have $(g,0,0) = (b,0,0)\phi_n(t,0)$ for some uniquely determined $b \in I_n$ and $t \in J_n$. Moreover, $(t,0,0)h_n \in H_n$. We also claim that

$$(b,0,0)f_n S_n S_n \subset F_n \quad \text{and} \quad (b,0,0)f_n S_n S_n^{-1} \subset F_n \tag{2.10}$$

if $n$ is large enough. To verify this, it suffices to show that

$$\frac{a_n}{n^2} + 2r_{n-1}\left(1 - \frac{1}{(n-1)^2}\right)\widetilde{a}_{n-1} + 4n\widetilde{a}_{n-1} < a_n,$$

which follows from (2.1) in a routine way. Hence

$$\begin{aligned} &(g,0,0)f_n s_n(h_n)\phi_n(h_n) = dc_{n+1}((t,0)+h_n) \quad \text{and} \\ &(g,0,0)f'_n s_n(h'_n)\phi_n(h'_n) = d'c_{n+1}((t,0)+h'_n), \end{aligned}$$

where $d := (b,0,0)f_n s_n(h_n) s_n((t,0)+h_n)^{-1}$ and

$$d' := (b,0,0)f'_n s_n(h'_n) s_n((t,0)+h'_n)^{-1}$$

belong to $F_n$ by (2.10). Now take any $B, B' \subset F_{n-1}$ and set $A := BC_n \subset F_n$ and $A' := B'C_n \subset F_n$. We have

$$\begin{aligned}
&\frac{\#\{g \in \Phi_n \mid (T_{(g,0,0)}x, T_{(g,0,0)}x') \in [A]_n \times [A']_n\}}{\#\Phi_n} \\
&\quad = \frac{1}{\#I_n} \sum_{b \in I_n} \frac{\#\{t \in J_n \mid d \in A, d' \in A'\}}{\#J_n} \\
&\quad = \frac{1}{\#I_n} \sum_{b \in I_n} \xi_n(A^{-1}(b,0,0)f_n s_n(h_n) \times {A'}^{-1}(b,0,0)f'_n s_n(h'_n)),
\end{aligned}$$

where $\xi_n := \mathrm{dist}_{t \in J_n}(s_n((t,0)h_n), s_n((t,0)h'_n))$. We consider two cases separately. Suppose first that $h_n = h'_n$ for all $n$ greater than some $N$. Then it is easy to see that $x' = T_k x$, where $k = f_N {f'_N}^{-1}$, and then it follows immediately that $(x, x')$ is generic for $\mu_{T_k}$.

Now consider the second case, where $h_n \neq h'_n$ for infinitely many, say *bad* $n$. It follows from Lemma 2.1 that $\|\xi_n - \lambda_{S_n} \times \lambda_{S_n}\| < \epsilon_n$ for all such $n$. Hence

$$\begin{aligned}
&\frac{1}{\#I_n} \sum_{b \in I_n} \xi_n(A^{-1}(b,0,0)f_n s_n(h_n) \times {A'}^{-1}(b,0,0)f'_n s_n(h'_n)) \\
&\quad = \frac{1}{\#I_n} \sum_{b \in I_n} \lambda_{S_n}(A^{-1}(b,0,0)f_n s_n(h_n))\lambda_{S_n}({A'}^{-1}(b,0,0)f'_n s_n(h'_n)) \pm \epsilon_n.
\end{aligned}$$

Now we derive from Lemma 2.3(ii) and (2.10) that

$$\lambda_{S_n}(A^{-1}(b,0,0)f_n s_n(h_n)) = \frac{\#(A \cap (b,0,0)f_n s_n(h_n)S_n^{-1})}{\#S_n} = \lambda_{F_{n-1}}(B) + \overline{o}(1)$$

and, in a similar way, $\lambda_{S_n}({A'}^{-1}(b,0,0)f'_n s_n(h'_n)) = \lambda_{F_{n-1}}(B') + \overline{o}(1)$. Hence

$$\nu([B]_{n-1} \times [B']_{n-1}) = \mu([B]_{n-1})\mu([B']_{n-1}) + \overline{o}(1)$$

provided that $n$ is bad. It remains to note that (1.6) holds along any subsequence, in particular along the subsequence of bad $n$. Hence $\nu = \mu \times \mu$. □

*Proof of Theorem 2.2.* The proof now follows from Proposition 2.4, Theorem 2.5 and the fact that $\mathfrak{F}_{G_b}$ and $\mathfrak{F}_{G_{b'}}$ are isomorphic if and only if the subgroups $G_b$ and $G_{b'}$ are conjugate in $C(T_{(1,0,0)})$ [dJR]. □

Notice that with some additional conditions on $s_n$ (cf. [Da4, Section 6]) one can show that $T_{(1,0,0)}$ is actually simple of all orders. For the definitions of higher order simplicity we refer to [dJR]. (In fact, 3-fold simplicity implies the simplicity of any order [GHR].) This would imply in turn that $T_{(1,0,0)}$ is mixing of any order whenever it is mixing.

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