

PACKING DIMENSION OF THE RANGE OF A LÉVY PROCESS

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ABSTRACT. Let $\{X(t)\}_{t \geq 0}$ denote a Lévy process in \mathbf{R}^d with exponent Ψ . Taylor (1986) proved that the packing dimension of the range $X([0, 1])$ is given by the index

$$(0.1) \quad \gamma' = \sup \left\{ \alpha \geq 0 : \liminf_{r \rightarrow 0^+} \int_0^1 \frac{\mathbb{P}\{|X(t)| \leq r\}}{r^\alpha} dt = 0 \right\}.$$

We provide an alternative formulation of γ' in terms of the Lévy exponent Ψ . Our formulation, as well as methods, are Fourier-analytic, and rely on the properties of the Cauchy transform. We show, through examples, some applications of our formula.

1. INTRODUCTION

Let $X := \{X(t)\}_{t \geq 0}$ denote a d -dimensional Lévy process [2], [20] which starts at the origin. Define Ψ to be the Lévy exponent of X , normalized so that $\mathbb{E}[\exp(iz \cdot X(t))] = \exp(-t\Psi(z))$ for all $t \geq 0$ and $z \in \mathbf{R}^d$, and let $\dim_{\mathbb{P}}$ denote the packing dimension [24], [22]. S. J. Taylor [23] has proved that with probability one, $\dim_{\mathbb{P}} X([0, 1]) = \gamma'$, where γ' is the index of Hendricks [8]; see (0.1).

Usually, one defines a Lévy process by constructing its Lévy exponent Ψ . From this perspective, formula (0.1) is difficult to apply in concrete settings. Primarily this is because the small- r behavior of $\int_0^1 \mathbb{P}\{|X(t)| \leq r\} dt$ is only well-understood when X is a nice Lévy process. For instance, when X is a subordinator γ' can be shown to be equal to the Blumenthal and Gettoor [4] upper index β [7], [3]; see also Theorem 3.3 below. When X is a general Lévy process, Pruitt and Taylor [18] find several quantitative relationships between γ' and other known fractal indices of Lévy processes.

The principal goal of this article is to describe $\gamma' = \dim_{\mathbb{P}} X([0, 1])$ more explicitly than (0.1), and solely in terms of the Lévy exponent Ψ . For all $r > 0$ define

$$(1.1) \quad W(r) := \int_{\mathbf{R}^d} \frac{\kappa(x/r)}{\prod_{j=1}^d (1 + x_j^2)} dx,$$

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where κ is the following well-known function [16], [11]:

$$(1.2) \quad \kappa(z) := \operatorname{Re} \left(\frac{1}{1 + \Psi(z)} \right) \quad \text{for all } z \in \mathbf{R}^d.$$

This function is symmetric (i.e., $\kappa(-z) = \kappa(z)$ for all $z \in \mathbf{R}^d$) and satisfies the pointwise bounds $0 \leq \kappa \leq 1$, which we use tacitly throughout. The following contains our formula for γ' .

Theorem 1.1. *For all d -dimensional Lévy processes X ,*

$$(1.3) \quad \dim_{\mathbb{P}} X([0, 1]) = \sup \left\{ \alpha \geq 0 : \liminf_{r \rightarrow 0^+} \frac{W(r)}{r^\alpha} = 0 \right\} = \limsup_{r \rightarrow 0^+} \frac{\log W(r)}{\log r},$$

almost surely, where $\sup \emptyset := 0$.

Xiao [25, Question 4.16] has asked if we can write $\dim_{\mathbb{P}} X([0, 1])$ explicitly in terms of Ψ . Theorem 1.1 answers this question in the affirmative.

The following is one of the many consequences of Theorem 1.1.

Theorem 1.2. *Let X be a d -dimensional Lévy process that has a non-trivial, non-degenerate Gaussian part. That is, $X = G + Y$, where G is a non-degenerate Gaussian Lévy process, and Y is an independent pure-jump Lévy process. Then, $\dim_{\mathbb{P}} X([0, 1]) = \dim_{\mathbb{P}} G([0, 1])$ and $\dim_{\mathbb{H}} X([0, 1]) = \dim_{\mathbb{H}} G([0, 1])$ a.s., where $\dim_{\mathbb{H}}$ denotes the Hausdorff dimension.*

We do not know of a direct proof of this result, although it is a very natural statement. However, some care is needed, as the result can fail when G is degenerate (Example 4.1). Our methods will make clear that in general we can say only that $\dim X([0, 1]) \geq \dim G([0, 1])$ a.s., where “dim” stands for either “ $\dim_{\mathbb{P}}$ ” or “ $\dim_{\mathbb{H}}$.”

We also mention the following ready consequence of Theorem 1.1:

Corollary 1.3. *Let X be a Lévy process in \mathbf{R}^d and $X'(t) := X(t) - X''(t)$, where X'' is an independent copy of X . Then, $\dim_{\mathbb{P}} X([0, 1]) \geq \dim_{\mathbb{P}} X'([0, 1])$ a.s.*

It has been shown that Corollary 1.3 continues to hold if we replace $\dim_{\mathbb{P}}$ by $\dim_{\mathbb{H}}$ everywhere; see Khoshnevisan, Xiao, and Zhong [13] and/or (1.4) below. Thus we have further confirmation of the somewhat heuristic observation of Kesten [11, p. 7] that the range of X is larger than the range of its symmetrization.

Theorem 1.1 is proved in Section 2. Our proof also yields the following almost-sure formula for the Hausdorff dimension of $X([0, 1])$:

$$(1.4) \quad \dim_{\mathbb{H}} X([0, 1]) = \sup \left\{ \alpha \geq 0 : \limsup_{r \rightarrow 0^+} \frac{W(r)}{r^\alpha} = 0 \right\} = \liminf_{r \rightarrow 0^+} \frac{\log W(r)}{\log r};$$

see Remark 2.4. Recently, Khoshnevisan, Xiao, and Zhong [13] established an equivalent formulation of this formula. Whereas their derivation is long and complicated, ours is direct and fairly elementary. Section 3 contains non-trivial examples wherein we compute $\dim_{\mathbb{P}} X([0, 1])$ for anisotropic Lévy processes X . Finally, Theorem 1.2 is proved in Section 4.

2. THE INCOMPLETE RENEWAL MEASURE

Define U to be the *incomplete renewal measure* of X . That is, for all Borel sets $A \subset \mathbf{R}^d$,

$$(2.1) \quad U(A) := \int_0^1 \mathbb{P}\{X(t) \in A\} dt.$$

We may deduce from (0.1) that

$$(2.2) \quad \dim_{\mathbb{P}} X([0, 1]) = \limsup_{r \rightarrow 0^+} \frac{\log U(B(0, r))}{\log r},$$

where $B(a, r) := \{z \in \mathbf{R}^d : |z - a| \leq r\}$ for all $a \in \mathbf{R}^d$ and $r \geq 0$ so that $\int_0^1 \mathbb{P}\{|X(t)| \leq r\} dt = U(B(0, r))$, and where $|y| := \max_{1 \leq j \leq d} |y_j|$ is the box norm of $y \in \mathbf{R}^d$.

Let ζ denote an independent, mean-one exponential random variable. The killed occupation measure of $B(0, r)$ can then be defined by

$$(2.3) \quad T(r) := \int_0^\zeta \mathbf{1}_{B(0,r)}(X(t)) dt \quad \forall r > 0,$$

where $\mathbf{1}_A$ denotes the indicator function of A .

Proposition 2.1. *For all $r > 0$,*

$$(2.4) \quad \frac{e}{4^d(e - 1)} \mathbb{E}[T(r)] \leq U(B(0, r)) \leq e \mathbb{E}[T(r)].$$

In order to prove this we first recall the notion of weak unimodality [12].

Definition 2.2. A Borel measure μ on \mathbf{R}^d is *c-weakly unimodal* if $c > 0$ is a constant that satisfies $\sup_{a \in \mathbf{R}^d} \mu(B(a, r)) \leq c\mu(B(0, r))$ for all $r > 0$.

The following is a variant of Lemma 4.1 of Khoshnevisan and Xiao [12].

Lemma 2.3. *U is 4^d -weakly unimodal.*

Proof. Let us fix $a \in \mathbf{R}^d$ and $r > 0$, and define $\sigma := \inf\{s > 0 : |X(s) - a| \leq r\}$, where $\inf \emptyset := \infty$. Clearly, σ is a stopping time, and

$$(2.5) \quad U(B(a, r)) = \mathbb{E} \left[\int_\sigma^1 \mathbf{1}_{B(a,r)}(X(s)) ds ; \sigma \leq 1 \right].$$

Thanks to the triangle inequality, the strong Markov property implies that

$$(2.6) \quad \begin{aligned} U(B(a, r)) &= \mathbb{E} \left[\int_0^{(1-\sigma)^+} \mathbf{1}_{B(0,r)}(X(u + \sigma) - a) du \right] \\ &\leq \mathbb{E} \left[\int_0^1 \mathbf{1}_{B(0,2r)}(X(u + \sigma) - X(\sigma)) du \right] = U(B(0, 2r)). \end{aligned}$$

Euclidean topology in the box norm dictates that there are points $z_1, \dots, z_{4^d} \in B(0, 2r)$ that have the property that $\bigcup_{i=1}^{4^d} B(z_i, r/2) = B(0, 2r)$. According to (2.6), we have the following “volume doubling” property:

$$(2.7) \quad U(B(0, 2r)) \leq \sum_{i=1}^{4^d} U(B(z_i, r/2)) \leq 4^d U(B(0, r)).$$

The desired result follows from this and (2.6). □

Proof of Proposition 2.1. Note that

$$(2.8) \quad U(B(0, r)) \leq e \int_0^1 \mathbb{P}\{|X(t)| \leq r\} e^{-t} dt \leq e \mathbb{E}[T(r)].$$

This proves the upper bound in (2.4). To prove the other half we note that

$$(2.9) \quad \mathbb{E}[T(r)] = \int_0^\infty \mathbb{P}\{|X(t)| \leq r\} e^{-t} dt \leq \sum_{j=0}^\infty e^{-j} \mathbb{E}[U(B(X(j), r))],$$

thanks to the Markov property. By Lemma 2.3, $\mathbb{E}[U(B(X(j), r))] \leq 4^d U(B(0, r))$ for every $j \geq 0$. The lower bound in (2.4) follows from this and (2.9). \square

Proof of Theorem 1.1. We derive only the second identity of (1.3); the first is manifestly an equivalent statement.

Let $(\mathcal{F}f)(z) := \int_{\mathbf{R}^d} e^{iz \cdot x} f(x) dx$ denote the Fourier transform of $f \in L^1(\mathbf{R}^d)$. For all fixed $r > 0$ and $x \in \mathbf{R}^d$ define

$$(2.10) \quad \phi_r(x) = \prod_{j=1}^d \frac{1 - \cos(2rx_j)}{2\pi r x_j^2}.$$

Then $\phi_r(x) \geq 0$, and $(\mathcal{F}\phi_r)(z) = \prod_{j=1}^d (1 - |z_j|/(2r))^+$ for all $z \in \mathbf{R}^d$ [6, p. 94]. As usual, $a^+ := \max(a, 0)$ for all $a \in \mathbf{R}$. Evidently, $\phi_r \in L^1(\mathbf{R}^d)$, and $0 \leq \mathcal{F}\phi_r \leq 1$ pointwise.

Note that $z \in B(0, r)$ implies that $1 - (2r)^{-1}|z_j| \geq \frac{1}{2}$. This implies that $\mathbf{1}_{B(0, r)}(z) \leq 2^d (\mathcal{F}\phi_r)(z)$ for all $z \in \mathbf{R}^d$. Therefore, by the Fubini–Tonelli theorem,

$$(2.11) \quad \mathbb{E}[T(r)] \leq 2^d \int_{\mathbf{R}^d} \kappa(x) \phi_r(x) dx \leq 2^d W(r).$$

The last inequality follows from the elementary bound

$$(2.12) \quad \frac{1 - \cos(2u)}{2\pi u^2} \leq \frac{1}{1 + u^2} \quad \text{for all } u \in \mathbf{R}.$$

This can be verified by considering $|u| \leq (\pi - 1)^{-1/2}$ and $|u| > (\pi - 1)^{-1/2}$ separately. Thanks to (2.2) and Proposition 2.1,

$$(2.13) \quad \dim_{\mathbb{P}} X([0, 1]) \geq \limsup_{r \rightarrow 0^+} \frac{\log W(r)}{\log r} \quad \text{a.s.}$$

In order to establish the converse inequality we introduce the process $\{S(t)\}_{t \geq 0}$ defined by $S(t) := (S_1(t), \dots, S_d(t))$, where S_1, \dots, S_d are independent symmetric Cauchy processes in \mathbf{R} , all with the same characteristic function $\mathbb{E}[e^{izS_1(t)}] = e^{-t|z|}$. We assume further that S is independent of X . Then for all $\lambda > 0$,

$$\mathbb{E}[\exp\{iX(t) \cdot S(\lambda)\}] = \mathbb{E}[\exp\{-\lambda \sum_{j=1}^d |X_j(t)|\}].$$

On the other hand, the scaling property of S implies

$$(2.14) \quad \mathbb{E} \left[e^{iX(t) \cdot S(\lambda)} \right] = \mathbb{E} \left[e^{-t\Psi(S(\lambda))} \right] = \frac{1}{\pi^d} \int_{\mathbf{R}^d} \frac{e^{-t\Psi(\lambda x)}}{\prod_{j=1}^d (1 + x_j^2)} dx.$$

For all $r, k > 0$ and $x \in \mathbf{R}^d$, $\exp\{-(k/r) \sum_{j=1}^d |x_j|\} \leq \mathbf{1}_{B(0,r)}(x) + e^{-k} \mathbf{1}_{B(0,r)^c}(x)$. Therefore, $\mathbf{1}_{B(0,r)}(x) \geq \mathbb{E}[\exp\{ix \cdot S(k/r)\}] - e^{-k} \mathbf{1}_{B(0,r)^c}(x)$, whence

$$(2.15) \quad \mathbb{E}[T(r)] \geq \int_0^\infty \mathbb{E}[\exp\{iX(t) \cdot S(k/r)\}] e^{-t} dt - e^{-k}(1 - \mathbb{E}[T(r)]).$$

This and (2.14) together with the fact that the quantity in (2.14) is real imply that

$$(2.16) \quad (1 - e^{-k})\mathbb{E}[T(r)] \geq -e^{-k} + \frac{1}{\pi^d} W(r/k).$$

Now we choose $k = r^{-\varepsilon}$, for an arbitrary small $\varepsilon > 0$, to find that the inequality in (2.13) is an equality. This completes our proof. \square

Remark 2.4. From the proof of Theorem 1.1 we see that $\mathbb{E}[T(r)]$ and $W(r)$ are roughly comparable; i.e., for all $\varepsilon, r > 0$ sufficiently small,

$$(2.17) \quad \frac{1}{\pi^d} W(r^{1+\varepsilon}) - \exp(-r^{-\varepsilon}) \leq \mathbb{E}[T(r)] \leq 2^d W(r).$$

Thanks to Proposition 2.1 this yields (1.4).

3. SOME EXAMPLES

We illustrate the utility of Theorem 1.1 by specializing it to a large class of examples.

3.1. Anisotropic examples. It is possible to construct examples of anisotropic Lévy “stable-like” processes whose $\dim_{\mathbb{P}} X([0, 1])$ are computable. The following furnishes most of the basic technical background that we shall need.

Theorem 3.1. *Let X be a Lévy process in \mathbf{R}^d with Lévy exponent Ψ . Suppose there exist constants β_1, \dots, β_d such that*

$$(3.1) \quad 2 \geq \beta_1 \geq \dots \geq \beta_d > 0 \quad \text{and} \quad \lim_{\|z\| \rightarrow \infty} \frac{1}{\ln \|z\|} \ln \operatorname{Re} \left(\frac{\sum_{j=1}^d |z_j|^{\beta_j}}{1 + \Psi(z)} \right) = 0.$$

If $N := \max\{1 \leq j \leq d : \beta_j = \beta_1\}$, then almost surely,

$$(3.2) \quad \dim_{\mathbb{P}} X([0, 1]) = \begin{cases} \beta_1, & \text{if } \beta_1 \leq N, \\ 1 + \beta_2 \left(1 - \frac{1}{\beta_1}\right), & \text{otherwise.} \end{cases}$$

Proof. Throughout this proof we write c and C for generic constants whose values can change between lines. In order to simplify the exposition somewhat, we note that it is sufficient to prove (3.2) under the following [slightly] stronger form of (3.1):

$$(3.3) \quad \frac{c}{1 + \sum_{j=1}^d |z_j|^{\beta_j}} \leq \operatorname{Re} \left(\frac{1}{1 + \Psi(z)} \right) \leq \frac{C}{1 + \sum_{j=1}^d |z_j|^{\beta_j}} \quad \text{for all } z \in \mathbf{R}^d.$$

First we consider the case when $\beta_1 \leq N$. Condition (3.3) implies that if $r \in (0, 1)$, then

$$(3.4) \quad W(r) \geq c r^{\beta_1} \int_{\mathbf{R}^d} \frac{dx}{(1 + \sum_{j=1}^d |x_j|^{\beta_j}) \prod_{j=1}^d (1 + x_j^2)} = C r^{\beta_1}.$$

Hence we have $\lim_{r \rightarrow 0} r^{-\alpha} W(r) = \infty$ for all $\alpha > \beta_1$. It follows from this and Theorem 1.1 that $\dim_p X([0, 1]) \leq \beta_1$ a.s.

Recall that $N := \max\{1 \leq j \leq d : \beta_j = \beta_1\}$. From this it follows that

$$(3.5) \quad \begin{aligned} W(r) &\leq c \int_{\mathbf{R}^N} \frac{dx}{(1 + \|x/r\|^{\beta_1}) \prod_{j=1}^N (1 + x_j^2)} \\ &= c r^{\beta_1} \left[\int_{\|x\| \leq 1} \frac{dx}{r^{\beta_1} + \|x\|^{\beta_1}} + C \right] \leq c r^{\beta_1} \log(1/r). \end{aligned}$$

In the above, $\log(1/r)$ accounts for the case that $\beta_1 = N$. The preceding bound implies that $\lim_{r \rightarrow 0} r^{-\alpha} W(r) = 0$ for every $\alpha < \beta_1$. This leads to the lower bound, $\dim_p X([0, 1]) \geq \beta_1$ a.s.

Next we consider the case when $\beta_1 > N$. This implies $N = 1$ and $\beta_1 > \beta_2$. In order to prove that $\dim_p X([0, 1]) \leq 1 + \beta_2(1 - \beta_1^{-1})$ a.s. we first derive a lower bound for $W(r)$. We do this by restricting the integral to the domain $D := \{x \in \mathbf{R}^d : |x_j| \leq 1 \text{ for } 3 \leq j \leq d\}$. More precisely, by (3.3) we have

$$(3.6) \quad \begin{aligned} W(r) &\geq c \int_D \frac{dx}{(1 + \sum_{j=1}^d |x_j/r|^{\beta_j}) \prod_{j=1}^d (1 + x_j^2)} \\ &\geq c \int_D \frac{dx}{(1 + \sum_{j=1}^d |x_j/r|^{\beta_j}) \prod_{j=1}^2 (1 + x_j^2)}. \end{aligned}$$

Let $(x_3, \dots, x_d) \in [-1, 1]^{d-2}$ be fixed and let $A := 1 + \sum_{j=3}^d |x_j/r|^{\beta_j}$. Consider

$$(3.7) \quad \begin{aligned} \mathcal{I} &:= \int_{\mathbf{R}^2} \frac{dx_1 dx_2}{(A + |x_1/r|^{\beta_1} + |x_2/r|^{\beta_2}) \prod_{j=1}^2 (1 + x_j^2)} \\ &\geq r^{\beta_2} \int_{r^{1-(\beta_2/\beta_1)}}^{\infty} dx_1 \int_0^{\infty} \frac{dx_2}{(r^{\beta_2} A + r^{\beta_2-\beta_1} x_1^{\beta_1} + x_2^{\beta_2}) \prod_{j=1}^2 (1 + x_j^2)}. \end{aligned}$$

Observe that $r^{\beta_2-\beta_1} x_1^{\beta_1} \geq 1$ for all $x_1 \geq r^{1-(\beta_2/\beta_1)}$. On the other hand, $r^{\beta_2} A \leq d-1$ for all $r \in (0, 1)$. It follows from these facts, and a change of variables, that

$$(3.8) \quad \begin{aligned} \mathcal{I} &\geq c r^{\beta_2} \int_{r^{1-(\beta_2/\beta_1)}}^{\infty} \frac{dx_1}{1 + x_1^2} \int_1^{\infty} \frac{1}{(r^{\beta_2-\beta_1} x_1^{\beta_1} + x_2^{\beta_2}) x_2^2} dx_2 \\ &\geq c r^{\beta_2} \int_{r^{1-(\beta_2/\beta_1)}}^{\infty} \frac{dx_1}{1 + x_1^2} \cdot \frac{1}{r^{\beta_2-\beta_1} x_1^{\beta_1}} \geq c r^{\beta_1} \int_{r^{1-(\beta_2/\beta_1)}}^1 \frac{dx_1}{x_1^{\beta_1}} \\ &\geq C r^{1+\beta_2(1-\beta_1^{-1})}. \end{aligned}$$

Combine this with (3.6) and (3.7) to deduce that $W(r) \geq c r^{1+\beta_2(1-\beta_1^{-1})}$ for all $r \in (0, 1)$, whence $\lim_{r \rightarrow 0} r^{-\alpha} W(r) = \infty$ for all $\alpha > 1 + \beta_2(1 - \beta_1^{-1})$. This implies that $\dim_p X([0, 1]) \leq 1 + \beta_2(1 - \beta_1^{-1})$ a.s.

Now we derive the lower bound for $\dim_p X([0, 1])$. It follows from (3.3) that

$$\begin{aligned}
 (3.9) \quad W(r) &\leq c \int_{\mathbf{R}^2} \frac{dx_1 dx_2}{(1 + |x_1/r|^{\beta_1} + |x_2/r|^{\beta_2}) \prod_{j=1}^2 (1 + x_j^2)} \\
 &= 4cr^{\beta_2} \int_0^\infty \frac{dx_1}{1 + x_1^2} \int_0^\infty \frac{dx_2}{(B + x_2^{\beta_2})(1 + x_2^2)},
 \end{aligned}$$

where $B := r^{\beta_2} + r^{\beta_2 - \beta_1} x_1^{\beta_1}$. It remains to verify that the last expression in (3.9) is at most $cr^{1 + \beta_2(1 - \beta_1^{-1})} \log(1/r)$, where $\log(1/r)$ appears because of the possibility that $\beta_2 = 1$.

By breaking the dx_2 -integral according to whether $|x_2| \leq 1$ or $|x_2| > 1$, and after a change of variables, we can verify the following elementary inequalities:

- (i) If $B \leq 1$, then $\int_0^\infty (B + x_2^{\beta_2})^{-1} (1 + x_2^2)^{-1} dx_2 \leq cg(x_1)$, where: $g(x_1) = 1$ if $\beta_2 < 1$, $\log(B^{-1})$ if $\beta_2 = 1$, and $B^{(1/\beta_2) - 1}$ if $\beta_2 < 1$.
- (ii) If $B > 1$, then $\int_0^\infty (B + x_2^{\beta_2})^{-1} (1 + x_2^2)^{-1} dx_2 \leq c/B$.

We return to (3.9) and split the dx_1 -integral respectively over the intervals $\{x_1 : B \leq 1\}$ and $\{x_1 : B > 1\}$. It follows from (3.9), (i) and (ii), and a direct computation, that $W(r) \leq cr^{1 + \beta_2(1 - \beta_1^{-1})} \log(1/r)$. Hence, Theorem 1.1 implies that $\dim_p X([0, 1]) \geq 1 + \beta_2(1 - \beta_1^{-1})$ a.s. This finishes the proof Theorem 3.1. \square

In the following, we apply Theorem 3.1 to operator-stable Lévy processes in \mathbf{R}^d with $d \geq 2$. Let us first recall from [21] that a non-degenerate distribution μ on \mathbf{R}^d is called *operator-stable* if there exist sequences of independent identically distributed random vectors $\{X_n\}$ in \mathbf{R}^d , non-singular linear operators $\{A_n\}$, and vectors $\{a_n\}$ in \mathbf{R}^d such that $\{A_n \sum_{k=1}^n X_k - a_n\}$ converges in law to μ . A distribution μ on \mathbf{R}^d is called *full* if it is not supported on any $(d - 1)$ -dimensional hyper-plane. Sharpe [21] proves that a full distribution μ in \mathbf{R}^d is operator-stable if and only if there exists a non-singular linear operator B on \mathbf{R}^d such that $\mu^t = t^B \mu * \delta(b(t))$ for all $t > 0$ and some $b(t) \in \mathbf{R}^d$. Here, μ^t denotes the t -fold convolution power of μ , and $t^B \mu(dx) := \mu(t^{-B} dx)$ is the image measure of μ under the action of the linear operator $t^B := \sum_{n=0}^\infty (\log t)^n B^n / n!$. In the above, B and $\{b(t), t > 0\}$ are called a *stability exponent* and the family of shifts of μ , respectively. The set of all possible exponents of an operator stable law is characterized by Holmes et al. [9]; see also [14, Theorem 7.2.11]. By analogy with the one-dimensional case, an operator-stable distribution μ satisfying $\mu^t = t^B \mu$ will be called strictly operator-stable; see [21, p. 64].

A stochastic process $Y := \{Y(t)\}_{t \in \mathbf{R}_+}$ with values in \mathbf{R}^d is said to be *operator self-similar* if there exists a linear operator B on \mathbf{R}^d such that for every $c > 0$, $\{Y(ct)\}_{t \geq 0} \stackrel{d}{=} \{c^B Y(t)\}_{t \geq 0}$, where “ $\stackrel{d}{=}$ ” denotes the equality of finite-dimensional distributions. The linear operator B is called a *self-similarity exponent* of Y . Let $X = \{X(t)\}_{t \geq 0}$ be a Lévy process in \mathbf{R}^d starting from 0 such that the distribution of $X(t)$ is full for every $t > 0$. Hudson and Mason [10, Theorem 7] proved that X is operator self-similar if and only if the distribution of $X(1)$, $\nu := P \circ (X(1))^{-1}$, is strictly operator stable. In this case, every stability exponent B of ν is also a self-similarity exponent of X . Hence, from now on we will call a Lévy process X in \mathbf{R}^d operator-stable if the distribution of $X(1)$ is full and strictly operator-stable, and we will refer to B simply as an *exponent* of X .

Operator-stable Lévy processes are scaling limits of d -dimensional random walks that are normalized by linear operators [14, Chapter 11]. All d -dimensional strictly stable Lévy processes of index α are operator-stable with exponent $B := \alpha^{-1}I$, where I denotes the $(d \times d)$ identity matrix.

More generally, let X_1, \dots, X_d be independent stable Lévy processes in \mathbf{R} with respective indices $\alpha_1, \dots, \alpha_d \in (0, 2]$. Define $X(t) := (X_1(t), \dots, X_d(t))$. One can then verify that X is an operator-stable Lévy process whose exponent B is the $(d \times d)$ diagonal matrix $\text{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_d^{-1})$. These processes were first introduced by Pruitt and Taylor [19] under the title of *Lévy processes with stable components*. These processes have been used to construct various counterexamples [8].

Let X be an operator-stable Lévy process in \mathbf{R}^d with exponent B . Factor the minimal polynomial of B into $q_1(x), \dots, q_p(x)$ where all roots of $q_i(x)$ have real part a_i , and $a_i < a_j$ for $i < j$. Define $\alpha_i := a_i^{-1}$, so that $\alpha_1 > \dots > \alpha_p$, and note that $0 < \alpha_i \leq 2$ [14, Theorem 7.2.1]. Define $V_i := \text{Ker}(q_i(B))$ and $d_i := \dim(V_i)$. Then $d_1 + \dots + d_p = d$, and $V_1 \oplus \dots \oplus V_p$ is a direct-sum decomposition of \mathbf{R}^d into B -invariant subspaces. We may write B as $B = B_1 \oplus \dots \oplus B_p$, where $B_i : V_i \rightarrow V_i$ and every eigenvalue λ of B_i has the property that $\text{Re } \lambda = a_i$. We can apply Theorem 3.1 for operator stable Lévy processes to obtain a wholly different proof of the following theorem of Meerschaert and Xiao [15, Theorem 3.2].

Theorem 3.2 (Meerschaert and Xiao [15]). *Let X be an operator-stable Lévy process in \mathbf{R}^d . Then almost surely,*

$$(3.10) \quad \dim_{\mathbf{P}} X([0, 1]) = \begin{cases} \alpha_1 & \text{if } \alpha_1 \leq d_1, \\ 1 + \alpha_2 \left(1 - \frac{1}{\alpha_1}\right) & \text{otherwise.} \end{cases}$$

Proof. Define $\beta_j := \alpha_\ell$ where ℓ is determined by $\sum_{i=0}^{\ell-1} d_i < j \leq \sum_{i=\ell-1}^{\ell} d_i$, and $d_0 := 0$. Because Meerschaert and Xiao [15] have established (3.1), Theorem 3.1 implies (3.10) with $N := d_1$. □

3.2. Subordinators. Let us consider the special case that X is a [non-negative] subordinator. We conclude this section by showing that our Theorem 1.1 includes the well known formula for $\dim_{\mathbf{P}} X([0, 1])$; see [7] and [3, §5.1.2]. Let Φ denote the Laplace exponent of X , normalized so that $\mathbb{E}[\exp(-\lambda X(t))] = \exp(-t\Phi(\lambda))$ for all $\lambda, t \geq 0$. The following is an immediate consequence of Theorem 1.1.

Theorem 3.3 (Fristedt and Taylor [7]; Bertoin [3]). *With probability one,*

$$(3.11) \quad \dim_{\mathbf{P}} X([0, 1]) = \limsup_{\lambda \rightarrow \infty} \frac{\log \Phi(\lambda)}{\log \lambda}.$$

Proof. Let $S = \{S(t)\}_{t \geq 0}$ denote an independent Cauchy process in \mathbf{R} such that $\mathbb{E}[\exp(i\xi S(t))] = \exp(-t|\xi|)$ for all $t \geq 0$ and $\xi \in \mathbf{R}$. Then,

$$(3.12) \quad e^{-t\Phi(\lambda)} = \mathbb{E}[e^{-\lambda X(t)}] = \mathbb{E}[e^{iX(t)S(\lambda)}] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t\Psi(\lambda z)}}{1 + z^2} dz.$$

Multiply both sides by e^{-t} and integrate $[dt]$ to find that

$$(3.13) \quad \frac{1}{1 + \Phi(\lambda)} = \frac{1}{\pi} W(1/\lambda).$$

A direct appeal to Theorem 1.1 finishes the proof. □

4. PROOF OF THEOREM 1.2

Throughout, $\|x\| := (x_1^2 + \dots + x_d^2)^{1/2}$ for all $x \in \mathbf{R}^d$. This is the usual ℓ^2 -norm on \mathbf{R}^d , and should not be confused with the ℓ^∞ -norm $|x| = \max_{1 \leq j \leq d} |x_j|$ that we have used so far.

Because of the Lévy–Khintchine formula [2], we can write X as $X = G + Y$, where G is a Gaussian Lévy process and Y is an independent pure-jump Lévy process. Thanks to the centered-ball inequality of Anderson [1], $a \mapsto P\{|G(t) - a| \leq r\}$ is maximized at the origin. Apply this, conditionally on Y , to find that

$$(4.1) \quad P\{|X(t)| \leq r\} \leq P\{|G(t)| \leq r\} \quad \text{for all } t, r > 0.$$

It follows from (0.1) and (2.2) that $\dim_p G([0, 1]) \leq \dim_p X([0, 1])$ a.s. The analogous bound for \dim_H follows from this and the formula of Pruitt [17].

In order to prove the converse bound we appeal to Theorem 1.1. Recall that

$$(4.2) \quad \Psi_X(z) = O(\|z\|^2) \quad \text{as } \|z\| \rightarrow \infty$$

Bochner [5, eq. (3.4.14), p. 67]. The subscript X refers to the process X , the subscript G to the process G , etc. Therefore, there exists a constant C such that

$$(4.3) \quad \kappa_X(z) \geq \frac{C}{1 + \|z\|^2} \quad \text{for all } z \in \mathbf{R}^d.$$

The non-degeneracy of G implies that $\|z\|^{-2}\Psi_G(z)$ is bounded below uniformly for all $z \in \mathbf{R}^d$. Because $\text{Re } \Psi_Y(z) \geq 0$, it follows that there exists a constant c such that $\kappa_X(z) \geq c\kappa_Y(z)$ for all $z \in \mathbf{R}^d$. Therefore, $W_X(r) \geq cW_G(r)$, and Theorem 1.1 shows that $\dim_p X([0, 1]) \leq \dim_p G([0, 1])$ a.s. The analogous bound for \dim_H follows from (1.4), Remark 2.4 and Proposition 2.1. This completes the proof. \square

We conclude this section by mentioning a simple example wherein Theorem 1.2 fails because G is degenerate.

Example 4.1. Let Y be an isotropic stable Lévy process in \mathbf{R}^2 with index $\alpha \in (1, 2]$. Let G_1 be an independent one-dimensional Brownian motion, and define $G(t) := (G_1(t), 0)$. Then, $X := G + Y$ has the form of the process in Theorem 1.2, but now G is degenerate. Direct calculations show that $\Psi_Y(z) = c\|z\|^\alpha$ for some $c > 0$, and $\Psi_G(z) = c'z_1^2$ for some $c' > 0$. It follows readily from this discussion that $\Psi_X(z) = c\|z\|^\alpha + c'z_1^2$, whence it follows that

$$(4.4) \quad \frac{A_1}{1 + |z_1|^2 + |z_2|^\alpha} \leq \kappa_X(z) \leq \frac{A_2}{1 + |z_1|^2 + |z_2|^\alpha} \quad \text{for all } z := (z_1, z_2) \in \mathbf{R}^2,$$

where A_1 and A_2 are universal constants. Theorem 3.1 implies that with probability one, $\dim_p X([0, 1]) = 1 + \alpha(1 - \frac{1}{2}) = 1 + (\alpha/2)$. On the other hand, according to Theorem 1.1, with probability one, $\dim_p Y([0, 1]) = \alpha$, whereas $\dim_p G([0, 1]) = 1$. Therefore, if $\alpha \in (1, 2)$, then $\dim_p X([0, 1])$ is almost surely *strictly* greater than both $\dim_p Y([0, 1])$ and $\dim_p G([0, 1])$.

Despite the preceding, it is not always necessary that G is non-degenerate, viz.,

Example 4.2. Let $Y := \{Y(t)\}_{t \in \mathbf{R}_+}$ be a Lévy process in \mathbf{R}^d with characteristic exponent $\Psi(\xi) = \|\xi\|^2 L(\xi)$, where $L : \mathbf{R}^d \rightarrow \mathbf{C}$ is slowly varying at infinity. Such exponents can be constructed via the Lévy–Khintchine formula. For any Gaussian process $G := \{G(t)\}_{t \in \mathbf{R}_+}$ in \mathbf{R}^d define $X := Y + G$. It follows from Theorem 3.1 that $\dim_p X([0, 1]) = \dim_p Y([0, 1]) = \min(2, d)$ almost surely.

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