PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 7, July 2008, Pages 2609–2613
S 0002-9939(08)09183-1
Article electronically published on February 29, 2008

ON THE SUPPORT OF THE SPECTRAL MEASURE OF A HARMONIZABLE SEQUENCE

ANDRZEJ MAKAGON AND AGNIESZKA WYŁOMAŃSKA

(Communicated by Richard C. Bradley)

Abstract. In this note we discuss a relationship between the correlation function of a harmonizable sequence and support of its spectral measure.

1. Introduction

Let Z denote the group of integers. The dual $\hat{Z}$ of Z is in this paper identified with the interval $[0, 2\pi)$ with addition modulo 2π.

A second-order zero-mean stochastic sequence is often viewed as a sequence $\{x(n),\ n \in Z\}$ of elements of a Hilbert space H over the field of complex numbers. The covariance function $R(n,m)$ of $x(n)$ is then defined as $R(n,m) = (x(n), x(m))$, where $(\cdot,\cdot)$ is an inner product in H. A sequence $x(n)$, $n \in Z$, is called *strongly harmonizable* if there is a measure F on $[0, 2\pi)^2$, called the *spectral measure* of the sequence $\{x(n),\ n \in Z\}$, such that

$$R(n,m) = \int_0^{2\pi}\int_0^{2\pi} e^{i(ns-mt)} F(ds, dt). \tag{1}$$

A sequence $\{x(n),\ n \in Z\}$ is *periodically correlated (PC)* with period T if for every $p \in Z$, the function

$$k \longrightarrow B_p(k) = R(p+k, k)$$

is periodic with the same period T. Every PC sequence is strongly harmonizable and its spectral measure F is supported on T lines (see [1]):

$$L_\lambda = \{(s,t) \in [0,2\pi)^2 : t = s + \lambda\}, \text{ where } \lambda \in \left\{0, \frac{2\pi}{T}, \frac{2\times 2\pi}{T}, \dots, \frac{(T-1)\times 2\pi}{T}\right\}.$$

Note that, since addition in $[0, 2\pi)$ is modulo 2π, each line L_λ, $\lambda \neq 0$, if drawn on the square $[0, 2\pi) \times [0, 2\pi)$, comprises two segments; for example, if $\lambda = 2\pi/T$, then $L_\lambda = \{(s, s + 2\pi/T) : s \in [0, 2\pi - 2\pi/T)\} \cup \{(s, s + 2\pi/T - 2\pi) : s \in [2\pi - 2\pi/T, 2\pi)\}$. If the functions $k \longrightarrow B_p(k)$ are merely almost-periodic and $\{x(n),\ n \in Z\}$ is strongly harmonizable, then its spectrum is supported on lines $L_\lambda = \{(s,t) \in [0,2\pi)^2 : t = s + \lambda\}$, $\lambda \in \Lambda$, where Λ is the set of nonzero frequencies of sequences $B_p(\cdot)$, $p \in Z$ (see [2]).

Received by the editors October 10, 2006, and, in revised form, March 23, 2007, and April 4, 2007.

2000 *Mathematics Subject Classification.* Primary 60G12, 42B10.

Key words and phrases. Harmonizable sequence, periodically correlated sequence, spectral measure.

©2008 American Mathematical Society
Reverts to public domain 28 years from publication

In both cases the location of spectral lines (that is, the set Λ) is a Borel support of measures whose Fourier transforms are sequences $k \longrightarrow B_p(k)$, $p \in Z$. The purpose of this note is to show that this phenomenon holds true for any harmonizable sequence $\{x(n),\ n \in Z\}$, namely, that a Borel support of the spectrum F of $\{x(n),\ n \in Z\}$ is on parallel to the diagonal stripes determined by the common support of measures which are the inverse Fourier transforms of sequences $k \longrightarrow B_p(k)$, $p \in Z$.

2. Theorem

In this paper, by a measure on a topological space G we will understand a finite complex σ-additive function defined on a Borel σ-algebra of G. Let μ be a measure on G and D be a Borel subset of G. We will say that μ *is concentrated on* D if $\mu(\Delta) = 0$ for every Borel set Δ disjoint with D (cf. [3]).

It turns out that it is easier to work with some transformation of the spectral measure F, namely with the measure

$$H(\Delta) = F\left(\Psi(\Delta)\right), \text{ where } \Psi(u, w) = (u, u + w). \tag{2}$$

Here and in the sequel an addition in $[0, 2\pi)$ will always be understood *modulo* 2π. Since Ψ is a homeomorphism of $[0, 2\pi)^2$ onto itself, H is a measure.

Theorem 1. *Let* $\{x(n),\ n \in Z\}$ *be a strongly harmonizable sequence,* F *be its spectral measure, and* Ψ *be the function defined in* (2). *Let* μ_p *denote the measure on* $[0, 2\pi)$ *such that*

$$R(p + k, k) = \int_0^{2\pi} e^{-iku} \mu_p(du),\ \ k \in Z \tag{3}$$

(the existence of μ_p *is proved below in* (6)*). Then for every Borel subset* D *of* $[0, 2\pi)$ *the following conditions are equivalent:*

(1) *for every* $p \in Z$*,* μ_p *is concentrated on* D*,*
(2) F *is concentrated on the set* $\Psi([0, 2\pi) \times D)$*.*

Proof. From (2) it follows that for any bounded complex measurable function ϕ,

$$\int_0^{2\pi}\int_0^{2\pi} \phi(s, t) F(ds, dt) = \int_0^{2\pi}\int_0^{2\pi} \phi(s, s + t) H(ds, dt). \tag{4}$$

In particular (1) and (4) imply that

$$R(p + k, k) = \int_0^{2\pi}\int_0^{2\pi} e^{ipu} e^{-iwk} H(du, dw). \tag{5}$$

From Fubini's theorem we therefore conclude that the measures μ_p, $p \in Z$, satisfying (3) exist and are given by

$$\mu_p(\Delta) = \int_0^{2\pi} e^{ipu} H(du, \Delta). \tag{6}$$

Since the Fourier transform determines a measure we conclude that

$$\mu_p(\Delta) = 0 \text{ iff } H(E \times \Delta) = 0 \text{ for every Borel } E \subset [0, 2\pi). \tag{7}$$

Suppose first that μ_p is concentrated on D for every $p \in Z$. Then from (7) it follows that for every Δ disjoint with D, $H(E \times \Delta) = 0$ for every E. Since rectangles determine the measure H, this implies that H is concentrated on the set $[0, 2\pi) \times D$; that is, $F = H \circ \Psi^{-1}$ sits on $\Psi([0, 2\pi) \times D)$. Conversely, if F is concentrated on

$\Psi([0,2\pi)\times D)$, then H is concentrated on $[0,2\pi)\times D$; that is, $H(E\times\Delta)=0$ for every Borel set $E\subset[0,2\pi)$ and for every Δ disjoint from D. From (7) we conclude that for every p, $\mu_p(\Delta)=0$ provided Δ is disjoint from D. □

Note that two different iterations of (5) give two representations of $R(p+k,k)$:

$$\text{(A)}\qquad R(p+k,k)=\int_0^{2\pi}e^{-ikw}\mu_p(dw),\text{ where }\mu_p(\Delta)=\int_0^{2\pi}e^{ipu}H(du,\Delta),$$

$$\text{(B)}\qquad R(p+k,k)=\int_0^{2\pi}e^{ipu}\nu_k(du),\text{ where }\nu_k(\Delta)=\int_0^{2\pi}e^{-ikw}H(\Delta,dw).$$

To visualize these relations, given a sequence $(a(n))$ we write that $\mu=\mathcal{F}_n^{-1}[a(n)]$ if $a(n)=\int_0^{2\pi}e^{-inx}\mu(dx)$, $n\in Z$ (for example, $\mu_p(\Delta)=\mathcal{F}_k^{-1}[R(p+k,k)](\Delta)$). With this notation the relations (A) and (B) take the form:

$$\text{(A}'\text{)}\qquad H(\Delta_1,\Delta_2)=\mathcal{F}_p^{-1}\left[\mathcal{F}_k^{-1}[R(-p+k,k)](\Delta_2)\right](\Delta_1),$$

$$\text{(B}'\text{)}\qquad H(\Delta_1,\Delta_2)=\mathcal{F}_k^{-1}\left[\mathcal{F}_p^{-1}[R(-p+k,k)](\Delta_1)\right](\Delta_2).$$

It is now clear that the behavior of the sequences $R(p+k,k)$ with respect to k determines the support (and possibly other properties) of $H(\Delta_1,\Delta_2)$ with respect to Δ_2, while the sequences $(p\to R(p+k,k))$ control properties of the measures $(\Delta_1\to H(\Delta_1,\Delta_2))$. In particular, the relation (B′) shows that:

If all $\nu_k=\mathcal{F}_p^{-1}[R(-p+k,k)]$ *vanish on a set* Δ_1, *then* H *and* F *vanish on* $\Delta_1\times[0,2\pi)$.

Recall that $F=H\circ\Psi^{-1}$ and $\Psi(u,w)=(u,u+w)$ with addition *modulo* 2π.

Example 1. Suppose that $R(p+k,k)$ is T-periodic in k for every p. Then $\mathcal{F}_k^{-1}[R(p+k,k)]$ is a measure and $\mu_p=\mathcal{F}_k^{-1}[R(p+k,k)]$ is concentrated on the set $\Lambda_T=\{(2\pi j/T):j=0,\dots,T-1\}$, i.e. $\mu_p=\sum_{j=1}^{T-1}r_p(j)\delta_{(2\pi j/T)}$, where δ_a denotes the unit measure concentrated at $\{a\}$. Since a PC sequence is harmonizable, from Theorem 1 we conclude that F is concentrated on "lines" $\ell_p=\{(s,t):t=s+2\pi j/T,\ s\in[0,2\pi)\}$. One can see what measures sit on those lines by taking $\Delta_2=\{2\pi j/T\}$ in (A′), which gives that

$$F(\Delta_1\cap\ell_j)=H(\Delta_1,\{2\pi j/T\})=\mathcal{F}_p^{-1}\left[\mu_{-p}(\{2\pi j/T\})\right](\Delta_1)=\mathcal{F}_p^{-1}\left[r_{-p}(j)\right](\Delta_1).$$

In particular, if $\sum_p|r_p(j)|<\infty$, then the measure $F(du\cap\ell_j)$ has a density $f_j(u)=\sum_p r_{-p}(j)e^{ipj}$.

Example 2. Suppose that $R(n,m)=g(n)\overline{g(m)}$, where $g(n)=\int_0^{2\pi}e^{int}f(t)dt$, and $f\in L^1([0,2\pi))$. Then $R(n,m)$ is the correlation function of a harmonizable sequence and

$$R(p+k,k)=\int_0^{2\pi}\int_0^{2\pi}e^{-ik(s-t)}e^{ipt}f(t)\overline{f(s)}dsdt=\int_0^{2\pi}\int_0^{2\pi}e^{-ikv}e^{ipu}f(u)\overline{f(v+u)}dudv.$$

Therefore

$$(8)\qquad \mu_p(dv)=\mathcal{F}_k^{-1}[R(p+k,k)](dv)=\left[\int_0^{2\pi}e^{ipu}f(u)\overline{f(v+u)}du\right]dv,$$

$$(9)\qquad \nu_k(du)=\mathcal{F}_p^{-1}[R(-p+k,k)](du)=\left[\int_0^{2\pi}e^{-ikv}\overline{f(v+u)}dv\right]f(u)du.$$

Suppose, for example, that the function f is supported on the set $[0,a]$, $0<a<\pi$. Then $f(u)\overline{f(v+u)}=0$ if $a<v<2\pi-a$, and hence by (8) each μ_p vanishes outside $[0,a]\cup[2\pi-a,2\pi)$. From Theorem 1 we conclude that F vanishes outside the strip $S_d=\{(t,s)\in[0,2\pi)^2: t-a\le s\le t+a\}$. On the other hand, from (9) it follows that each ν_k vanishes outside of $[0,a]$, and hence F is concentrated on S_h and, using the fact that $F(\Delta_1,\Delta_2)=\overline{F(\Delta_2,\Delta_1)}$, we conclude that F must be concentrated on $[0,a]\times[0,a]$. This, of course, was clear from the beginning since $F(dt,ds)=f(t)\overline{f(s)}dsdt$, and the example was intended just to illustrate how the theorem works in the case when F is absolutely continuous with respect to the Lebesgue measure $dtds$. The authors are grateful to the reviewer for suggesting this example.

3. Comment on harmonizability

The main assumption in Section 2 was that $\{x(n),\ n\in Z\}$ is strongly harmonizable, which yielded the existence of measures μ_p in (3). However, in some instances, for example if $\{x(n),\ n\in Z\}$ is PC (see Example 1), the existence of these measures can be deduced from the form of the sequences $R(p+k,k)$. The immediate question is whether or not the existence of measures μ_p in (3) itself implies harmonizability of $\{x(n),\ n\in Z\}$. Following Hurd [2] one can show that the answer is affirmative if the common support of all μ_p is a finite set.

Theorem 2 (cf. [2], Proposition 3)**.** *Suppose that $\{x(n),\ n\in Z\}$ is a stochastic sequence such that for every $p\in Z$ there is a measure μ_p satisfying*

$$R(p+k,k)=\int_0^{2\pi}e^{-iku}\mu_p(du),\ k\in Z. \tag{10}$$

Suppose additionally that all measures μ_p are concentrated on the same finite set $\Lambda\subset[0,2\pi)$ containing 0. Then the sequence $\{x(n),\ n\in Z\}$ is strongly harmonizable.

We sketch the proof, referring for details to Hurd's paper [2].

First note that since $R(p+k,k)=\int_0^{2\pi}e^{-ikw}\mu_p(dw)$, from Lebesgue's Theorem it follows that for every $\lambda\in[0,2\pi)$, the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}e^{ij\lambda}R(p+j,j)=\lim_{N\to\infty}\int_0^{2\pi}\frac{1}{N}\frac{1-e^{i(\lambda-w)N}}{1-e^{i(\lambda-w)}}\mu_p(dw)$$

exists and equals $\mu_p(\{\lambda\})$, if $\lambda\in\Lambda$, and 0 otherwise.

We first examine the sequence $\mu_p(\{0\})$, $p\in Z$. Since

$$\mu_{p-q}(\{0\})=\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}R(p-q+j,j)=\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}R(p+k,q+k)$$

and $R(n,m)=(x(n),x(m))$,

$$\sum_i\sum_j c_i\overline{c_j}\mu_{p_i-p_j}(\{0\})=\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\left\|\sum_i c_ix(p_i+k)\right\|^2\ge 0;$$

that is, $\mu_p(\{0\})$, $p\in Z$, is nonnegative definite. From Bochner's Theorem ([4], Section 1.4.3) it follows that there is a nonnegative measure γ_0 on $[0,2\pi)$ such that

$$\mu_p(\{0\})=\int_0^{2\pi}e^{ipu}\gamma_0(du).$$

Now, repeating arguments presented in [2, p. 32], one can show that for every $\lambda \in \Lambda$ there is a constant M such that for any selection of complex numbers $c_1, \dots, c_n$ and integers $p_1, \dots, p_n$,

$$\left| \sum_{j=1}^{n} c_j \mu_{p_j}(\{\lambda\}) \right| \leq M \times \sup_t \left| \sum_{j=1}^{n} c_j e^{itp_j} \right|.$$

Using [4, Section 1.9.1], we conclude that for every $\lambda \in \Lambda$ there is a measure γ_λ of $[0, 2\pi)$ such that

$$\mu_p(\{\lambda\}) = \int_0^{2\pi} e^{ipu} \gamma_\lambda(du), \; p \in Z. \tag{11}$$

For each $\lambda \in \Lambda$ let H_λ be the image of γ_λ through the mapping $[0, 2\pi) \ni s \to (s, \lambda) \in [0, 2\pi)^2$, and let $H = \sum_{\lambda \in \Lambda} H_\lambda$. Since Λ is finite, H is a measure on $[0, 2\pi)^2$. Using (11) and (10) it is easy to verify that

$$\int_0^{2\pi} \int_0^{2\pi} e^{ipu} e^{-iwk} H(du, dw) = \sum_\lambda e^{-i\lambda k} \int_0^{2\pi} e^{ipu} \gamma_\lambda(du) = R(p + k, k).$$

Hence $F = H \circ \Psi^{-1}$ is the spectral measure of $\{x(n), \; n \in Z\}$, and so $x(n)$ is strongly harmonizable. □

References

[1] E. G. Gladyshev, "Periodically correlated random sequences", *Soviet Math.* **2** (1961), 385-388.

[2] H. L. Hurd, "Correlation Theory of Almost Periodically Correlated Processes", *J. Mult. Anal.* **37** (1) (1991), 24-45. MR1097303 (92e:60074)

[3] W. Rudin, *Real and Complex Analysis*, McGraw-Hill (1987). MR924157 (88k:00002)

[4] W. Rudin, *Fourier Analysis on Groups*, John Wiley & Sons (1990). MR1038803 (91b:43002)

Wrocław College of Management and Finance, Wrocław, Poland; and Department of Mathematics, Hampton University, Hampton, Virginia 23668

Institute of Mathematics and Computer Science, Wrocław University of Technology, Wrocław, Poland