DETECTING COMPLETENESS FROM EXT-VANISHING

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ABSTRACT. Motivated by work of C. U. Jensen, R.-O. Buchweitz, and H. Flenner, we prove the following result. Let $R$ be a commutative noetherian ring and $\mathfrak{a}$ an ideal in the Jacobson radical of $R$. Let $\hat{R}_{\mathfrak{a}}$ be the $\mathfrak{a}$-adic completion of $R$. If $M$ is a finitely generated $R$-module such that $\text{Ext}_R^i(M) = 0$ for all $i \neq 0$, then $M$ is $\mathfrak{a}$-adically complete.

INTRODUCTION

A result of Jensen [13, (8.1)] characterizes the completeness property of a semilocal ring in terms of Ext-vanishing: If $R$ is a commutative noetherian ring, then it is a finite product of complete local rings if and only if $\text{Ext}_R^i(B, M) = 0$ for $i \neq 0$ whenever $B$ is flat and $M$ is finitely generated over $R$. In their investigation of Hochschild homology, Buchweitz and Flenner [3, (2.3)] recover one implication of the local case of this result: Let $R$ be a ring and $\mathfrak{m} \subset R$ a maximal ideal; if $M$ is an $\mathfrak{m}$-adically complete $R$-module, then $\text{Ext}_R^i(B, M) = 0$ for all $i \neq 0$ and each flat $R$-module $B$; see also [8, (3.7)] for the local case.

In this paper, we investigate converses to the Buchweitz-Flenner result: If $M$ is an $R$-module such that $\text{Ext}_R^i(B, M) = 0$ for all $i \neq 0$ and each flat $R$-module $B$, must $M$ be $\mathfrak{m}$-adically complete? One readily sees that this need not be the case when $M$ is not finitely generated. If $R$ is a local domain with $\dim(R) > 0$ and $M$ is the quotient field of $R$, then $M$ is not $\mathfrak{m}$-adically complete. However, $M$ is injective so $\text{Ext}_R^i(B, M) = 0$ for all $i \neq 0$ and each $R$-module $B$.

The following result is proved in 3.1. When $M$ finitely generated, it shows that the completeness of $M$ can be ascertained from the vanishing of the Ext-modules against a single flat module, namely $\hat{R}$.

Theorem A. Let $R$ be a commutative noetherian ring and $\mathfrak{a}$ an ideal in the Jacobson radical of $R$. Let $\hat{R}_{\mathfrak{a}}$ be the $\mathfrak{a}$-adic completion of $R$ and let $M$ be a finitely generated $R$-module. The following conditions are equivalent:

(i) $M$ is $\mathfrak{a}$-adically complete.

(ii) $\text{Ext}_R^i(\hat{R}_{\mathfrak{a}}, M) = 0$ for all $i \neq 0$.

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(iii) $\operatorname{Ext}^i_R(\hat{R}^a, M) = 0$ for all $i = 1, \ldots, \dim_R(M)$.

As a consequence of this theorem we obtain the following two results. The first is proved in [3,3] and the second is contained in Corollary [4,9]

**Theorem B.** The ring $R$ is $a$-adically complete if and only if the completion $\hat{R}^a$ is module-finite over $R$.

**Theorem C.** Let $M, N$ be finitely generated $R$-modules and $t$ an integer such that $\operatorname{Ext}^i_R(N, M) = 0$ for each $i < t$. If $\operatorname{Ext}^i_R(\hat{N}^a, M) = 0$ for each $i \neq t$, then $\operatorname{Ext}^i_R(N, M) = 0$ for each $i \neq t$ and $\operatorname{Ext}^i_R(N, M)$ is $a$-adically complete.

To prove these results, we employ a combination of classical module-theory and derived category techniques. Preliminary module-theoretic results are presented in Section [1]. Requisite derived category notions are discussed in Section [2].

## 1. Analytic Conductor Submodules

*Throughout this work, $R$ is a commutative noetherian ring and $a$ is an ideal contained in the Jacobson radical of $R$.*

**Lemma 1.1.** If $M$ is a finitely generated $R$-module, then $M$ admits a unique maximal $a$-adically complete submodule $C^a_M$.

**Proof.** Let $C^a(M)$ denote the collection of $a$-adically complete submodules of $M$ which is nonempty because it contains the zero submodule. Since $M$ is noetherian, this collection contains maximal elements, each of which is finitely generated. Let $N, N' \in C^a(M)$ be maximal elements and suppose that $N \neq N'$. By maximality, one has $N \nys N'$ and so $N \nys N + N'$. In particular, $N + N'$ is not $a$-adically complete. However, the module $N \oplus N'$ is finitely generated and $a$-adically complete. Hence, the homomorphic image $N + N'$ of $N \oplus N'$ is $a$-adically complete, a contradiction. Thus, $N = N'$, and the maximal element of $C^a(M)$ is unique. □

The submodule $C^a_M$ is the analytic conductor of $M$ with respect to $a$. It is the largest $R$-submodule of $M$ that is also an $\hat{R}^a$-module. Before presenting an important property of $C^a_M$ for this work, we introduce some frequently used maps.

### 1.2. Let $M$ be an $R$-module. The map $g^a_M: \operatorname{Hom}_R(\hat{R}^a, M) \to M$ is given by $g^a_M(\varphi) = \varphi(1)$, and $\varepsilon^a_M: M \to \hat{M}^a$ is the natural inclusion. Assume now that $M$ is finitely generated, so that $C^a_M$ is defined. Let $i^a_M: C^a_M \to M$ denote the natural inclusion. The map $f^a_M: C^a_M \to \operatorname{Hom}_R(\hat{R}^a, M)$ is given by $f^a_M(m)(r) = rm$.

The next result yields a well-defined map $k^a_M: \operatorname{Hom}_R(\hat{R}^a, M) \to C^a_M$, given by $k^a_M(\varphi) = \varepsilon(1)$, such that $g^a_M = i^a_M k^a_M$.

**Lemma 1.3.** If $M$ is a finitely generated $R$-module, then the natural inclusion $\operatorname{Hom}_R(\hat{R}^a, i^a_M): \operatorname{Hom}_R(\hat{R}^a, C^a_M) \to \operatorname{Hom}_R(\hat{R}^a, M)$ is bijective.

**Proof.** By left-exactness of $\operatorname{Hom}_R(\hat{R}^a, -)$ the given map is injective. To see that this map is surjective, fix $\varphi \in \operatorname{Hom}_R(\hat{R}^a, M)$; it suffices to show $\operatorname{Im}(\varphi) \subseteq C^a_M$. The image $\operatorname{Im}(\varphi)$ is finitely generated over $R$ and a homomorphic image of the $a$-adically complete $R$-module $\hat{R}^a$. Hence, $\operatorname{Im}(\varphi)$ is $a$-adically complete, and the desired conclusion follows from Lemma [1,1] □
2. Derived local homology and cohomology

We work in the derived category $D(R)$ of complexes of $R$-modules, indexed homologically. References on the subject include [9][11]. A complex $X$ is homologically bounded to the right if $H_i(X) = 0$ for all $i < 0$; it is homologically degreewise finite if $H_i(X)$ is finitely generated for each $i$; it is homologically finite if $\bigoplus_i H_i(X)$ is finitely generated; and it is homologically concentrated in degree $s$ if $H_i(X) = 0$ for all $i \neq s$. Isomorphisms in $D(R)$ are identified by the symbol $\simeq$, as are quasiisomorphisms in the category of complexes. For $X, Y \in D(R)$ set $\inf(X)$ and $\sup(X)$ to be the infimum and supremum, respectively, of the set $\{ n \in \mathbb{Z} \mid H_n(X) \neq 0 \}$. Let $X \otimes^L_R Y$ and $R\text{Hom}_R(X, Y)$ denote the left-derived tensor product and right-derived homomorphism complexes, respectively.

The left-derived local homology and right-derived local cohomology functors with support in an ideal $a$ are denote $L\Lambda^a(-)$ and $R\Gamma_a(-)$, respectively; see [1][10]. These are computed as follows. If $P \xrightarrow{\sim} X \xrightarrow{\sim} J$ are K-projective and K-injective resolutions, respectively, as in [2][10], then

$$\Lambda^a(-) = \lim_n (R/a^n \otimes_R -), \quad \Gamma_a(-) = \text{colim}_n \text{Hom}_R(R/a^n, -),$$

$$L\Lambda^a(X) = \Lambda^a(P), \quad R\Gamma_a(X) = \Gamma_a(J).$$

Note that the functor $\Gamma_a(-)$ is left-exact while $\Lambda^a(-)$ is neither left- nor right-exact.

2.1. Here is a catalog of properties of $L\Lambda^a(-)$ and $R\Gamma_a(-)$ that we will utilize.

(a) There are natural transformations of functors on $D(R)$ [11] (0.3)*:

$$R\Gamma_a(-) \xrightarrow{?} 1_{D(R)}(-) \xrightarrow{\sim} L\Lambda^a(-).$$

(b) The following are equivalences of functors on $D(R)$ [11] Cor. to (0.3)*:

$$L\Lambda^a(R\Gamma_a(-)) \xrightarrow{L\Lambda^a(\gamma)} L\Lambda^a(-) \quad \text{and} \quad R\Gamma_a(-) \xrightarrow{R\Gamma_a(\nu)} R\Gamma_a(L\Lambda^a(-)).$$

(c) One has natural equivalences of functors on $D(R)$ ([11] (0.3)] and [14] (3.1.2)):

$$L\Lambda^a(-) \simeq R\text{Hom}_R(R\Gamma_a(R), -) \quad \text{and} \quad R\Gamma_a(-) \simeq R\Gamma_a(R) \otimes^L_R -.$$

(d) (Adjointness) There is a natural equivalence of bifunctors on $D(R)$,

$$R\text{Hom}_R(R\Gamma_a(-), -) \xrightarrow{\theta} R\text{Hom}_R(-, L\Lambda^a(-)),$$

such that, for all complexes $X$ and $Y$ the next diagram commutes [11] (0.3)].

\[
\begin{array}{ccc}
R\text{Hom}_R(X, Y) & \xrightarrow{\text{RHom}_R(-, \nu_Y)} & R\text{Hom}_R(-, \nu_Y) \\
\text{RHom}_R(\gamma_X, Y) & & \text{RHom}_R(\gamma_X, Y) \\
& \xrightarrow{\theta_{XY}} & \text{RHom}_R(X, L\Lambda^a(Y))
\end{array}
\]

In particular, the morphism $R\text{Hom}_R(\gamma_X, Y)$ is an isomorphism in $D(R)$ if and only if $R\text{Hom}_R(-, \nu_Y)$ is so.

(e) One has a natural equivalence of functors on the full subcategory of $D(R)$ of complexes that are homologically finite and bounded to the right [5] (2.8)],

$$L\Lambda^a(-) \simeq - \bigotimes_R \tilde{R}^a.$$
Proof. \( \epsilon(1) \) is adjunction and maps in the following commutative diagram are isomorphisms:

\[
\begin{array}{c}
\phantom{\text{Diagram}} \\
\phantom{\text{Diagram}} \\
\phantom{\text{Diagram}} \\
\end{array}
\]

We now verify facts about \( LA^a(-) \) and \( R\Gamma_a(-) \) for the sequel. Fix \( M \in D(R) \) with \( K \)-injective resolution \( M \xrightarrow{\sim} J \). The map \( g_\varphi^2 : \text{Hom}_R(\hat{R}^a, J) \to J \) given by \( \varphi \mapsto \varphi(1) \) describes a well-defined morphism \( h_M^a : \text{RHom}_R(\hat{R}^a, M) \to M \) in \( D(R) \).

**Lemma 2.2.** If \( M \) is an \( R \)-complex, then the induced morphisms

\[
\begin{align*}
LA^a(h_M^a) : \ & LA^a(\text{RHom}_R(\hat{R}^a, M)) \to LA^a(M), \\
R\Gamma_a(h_M^a) : \ & R\Gamma_a(\text{RHom}_R(\hat{R}^a, M)) \to R\Gamma_a(M)
\end{align*}
\]

are isomorphisms in \( D(R) \). In particular, if \( LA^a(M) \neq 0 \) or \( R\Gamma_a(M) \neq 0 \), then \( \text{RHom}_R(\hat{R}^a, M) \neq 0 \).

**Proof.** For the first isomorphism, it suffices to check that the morphism

\[
\text{RHom}_R(R\Gamma_a(R), \text{RHom}_R(\hat{R}^a, M)) \xrightarrow{\text{RHom}_R(R\Gamma_a(R), h_M^a)} \text{RHom}_R(R\Gamma_a(R), M)
\]

is an isomorphism in \( D(R) \); see 2.1. In the commutative diagram

\[
\begin{array}{c}
\text{RHom}_R(R\Gamma_a(R), \text{RHom}_R(\hat{R}^a, M)) \\
\downarrow \\
\text{RHom}_R(R\Gamma_a(R), M)
\end{array}
\]

(1) is adjunction and \( \epsilon_M^a : R \to \hat{R}^a \) is the natural inclusion. Since \( R\Gamma_a(R) \otimes_R h_M^a \) is an isomorphism by 2.1, the same is true of \( \text{RHom}_R(R\Gamma_a(R) \otimes_R \epsilon_M^a, M) \). The diagram implies that \( \text{RHom}_R(R\Gamma_a(R), h_M^a) \) is an isomorphism.

For the second isomorphism, use the equivalence of 2.1.1 to see that the vertical maps in the following commutative diagram are isomorphisms:

\[
\begin{array}{c}
R\Gamma_a(\text{RHom}_R(\hat{R}^a, M)) \\
\downarrow \\
R\Gamma_a(\text{LA}^a(\text{RHom}_R(\hat{R}^a, M)))
\end{array}
\]

The morphism \( R\Gamma_a(\text{LA}^a(h_M^a)) \) is an isomorphism in \( D(R) \) because we have shown that \( \text{LA}^a(h_M^a) \) is so. The diagram shows that \( R\Gamma_a(h_M^a) \) is an isomorphism as well.

The final statement follows from the additivity of \( \text{LA}^a(-) \) and \( R\Gamma_a(-) \). \( \Box \)

**Lemma 2.3.** If \( M, N \) are homologically finite \( R \)-complexes, then the complex \( X = \text{RHom}_R(N, M) \) is homologically degreewise finite and \( \text{LA}^a(X) \simeq X \otimes_R \hat{R}^a \). In particular, one has \( \inf(\text{LA}^a(X)) = \inf(X) \) and \( \sup(\text{LA}^a(X)) = \sup(X) \).
Proof. The finiteness of each $H_i(X)$ is standard. A verification of the isomorphism is essentially in [6, Proof of (5.9)]. The flatness of $R \to \widehat{R}$ implies $H_i(L\Lambda^a(X)) \cong H_i(X) \otimes_R \widehat{R}$, and the equalities follow from the faithful flatness of $R \to \widehat{R}$. □

We next prove a vanishing result akin to [3, (2.3)]. Note that $M$ is not assumed to be finitely generated.

**Proposition 2.4.** Let $M$ be an $R$-module such that the morphism $\nu_M: M \to L\Lambda^a(M)$ is an isomorphism in $D(R)$. Then $\text{Ext}^i_R(\widehat{R}, M) = 0$ for each $i \neq 0$, and the evaluation map $g^*_M: \text{Hom}_R(\widehat{R}, M) \to M$ is an isomorphism.

Proof. Because the morphism $\nu_M: M \to L\Lambda^a(M)$ is an isomorphism in $D(R)$, the same is true of $R\text{Hom}_R(X, \nu_M): R\text{Hom}_R(X, M) \to R\text{Hom}_R(X, L\Lambda^a(M))$ for each $R$-complex $X$. From [2, 1(0)] it follows that the morphism $R\text{Hom}_R(\gamma_X, M): R\text{Hom}_R(X, M) \to R\text{Hom}_R(R\Gamma_a(X), M)$ is an isomorphism in $D(R)$.

The naturality of $\gamma$ provides the following commutative diagram in $D(R)$.

$$
\begin{array}{ccc}
R\Gamma_a(R) & \xrightarrow{R\Gamma_a(\nu_R)} & R\Gamma_a(L\Lambda^a(R)) \\
\gamma_R & & \gamma_{L\Lambda^a(R)} \\
R & \xrightarrow{\nu_R} & L\Lambda^a(R)
\end{array}
$$

An application of $R\text{Hom}_R(-, M)$ yields the following commutative diagram in $D(R)$:

$$
\begin{array}{ccc}
R\text{Hom}_R(L\Lambda^a(R), M) & \xrightarrow{R\text{Hom}_R(\nu_R, M)} & R\text{Hom}_R(R, M) \\
R\text{Hom}_R(\gamma_{L\Lambda^a(R)}, M) & \simeq & R\text{Hom}_R(\gamma_R, M) \\
R\text{Hom}_R(R\Gamma_a(L\Lambda^a(R)), M) & \xrightarrow{R\text{Hom}_R(R\Gamma_a(\nu_R), M)} & R\text{Hom}_R(R\Gamma_a(R), M),
\end{array}
$$

where the vertical morphisms are isomorphisms because of the argument of the previous paragraph. Hence, the morphism $R\text{Hom}_R(\nu_R, M)$ is also an isomorphism.

Next consider the commutative triangle

$$
\begin{array}{ccc}
R & \xrightarrow{\nu_R} & \widehat{R} \\
\kappa \downarrow & & \downarrow \\
L\Lambda^a(R) & \xrightarrow{\kappa} & \widehat{R},
\end{array}
$$

where $\kappa$ is obtained by taking degree 0 homology; see, e.g., [2, 1(0)]. Apply $R\text{Hom}_R(-, M)$ to produce the following commutative diagram in $D(R)$:

$$
\begin{array}{ccc}
R\text{Hom}_R(\widehat{R}, M) & \xrightarrow{R\text{Hom}_R(\nu_R, M)} & R\text{Hom}_R(R, M) \\
R\text{Hom}_R(\kappa, M) & \simeq & R\text{Hom}_R(\nu_R, M) \\
R\text{Hom}_R(L\Lambda^a(R), M) & \xrightarrow{R\text{Hom}_R(\nu_R, M)} & R\text{Hom}_R(R, M),
\end{array}
$$

which implies that $R\text{Hom}_R(\nu_R, M)$ is an isomorphism in $D(R)$.
In the final commutative diagram,

\[
\begin{array}{ccc}
\text{RHom}_R(R, M) & \xrightarrow{\xi} & \text{RHom}_R(C^a, M) \\
\text{RHom}_R(C^a, M) & \xrightarrow{g^a_M} & \text{RHom}_R(R^a, M)
\end{array}
\]

the morphism $\xi$ is the natural evaluation isomorphism. The diagram shows that $h^a_M$ is an isomorphism in $D(R)$. Since $M$ is a module, this implies $\text{Ext}^i_R(R^a, M) = 0$ for each $i \neq 0$ and further that the induced map $\text{Hom}_R(P, M) \to M$ is bijective. The definitions yield an equality $H^0(h^a_M) = g^a_M$, completing the proof. \qed

**Remark 2.5.** If $M$ is an $R$-module such that $M \cong \hat{M}^a$, then $M \simeq \Lambda^a(M)$. Indeed, the isomorphism $M \cong \hat{M}^a$ shows that $M$ is an $R^a$-module. Let $P$ be an $R^a$-free resolution of $M$. Then $P$ is an $R$-flat resolution of $M$ consisting of $a$-adically complete modules. Thus, one has $\Lambda^a(M) \simeq \Lambda^a(P) \cong P \simeq M$.

We are now in a position to give a useful alternate description of the analytic conductor submodule $C^a_M$; see \[\text{3.2}\] for the definitions of the maps.

**Proposition 2.6.** Let $M$ be a finitely generated $R$-module. The homomorphisms $f^a_M : C^a_M \to \text{Hom}_R(R^a, M)$ and $k^a_M : \text{Hom}_R(R^a, M) \to C^a_M$ are inverse isomorphisms. In particular, $\text{Hom}_R(R^a, M)$ is finitely generated over $R$.

**Proof.** One checks from the definitions that the composition $k^a_M f^a_M$ is the identity on $C^a_M$. Hence, the first conclusion will be verified once we show that $k^a_M$ is bijective; the second conclusion will then follow, as $C^a_M$ is finitely generated over $R$.

The module $C^a_M$ is $a$-adically complete, so Proposition \[\text{2.3}\] implies that the evaluation map $g^a_{C^a_M} : \text{Hom}_R(R^a, C^a_M) \to C^a_M$ is bijective. By Lemma \[\text{1.3}\] the map $\text{Hom}_R(R^a, i^a_M) : \text{Hom}_R(R^a, C^a_M) \to \text{Hom}_R(R^a, M)$ is an isomorphism. In particular, the composition $k^a_M = g^a_{C^a_M} \circ \text{Hom}_R(R^a, i^a_M)^{-1}$ is bijective, as desired. \qed

3. Detecting completeness

3.1. **Proof of Theorem \[\text{A}\]** The implication $(\text{ii}) \implies (\text{iii})$ follows from Proposition \[\text{2.3}\] and Remark \[\text{2.5}\] and $(\text{iii}) \implies (\text{ii})$ is trivial.

For the implication $(\text{iii}) \implies (\text{i})$, set $S = R / \text{Ann}_R(M)$. A result of Gruson and Raynaud \[\text{[15]}\] Seconde Partie, Thm. (3.2.6)], and Jensen \[\text{[12]}\] Prop. 6] provides the following bound on the projective dimension of $\hat{S}^a$ as an $S$-module:

\[(*) \quad \text{pd}_S(\hat{S}^a) \leq \dim(S) = \dim_R(M).\]

Consider the following sequence of isomorphisms in $D(R)$:

\[
\text{RHom}_R(R^a, M) \simeq \text{RHom}_R(R^a, \text{RHom}_S(S, M))
\]

\[
\simeq \text{RHom}_S(\hat{S}^a \otimes_R S, M)
\]

\[
\simeq \text{RHom}_S(\hat{S}^a, M).
\]

The first isomorphism follows from the fact that $M$ is naturally an $S$-module. The second is adjunction, and the third is standard as $S$ is finitely generated over $R$. Combining $(*)$ with the displayed isomorphisms, the assumption $\text{Ext}^i_R(R^a, M) = 0$ for all $i = 1, \ldots, \dim_R(M)$ implies $\text{Ext}^i_R(R^a, M) = 0$ for all $i \neq 0$. 
It follows that the natural map \( \lambda : \text{Hom}_R(\hat{R}^a, M) \to \text{RHom}_R(\hat{R}^a, M) \) is an isomorphism in \( \text{D}(R) \). Proposition 2.6 implies that the composition \( \lambda \circ f^a_M : C^a_M \to \text{RHom}_R(\hat{R}^a, M) \) is also an isomorphism in \( \text{D}(R) \). Because \( M \) is finitely generated, the natural morphism \( \mu : \text{L}A^a(M) \to \hat{M}^a \) is also an isomorphism in \( \text{D}(R) \). These data yield the following commutative diagram.

\[
\begin{array}{cccc}
C^a_M \xrightarrow{\epsilon_{C^a_M}} \text{L}A^a(C^a_M) & \xrightarrow{\text{L}A^a(\lambda \circ f^a_M)} & \text{L}A^a(\text{RHom}_R(\hat{R}^a, M)) \\
\downarrow \epsilon^a_M & & \downarrow \text{L}A^a(\hat{f}^a_M) \\
M \xrightarrow{\epsilon^a_M} \hat{M}^a & \xrightarrow{\mu} & \text{L}A^a(M)
\end{array}
\]

One sees that the composition of natural maps \( C^a_M \xrightarrow{\epsilon^a_M} M \xrightarrow{\epsilon^a_M} \hat{M}^a \) is bijective. Since \( \epsilon^a_M \) is also injective, the result now follows. \( \square \)

**Remark 3.2.** As the referee indicated, one can interpret Theorem [A] as a statement about the \( a \)-adic completeness of \( R / \text{Ann}_R(M) \) because \( M \) is \( a \)-adically complete if and only if \( R / \text{Ann}_R(M) \) is \( a \)-adically complete. For the sake of completeness, we include a sketch of the proof.

For one implication, assume that \( M \) is \( a \)-adically complete. For each prime \( \mathfrak{p} \in \text{Ass}_R(M) \), the injection \( R / \mathfrak{p} \hookrightarrow M \) and the completeness of \( M \) imply that \( R / \mathfrak{p} \) is \( a \)-adically complete. In particular, this is true for each minimal prime \( \mathfrak{p} \) containing \( \text{Ann}_R(M) \), and it follows that the same is true for each nonminimal prime \( \mathfrak{p} \) containing \( \text{Ann}_R(M) \). A prime filtration argument applied to \( R / \text{Ann}_R(M) \) shows that \( R / \text{Ann}_R(M) \) is \( a \)-adically complete.

Conversely, if \( R / \text{Ann}_R(M) \) is \( a \)-adically complete, then there exists an integer \( r \) and a surjection \( (R / \text{Ann}_R(M))^r \to M \), and it follows that \( M \) is \( a \)-adically complete.

From this fact, one easily deduces the following: When \( N \) is a second finitely generated \( R \)-module, if \( M \) is \( a \)-adically complete and \( \text{Supp}_R(N) \subseteq \text{Supp}_R(M) \), then \( N \) is \( a \)-adically complete.

3.3. **Proof of Theorem [B]**. One implication is trivial. For the other, assume that \( \hat{R}^a \) is module-finite over \( R \). As \( \hat{R}^a \) is flat and module-finite over \( R \), it is projective, and so \( \text{Ext}_R^i(\hat{R}^a, R) = 0 \) for each \( i \neq 0 \). The completeness of \( R \) follows from Theorem [A]. \( \square \)

The next example shows that the nontrivial implication in Corollary [B] fails if \( a \) is not assumed to be in the Jacobson radical of \( R \).

**Example 3.4.** Let \( k \) be a field and set \( R = k \times k \) and \( b = k \times 0 \). The Jacobson radical of \( R \) is 0. One readily checks that \( \hat{R}^b \cong 0 \times k \), showing that \( R \) is not \( b \)-adically complete even though \( \hat{R}^b \) is module-finite over \( R \).

Theorem [A] provides the converse to \( \text{K} \) (2.3)] when \( R \) is local and \( M \) is finitely generated. This is the implication \( i \Rightarrow i \) in the next result. The implication \( i \Rightarrow \text{K} \) is in \( \text{K} \) (3.7) or \( \text{K} \) (2.3), while the implication \( \text{K} \Rightarrow \text{K} \) is trivial.

**Corollary 3.5.** Let \( (R, m) \) be a local ring. For a finitely generated \( R \)-module \( M \) the following conditions are equivalent:

(i) \( M \) is \( m \)-adically complete.

(ii) For each flat \( R \)-module \( B \) and each \( i \neq 0 \), one has \( \text{Ext}_R^i(B, M) = 0 \).
(iii) For each \( i \neq 0 \), one has \( \text{Ext}^i_R(\hat{R}^a, M) = 0 \).

With Theorem \([\mathbf{A}]\) and Corollary \([3.5]\) in mind, one may ask what the finitely generated complete \( R \)-modules look like, say, when \( R \) is not complete. Examples include the modules of finite length. We observe next that one can have complete \( R \)-modules of infinite length.

**Example 3.6.** Let \((S, n)\) be a non-Artinian complete local ring. Set \( R = S[X]_{(n, X)} \) with maximal ideal \( m = (n, X)R \). The ring \( R \) is not \( m \)-adically complete, while the module \( R/(X)R \cong S \) is \( m \)-adically complete and has infinite length.

A finitely generated \( R \)-module \( C \) is semidualizing if \( R \cong \text{RHom}_R(C, C) \).

**Corollary 3.7.** If \( C \) is a semidualizing \( R \)-module such that \( \text{Ext}^i_R(\hat{R}^a, C) = 0 \) for all \( i \neq 0 \), then \( R \) is \( a \)-adically complete.

**Proof.** Theorem \([\mathbf{A}]\) implies that \( C \) is \( a \)-adically complete and hence \( C \cong C \otimes_R \hat{R}^a \cong C \otimes_R \hat{R} \). By \([5, (5.8)]\) the complex \( C \otimes_R \hat{R} \) is \( \hat{R} \)-semidualizing. This provides (1) in the following sequence while (4) and (5) are by hypothesis:

\[
\begin{align*}
\hat{R} & \xrightarrow{(1)} \text{RHom}_{\hat{R}}(C \otimes_R \hat{R}^a, C) \\
& \cong \text{RHom}_R(C, \text{RHom}_{\hat{R}}(\hat{R}^a, C)) \\
& \cong \text{RHom}_R(C, C) \\
& \cong R.
\end{align*}
\]

(2) is adjunction \([5, (1.5.2)]\) and (3) is standard \([5, (1.5.5)]\). \(\square\)

Here is a version of Theorem \([\mathbf{A}]\) for complexes.

**Proposition 3.8.** Let \( M \) be a homologically degreewise finite \( R \)-complex such that \( \inf(\Lambda^a(M)) = \inf(M) \) and \( \sup(\Lambda^a(M)) = \sup(M) \), e.g., if \( M \) is homologically finite. Fix an integer \( s \geq \sup(M) \). If \( \text{RHom}_R(\hat{R}^a, M) \) is homologically concentrated in degree \( s \), then so is \( M \), and the module \( H_s(M) \) is \( a \)-adically complete.

**Proof.** Assume \( M \not\cong 0 \). Then \( \sup(\Lambda^a(M)) = \sup(M) > -\infty \), and Lemma \([2.2]\) implies \( \text{RHom}_R(\hat{R}^a, M) \not\cong 0 \). Our hypotheses provide (1) and (3) in the sequence

\[
\begin{align*}
s & \xrightarrow{(1)} \sup(M) \xrightarrow{(2)} \sup(\text{RHom}_R(\hat{R}^a, M)) \xrightarrow{(3)} s
\end{align*}
\]

and (2) is from \([7, (2.1)]\); this implies \( s = \sup(M) \). Since \( \text{RHom}_R(\hat{R}^a, M) \) is homologically concentrated in degree \( s \), one has \( \Sigma^s \text{Ext}^{-s}_R(\hat{R}^a, M) \cong \text{RHom}_R(\hat{R}^a, M) \), providing the first of the following isomorphisms:

\[
\Lambda^a(\Sigma^s \text{Ext}^{-s}_R(\hat{R}^a, M)) \cong \Lambda^a(\text{RHom}_R(\hat{R}^a, M)) \cong \Lambda^a(M),
\]

while the second one is from Lemma \([2.2]\). This provides (5) in the following sequence:

\[
\begin{align*}
\inf(M) & \xrightarrow{(4)} \inf(\Lambda^a(M)) \xrightarrow{(5)} \inf(\Lambda^a(\Sigma^s \text{Ext}^{-s}_R(\hat{R}^a, M))) \xrightarrow{(6)} s \xrightarrow{(7)} \sup(M) \xrightarrow{(8)} \inf(M)
\end{align*}
\]
while (4) is by assumption, (6) is by \[2.1(c)\]. (7) is proved above, and (8) is trivial. It follows that \(\inf(M) = \sup(M) = s\) and so \(M\) is homologically concentrated in degree \(s\). Finally, one has \(M \simeq \Sigma^s H_s(M)\) and so
\[
R\text{Hom}(\hat{\mathcal{R}}^s, M) \simeq R\text{Hom}(\hat{\mathcal{R}}^s, \Sigma^s H_s(M)) \simeq \Sigma^s R\text{Hom}(\hat{\mathcal{R}}^s, H_s(M)).
\]
Since this is homologically concentrated in degree \(s\), one has \(\text{Ext}^i_R(\hat{\mathcal{R}}^s, H_s(M)) = 0\) for each \(i \neq 0\). Theorem \[A\] implies that \(H_s(M)\) is \(a\)-adically complete. \(\square\)

The next result contains Theorem \[C\] from the Introduction.

Corollary 3.9. Let \(M, N\) be homologically finite \(R\)-complexes and \(s \in \mathbb{Z}\) such that \(s \geq \sup(R\text{Hom}(N, M))\). If \(R\text{Hom}(\hat{N}^s, M)\) is homologically concentrated in degree \(s\), then so is \(R\text{Hom}(N, M)\), and \(\text{Ext}^i_R(\hat{N}^s, M)\) is \(a\)-adically complete.

Proof. \[2.1(c)\] and adjunction provide the following sequence:
\[
R\text{Hom}(\hat{N}^s, M) \simeq R\text{Hom}(\hat{N}^s \otimes^L_R N, M) \simeq R\text{Hom}(\hat{N}^s, R\text{Hom}(N, M)).
\]
Lemma \[2.3\] shows that Proposition \[3.8\] applies to the complex \(R\text{Hom}(N, M)\), yielding the desired conclusion. \(\square\)

Corollary 3.10. Assume that \(R\) is local and \(M, N\) are nonzero finitely generated \(R\)-modules with \(\text{pd}_R(N) < \infty\). If \(\text{Ext}^i_R(\hat{N}^s, M) = 0\) for each \(i \neq 0\), then \(N\) is free and \(M\) is \(a\)-adically complete.

Proof. Using \(s = 0\) in Corollary \[3.9\] one concludes that \(\text{Ext}^i_R(N, M) = 0\) for each \(i \neq 0\) and that \(\text{Hom}(N, M)\) is \(a\)-adically complete. Since \(\text{pd}_R(N)\) is finite, one has
\[
R\text{Hom}(N, M) \simeq R\text{Hom}(N, R) \otimes^L_R M
\]
by tensor-evaluation \[2.1(4.4)\]. The next equalities are from \[7 (2.1)\] and \[5 (2.13)\]:
\[
0 = \inf(R\text{Hom}(N, M)) = \inf(R\text{Hom}(N, R)) + \inf(M) = -\text{pd}_R(N).
\]
Since \(R\) is local, the module \(N \neq 0\) is free and \(\text{Hom}(N, M) \cong M^n\) for some \(n > 0\). Because \(M^n\) is \(a\)-adically complete, the same is true of \(M\). \(\square\)

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