PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 7, July 2008, Pages 2505–2513
S 0002-9939(08)09200-9
Article electronically published on March 7, 2008

SHARP ESTIMATES FOR THE IDENTITY MINUS HARDY OPERATOR ON THE CONE OF DECREASING FUNCTIONS

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(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. It is shown that if we restrict the identity minus Hardy operator on the cone of nonnegative decreasing functions f in L^p, then we have the sharp estimate

$$\|(I-H)f\|_{L^p} \leq \frac{1}{(p-1)^{\frac{1}{p}}} \|f\|_{L^p}$$

for $p = 2, 3, 4, \ldots$. In other words,

$$\|f^{**} - f^*\|_{L^p} \leq \frac{1}{(p-1)^{\frac{1}{p}}} \|f\|_{L^p}$$

for each $f \in L^p$ and each integer $p \geq 2$.

It is also shown, via a connection between the operator $I - H$ and Laguerre functions, that

$$\|(1-\alpha)I + \alpha(I-H)\|_{L^2 \to L^2} = \|I - \alpha H\|_{L^2 \to L^2} = 1$$

for all $\alpha \in [0, 1]$.

1. Introduction

The importance of the Hardy operator

$$(Hf)(x) = \frac{1}{x} \int_0^x f(t)dt$$

in analysis and its applications is well known. In this paper we consider the related operator

$$(I-H)f(x) = f(x) - \frac{1}{x} \int_0^x f(t)dt,$$

which also seems to be quite interesting. For example, in [1] it was shown that this operator is bounded and has a bounded inverse in $L^p(t^\alpha, \frac{dt}{t})$ for $p \geq 1$ and $\alpha \in (-1, \infty) \setminus 0$. Moreover, in [2] (see also [3] for the weighted case) it was shown that if we take $e_0 = \chi_{(0,1)}$, the characteristic function of the unit interval, then the sequence

$$e_n = (I-H)^n e_0, \quad n \in \mathbb{Z} \tag{1.1}$$

Received by the editors February 9, 2006, and, in revised form, January 26, 2007, and March 30, 2007.

2000 *Mathematics Subject Classification.* Primary 26D10, 46E30.

Key words and phrases. The Hardy operator, cone of decreasing functions, sharp estimates.

forms an orthonormal basis in $L^2 = L^2(0,\infty)$. Therefore

$$(I - H)e_n = e_{n+1},$$

and we see that $I - H$ and its inverse $(I - H)^{-1}$ are shift isometries in L^2. In particular, we have

$$\|(I - H)f\|_{L^2} = \|f\|_{L^2}$$

for all $f \in L^2(0,\infty)$. With this interesting property in mind it is natural to consider estimates for the operator $I - H$ for the case $p \neq 2$.

This paper consists of two parts. In the first part we will show that if we consider the bijective isometry $(Uf)(x) = e^{\frac{x}{2}} f(e^x)$ between the spaces $L^2(0,\infty)$ and $L^2(\mathbb{R})$, then the operator $U(I-H)U^{-1}$ coincides with the shift operator in the orthonormal basis in $L^2(\mathbb{R})$ which consists of the Laguerre functions $l_n(x)$ and the functions $u_{-n}(x) = l_n(-x)$. This gives a new simple proof of (1.1) and also shows that

$$\|\alpha I + (1-\alpha)(H - I)\|_{L^2 \to L^2} = 1, \quad 0 \leq \alpha \leq 1;$$

i.e., the unit sphere of the space of all bounded linear operators on L^2 contains an interval with the endpoints I, $H - I$ and the midpoint $\frac{1}{2}H$.

The second part of the paper is devoted to the study of estimates for $H - I$ and its inverse $(H - I)^{-1}$ on certain cones of functions in $L^p = L^p(0,\infty)$. To formulate the result let us consider the cone C^p of decreasing nonnegative functions from L^p and introduce the notation

$$\|H - I\|_{C^p} = \sup_{f \in C^p,\ \|f\|_{L^p} \leq 1} \|(H - I)f\|_{L^p}\,. \tag{1.2}$$

We prove that

$$\|H - I\|_{C^p} = \frac{1}{(p-1)^{1/p}},\ p \in \{2, 3, 4, ...\}. \tag{1.3}$$

We give an example which shows that this estimate is not true if instead of C^p we consider the whole space L^p. Our proof does not work for other values of p and we do not know if (1.3) is true for all $p \geq 2$.

For each f in the cone of decreasing functions we have

$$(H - I)f(x) = f^{**}(x) - f^*(x)$$

for almost every x. So we see that (1.3) gives a sharp estimate in the following well-known equivalence in Lorentz $L^{p,q}$-spaces (see [4]):

$$\|f\|_{L_{p,q}} := \left(\int_0^\infty \left(x^{1/p} f^*(x) \right)^q \frac{dx}{x} \right)^{1/q} \approx \left(\int_0^\infty \left(x^{1/p} (f^{**}(x) - f^*(x)) \right)^q \frac{dx}{x} \right)^{1/q}$$

for the case $q = p \in \{2, 3, 4, ...\}$. In fact the functional $f^{**} - f^*$ is of considerable interest in a number of contexts. One notable instance is the paper [5] (see also [4]), and there are numerous more recent papers which also deal with it.

2. The identity minus Hardy operator and Laguerre polynomials

It is well known that the Laguerre polynomials $L_n(x)$, $n = 0, 1, ...$ (see [6]) have the following properties:

$$L_n(0) = 1,\ L_0(x) \equiv 1,\ \ L_n'(x) - L_n(x) = L_{n+1}'(x). \tag{2.1}$$

From (2.1) it follows that

$$\int_0^x (L_n'(s) - L_n(s))ds = \int_0^x L_{n+1}'(s)ds$$

and, therefore,

$$L_n(x) - \int_0^x L_n(s)ds = L_{n+1}(x). \tag{2.2}$$

It also follows from (2.1), using integration by parts, that

$$\int_x^\infty e^{-s} L_{m+1}(s)ds = e^{-x}(L_{m+1}(x) - L_m(x)) \quad \text{for all } m \geq 0. \tag{2.3}$$

It is well known that the Laguerre functions

$$l_n(x) = L_n(x)e^{-\frac{x}{2}}$$

form an orthonormal basis in $L^2(0, \infty)$. Therefore the family of functions

$$\begin{aligned} g_n(u) &= l_{n-1}(u)\chi_{(0,\infty)}, n \geq 1, \\ g_n(u) &= -l_{-n}(-u)\chi_{(-\infty,0)}, n \leq 0, \end{aligned}$$

form an orthonormal basis in $L^2(\mathbb{R})$. Let $\varphi(u) = e^{-u/2}\chi_{(0,\infty)}$. Then, by applying (2.2) for $n \geq 0$ and by applying (2.3) for $n \leq -1$, we see that the operator

$$Sf = f - f * \varphi,$$

where $*$ means convolution, has the property

$$Sg_n = g_{n+1}.$$

So S is a shift isometry in $L^2(\mathbb{R})$.

Let $f \in L^2(0, \infty)$ and consider the operator

$$(Uf)(x) = e^{x/2}f(e^x).$$

As

$$\int_0^\infty f(x)^2 dx = \int_{-\infty}^\infty (f(e^x)e^{\frac{x}{2}})^2 dx,$$

we obtain that $U : L^2(0, \infty) \to L^2(\mathbb{R})$ is a bijective isometry. Furthermore,

$$\begin{aligned} (UHf)(u) = e^{u/2}\frac{1}{e^u}\int_0^{e^u} f(s)ds &= \int_{-\infty}^u f(e^v)e^{v/2}e^{-(u-v)/2}dv \\ &= \int_{-\infty}^u (Uf)(v)e^{-(u-v)/2}dv. \end{aligned}$$

We get $(UHf)(u) = Uf * \varphi(u)$ and therefore $U(I - H)f = Uf - Uf * \varphi = SUf$, i.e.

$$U(I - H) = SU,$$

or equivalently

$$U(I - H)U^{-1} = S.$$

As $(Uf)(x) = e^{\frac{x}{2}}f(e^x)$ is a bijective isometry between the spaces $L^2(0, \infty)$ and $L^2(\mathbb{R})$ we immediately obtain from the last equality that the sequence

$$e_n = -U^{-1}g_n \quad (n \in \mathbb{Z})$$

forms an orthonormal basis in $L^2(0, \infty)$ and

$$(I - H)e_n = e_{n+1} \quad (n \in \mathbb{Z}).$$

Clearly $U(\chi_{(0,1)}) = -g_0$ and so $e_0 = \chi_{(0,1)}$. Thus we have obtained a new proof of the result of [2] which was mentioned in the introduction. (Cf. (1.1)).

Corollary 1. *For any* $0 \le \alpha \le 1$ *we have*

$$\|\alpha I + (1-\alpha)(H-I)\|_{L^2 \to L^2} = 1; \tag{2.4}$$

i.e. the unit sphere of the space of all bounded linear operators on L^2 *contains an interval with the endpoints* I *and* $H - I$ *and the midpoint* $\frac{1}{2}H$.

Proof. So far I has denoted the identity operator on $L^2(0,\infty)$. Here it will also unambiguously denote the identity operator on $\ell^2(\mathbb{Z})$.

Let V be the shift operator on $\ell^2(\mathbb{Z})$. Then it is easy to check that

$$\|\alpha I + (1-\alpha)V\|_{\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})} = 1 \tag{2.5}$$

for each $\alpha \in [0,1]$.

Now we let $\{\psi_n\}_{n\in\mathbb{Z}}$ be the canonical orthonormal basis of ℓ^2, i.e., $\psi_n = \{\delta_{mn}\}_{m\in\mathbb{Z}}$ and use the fact that, obviously, $\{(-1)^n e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(0,\infty)$. We define an isometry W of $\ell^2(\mathbb{Z})$ onto $L^2(0,\infty)$ by setting $W\psi_n = (-1)^n e_n$ for each $n \in \mathbb{Z}$. Since $(H-I)(-1)^n e_n = (-1)^{n+1} e_{n+1}$ we see that $H - I = WVW^{-1}$. Thus (2.4) follows immediately from (2.5). □

Remark 1. Using the same method of proof as in Corollary 1, we can also show that

$$\|(1-\alpha)I + \alpha(I-H)\|_{L^2 \to L^2} = \|I - \alpha H\|_{L^2 \to L^2} = 1$$

for all $\alpha \in [0,1]$.

3. Operator norms of $H - I$ and $(H-I)^{-1}$ on cones in $L^p(0,\infty)$

We start with the case $p = 3$.

Theorem 1. $\|H - I\|_{C^3} = \frac{1}{\sqrt[3]{2}}$.

Proof. Let us consider the subset consisting of all simple functions in C^3. Each simple function in C^3 can be written in the form

$$g(x) = \sum_{i=1}^{n} c_i \chi_{(0,a_i)}\, , c_i \ge 0, i = 1,2,...,n, 0 < a_1 < a_2 < ... < a_n, \tag{3.1}$$

where $\chi_{(a,b)}$ denotes the characteristic function of (a,b). This subset is dense in C^3; therefore to prove that $\|H - I\|_{C^3} \le \frac{1}{\sqrt[3]{2}}$, it is enough to show that

$$\|(H-I)g\|_{L^3} \le \frac{1}{\sqrt[3]{2}} \|g\|_{L^3}$$

holds for all simple functions $g \in C^3$. Straightforward calculations give:

$$(H-I)g(x) = \begin{cases} 0 & , 0 < x < a_1, \\ \frac{c_1 a_1}{x} & , a_1 < x < a_2, \\ \frac{c_1 a_1 + c_2 a_2}{x} & , a_2 < x < a_3, \\ \vdots & \vdots \\ \frac{c_1 a_1 + c_2 a_2 + ... + c_n a_n}{x} & , a_n < x. \end{cases}$$

Let us calculate the norm of g:

$$\begin{aligned} \|g\|_{L^3}^3 &= \int_0^\infty |g(x)|^3\,dx \\ &= (c_1+c_2+...+c_n)^3 a_1 + (c_2+c_3+...+c_n)^3(a_2-a_1)+... \\ &\quad + (c_{n-1}+c_n)^3(a_{n-1}-a_{n-2}) + c_n^3(a_n-a_{n-1}). \end{aligned} \tag{3.2}$$

Similarly, for the norm of $\sqrt[3]{2}(H-I)g$:

$$\begin{aligned} \left\|\sqrt[3]{2}(H-I)g\right\|_{L^3}^3 &= \int_0^\infty \left|\sqrt[3]{2}(H-I)g(x)\right|^3 dx \\ &= c_1^3 a_1^3\left(\frac{1}{a_1^2}-\frac{1}{a_2^2}\right) + (c_1a_1+c_2a_2)^3\left(\frac{1}{a_2^2}-\frac{1}{a_3^2}\right)+... \\ &\quad + (c_1a_1+c_2a_2+...+c_{n-1}a_{n-1})^3\left(\frac{1}{a_{n-1}^2}-\frac{1}{a_n^2}\right) \\ &\quad + (c_1a_1+c_2a_2+...+c_{n-1}a_{n-1}+c_na_n)^3\frac{1}{a_n^2}. \end{aligned} \tag{3.3}$$

The two expressions (3.2) and (3.3) are both homogeneous polynomials of order 3 in the variables $c_1, c_2, ..., c_n$. Thus, to compare the sizes of $\|g\|_{L^3}^3$ and $\left\|\sqrt[3]{2}(H-I)g\right\|_{L^3}^3$, we compare the coefficients of these two polynomials in Table 1.

TABLE 1

	$\|g\|_{L^3}^3$	$\left\|\sqrt[3]{2}(H-I)g\right\|_{L^3}^3$
c_i^3 :	a_i	a_i
$c_i^2c_j,\ i<j$:	$3a_i$	$3\frac{a_i^2}{a_j}$
$c_ic_j^2,\ i<j$:	$3a_i$	$3a_i$
$c_ic_jc_k,\ i<j<k$:	$6a_i$	$6\frac{a_ia_j}{a_k}$

To obtain these coefficients we use the multinomial theorem and note that in the case of $\|g\|_{L^3}^3$, when calculating the coefficient of $c_ic_jc_k$, we only have to consider the contributions of the first m terms of (3.2), where $m = \min\{i,j,k\}$.

Analogously, in the case of $\left\|\sqrt[3]{2}(H-I)g\right\|_{L^3}^3$, for the coefficient of $c_ic_jc_k$, we only have to consider the contribution of the mth term in (3.3) and those after it, where this time $m = \max\{i,j,k\}$.

We observe that the coefficients for $c_i^2c_j$ and $c_ic_jc_k$ differ between $\|g\|_{L^3}^3$ and $\left\|\sqrt[3]{2}(H-I)g\right\|_{L^3}^3$. For $c_i^2c_j$ we have

$$\frac{a_i^2}{a_j} < a_i$$

because $0 < a_i < a_j$ and for $c_ic_jc_k$ the inequality

$$\frac{a_ia_j}{a_k} < a_i$$

is true because $0 < a_j < a_k$. Hence, all coefficients for $\|g\|_{L^3}^3$ are greater than or equal to those for $\left\|\sqrt[3]{2}(H-I)g\right\|_{L^3}^3$. Altogether this gives the inequality

$$\|(H-I)g\|_{L^3} \leq \frac{1}{\sqrt[3]{2}} \|g\|_{L^3} . \tag{3.4}$$

To finish the proof of the theorem we only need to note that we obtain equality in (3.4) when $g(x) = \chi_{(0,a)}$. □

For a general integer $p \geq 2$ the calculations become more involved, but in principle the same method as in Theorem 1 works.

Theorem 2. $\|H-I\|_{C^p} = \frac{1}{(p-1)^{1/p}}$, $p \in \{2, 3, 4, ...\}$.

Proof. As in Theorem 1 it suffices to consider simple functions in C^p, which can be written in the form

$$g(x) = \sum_{i=1}^{n} c_i \chi_{(0,a_i)} , c_i \geq 0, i = 1, 2, ..., n, 0 < a_1 < a_2 < ... < a_n.$$

Calculations using the multinomial theorem show that we have the formulas shown in Table 2 for the coefficient for a general term $c_{i_1}^{j_1} c_{i_2}^{j_2} ... c_{i_m}^{j_m}$ in the polynomials $\|g\|_{L^p}^p$ and $\left\|(p-1)^{1/p}(H-I)g\right\|_{L^p}^p$,

TABLE 2

	$\|g\|_{L^p}^p$	$\left\|(p-1)^{1/p}(H-I)g\right\|_{L^p}^p$
$c_{i_1}^{j_1} c_{i_2}^{j_2} ... c_{i_m}^{j_m}$:	$\frac{p!}{j_1! j_2! ... j_m!} a_{i_1}$	$\frac{p!}{j_1! j_2! ... j_m!} \frac{a_{i_1}^{j_1} a_{i_2}^{j_2} ... a_{i_m}^{j_m}}{a_{i_m}^{p-1}}$

where m is an integer with $1 \leq m \leq n$ and $1 \leq i_1 < i_2 < ... < i_m \leq n$, $j_1 + j_2 + ... + j_m = p$, $j_1 > 0$ and $j_2, j_3, ..., j_m \geq 0$.

The fact that we have chosen m arbitrarily in the above calculations of the coefficients of the multiple powers $c_{i_1}^{j_1} c_{i_2}^{j_2} ... c_{i_m}^{j_m}$ means that we have accounted for all possible terms in the polynomials $\|g\|_{L^p}^p$ and $\left\|(p-1)^{1/p}(H-I)g\right\|_{L^p}^p$.

Since $j_1 > 0$, we see that

$$\frac{a_{i_1}^{j_1} a_{i_2}^{j_2} ... a_{i_m}^{j_m}}{a_{i_m}^{p-1}} \leq a_{i_1}$$

for all possible choices of $i_1, i_2, ..., i_m$, $j_1, j_2, ..., j_m$ and m. This gives the inequality

$$\|(H-I)g\|_{L^p} \leq \frac{1}{(p-1)^{1/p}} \|g\|_{L^p} .$$

The case of equality is obtained when $g(x) = \chi_{(0,a)}$. □

The restriction to the functions in $C^p \subset L^p(0,\infty)$ is essential for the estimate

$$\|(H-I)f\|_{L^p} \leq \frac{1}{(p-1)^{1/p}} \|f\|_{L^p}$$

to be true. We illustrate this with the following example:

Example 1. Let $f(x) = \chi_{(1,3/2)}$. Straightforward calculations give: $\frac{1}{\sqrt[3]{2}} \|f\|_{L^3} = \frac{1}{\sqrt[3]{4}}$ and $\|(H-I)f\|_{L^3} = \sqrt[3]{\frac{11}{36}}$. Hence, $\|(H-I)f\|_{L^3} > \frac{1}{\sqrt[3]{2}} \|f\|_{L^3}$.

We now turn to an estimate for the inverse operator $(H-I)^{-1}$. As shown in [1], this operator is given by

$$(H-I)^{-1}f(x)=\int_x^\infty \frac{f(s)}{s}ds-f(x). \tag{3.5}$$

Let us introduce the cone $\widetilde{C}^p$ defined by

Definition 1. $\widetilde{C}^p=\{f\in L^p(0,\infty): xf(x)$ is an increasing nonnegative function$\}$.

By analogy with C^p we define the operator norm on $\widetilde{C}^p$ to be:

$$\left\|(H-I)^{-1}\right\|_{\widetilde{C}^p}=\sup_{f\in\widetilde{C}^p,\ \|f\|_{L^p}\le 1}\left\|(H-I)^{-1}f\right\|_{L^p}.$$

The reason for considering the cone $\widetilde{C}^p$ is connected with the following fact.

Proposition 1. *The operator $H-I$ maps the cone C^p onto the cone $\widetilde{C}^p$ and therefore the operator $(H-I)^{-1}$ maps the cone $\widetilde{C}^p$ onto the cone C^p.*

Proof. First of all, if $f\in C^p$, then

$$(H-I)f(x)=\frac{1}{x}\int_0^x f(t)dt-f(x)=\frac{1}{x}\int_0^x (f(t)-f(x))\,dt\ge 0.$$

Moreover, if $0<x_1<x_2$, then

$$\begin{aligned}x_1(H-I)f(x_1) &= x_1(\frac{1}{x_1}\int_0^{x_1}(f(t)-f(x_1))\,dt)=x_2(\frac{1}{x_2}\int_0^{x_1}(f(t)-f(x_1))\,dt)\\ &\le x_2(\frac{1}{x_2}\int_0^{x_2}(f(t)-f(x_2))\,dt)=x_2(H-I)f(x_2).\end{aligned}$$

So, since the operator $H-I$ is bounded in L^p for $p>1$, we see that it maps the cone C^p into the cone $\widetilde{C}^p$. To prove that it maps onto the cone $\widetilde{C}^p$ it is enough to show that $(H-I)^{-1}$ maps the cone $\widetilde{C}^p$ into the cone C^p. Indeed, if $f\in\widetilde{C}^p$, then

$$(H-I)^{-1}f(x)=\int_x^\infty f(s)\frac{ds}{s}-f(x)=\int_x^\infty \frac{sf(s)-xf(x)}{s}\frac{ds}{s}\ge 0.$$

Moreover, if $0<x_1<x_2$, then

$$\begin{aligned}(H-I)^{-1}f(x_1) &= \int_{x_1}^\infty \frac{sf(s)-x_1f(x_1)}{s}\frac{ds}{s}\ge\int_{x_2}^\infty \frac{sf(s)-x_2f(x_2)}{s}\frac{ds}{s}\\ &= (H-I)^{-1}f(x_2).\end{aligned}$$

So $(H-I)^{-1}f$ is a decreasing nonnegative function. As $(H-I)^{-1}$ is bounded in L^p for $p>1$, we obtain that $(H-I)^{-1}$ maps $\widetilde{C}^p$ into C^p. □

Now we proceed to our main result for $\left\|(H-I)^{-1}\right\|_{\widetilde{C}^p}$:

Theorem 3. $\left\|(H-I)^{-1}\right\|_{\widetilde{C}^q}=(q-1)^{\frac{q-1}{q}}$*, where* $q=p'=\frac{p}{p-1}$ *and* $p\in\{2,3,4,...\}$.

Proof. Let $f\in\widetilde{C}^q$. Then $(H-I)^{-1}f\in C^q$. Therefore, for the function

$$g(x)=\lambda\left|(H-I)^{-1}f(x)\right|^{q-1}\operatorname{sgn}((H-I)^{-1}f(x))$$

where

$$\lambda=\left(\int_0^\infty\left|(H-I)^{-1}f(x)\right|^q dx\right)^{-1/p}$$

we have $\|g\|_{L^p} = 1$ and $\int_0^\infty ((H-I)^{-1}f(x))g(x)dx = \left\|(H-I)^{-1}f\right\|_{L^q}$.

Moreover, from (3.5) it follows that $(H-I)^{-1} = (H-I)^*$. Therefore, from Hölder's inequality we have

$$\left\|(H-I)^{-1}f\right\|_{L^q} = \int_0^\infty ((H-I)^{-1}f(x))g(x)dx = \int_0^\infty f(x)((H-I)g(x))dx$$
$$\leq \|f\|_{L^q}\|(H-I)g\|_{L^p} \leq \frac{1}{(p-1)^{1/p}}\|f\|_{L^q}\|g\|_{L^p} = (q-1)^{\frac{q-1}{q}}\|f\|_{L^q},$$

or equivalently

$$\left\|(H-I)^{-1}\right\|_{\widetilde{C}^q} \leq (q-1)^{\frac{q-1}{q}}.$$

We only need to note that

$$\left\|(H-I)^{-1}f\right\|_{L^q} = (q-1)^{\frac{q-1}{q}}\|f\|_{L^q}$$

for $f(x) = \frac{1}{x}\chi_{(a,\infty)} \in \widetilde{C}^p, a > 0$ to finish the proof. □

Corollary 2. *Let $f \in C^q$ where $q = \frac{p}{p-1}$ and $p \in \{2,3,4,...\}$. Then*

$$\|f\|_{L^q} \leq (q-1)^{\frac{q-1}{q}}\|(H-I)f\|_{L^q}.$$

Proof. From Theorem 3 above we have

$$\left\|(H-I)^{-1}g\right\|_{L^q} \leq (q-1)^{\frac{q-1}{q}}\|g\|_{L^q}, \forall g \in \widetilde{C}^q. \tag{3.6}$$

Because $H-I$ is a bijective mapping between C^q and $\widetilde{C}^q$ we can write $g = (H-I)f$. We only need to substitute this expression in (3.6). □

We would like to finish by stating two problems.

Problem 1. Is it true that $\|H-I\|_{C^p} = \frac{1}{(p-1)^{1/p}}$ for all $p \geq 2$?

Problem 2. Is it true that $\|H-I\|_{L^p\to L^p} = \frac{1}{p-1}$, $1 < p \leq 2$?

Remark 2. If $\|H-I\|_{L^p\to L^p} = \frac{1}{p-1}$, $1 < p \leq 2$, then from the well-known result

$$\|H\|_{L^p\to L^p} = 1 + \frac{1}{p-1}$$

we will have

$$\|H\|_{L^p\to L^p} = \|I\|_{L^p\to L^p} + \|H-I\|_{L^p\to L^p},$$

and therefore the unit sphere of the space of all bounded linear operators on L^p will contain an interval with the endpoints $(p-1)(H-I)$ and I (compare with Corollary 1).

Acknowledgements

The authors are very thankful to the referee for valuable comments which improved the initial version of this paper.

References

[1] N. Kruglyak, L. Maligranda and L.-E. Persson, *On an elementary approach to the fractional Hardy inequality*, Proc. Amer. Math. Soc. **128** (2000), pp. 727-734. MR1676324 (2000e:26020)

[2] A. Brown, P.R. Halmos and A.L. Shields, *Cesàro operators*, Acta Sci. Math. (Szeged) **26** (1965), pp. 125-137. MR0187085 (32:4539)

[3] N. Kaiblinger, L. Maligranda and L.-E. Persson, *Norms in weighted L^2-spaces and Hardy operators,* Function Spaces (Poznan, 1998), Lecture Notes in Pure and Appl. Math., vol. 213, Marcel Dekker, New York, 2000, pp. 205-216. MR1772126 (2001g:46061)

[4] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, MA, 1988. MR928802 (89e:46001)

[5] C. Bennett, R. DeVore and R. Sharpley, *Weak-L^∞ and BMO*, Ann. of Math. (2) **113** (1981), pp. 601-611. MR621018 (82h:46047)

[6] G. Sansone, *Orthogonal Functions* (Revised English Edition), Dover, New York, 1991, 411 pp. MR1118381 (92e:42015)

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