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A REMARK ON IRREGULARITY OF THE $\overline{\partial}$-NEUMANN PROBLEM ON NON-SMOOTH DOMAINS

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Abstract. It is an observation due to J. J. Kohn that for a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$ there exists $s > 0$ such that the $\overline{\partial}$-Neumann operator on Ω maps $W^s_{(0,1)}(\Omega)$ (the space of $(0,1)$-forms with coefficient functions in L^2-Sobolev space of order s) into itself continuously. We show that this conclusion does not hold without the smoothness assumption by constructing a bounded pseudoconvex domain Ω in $\mathbb{C}^2$, smooth except at one point, whose $\overline{\partial}$-Neumann operator is not bounded on $W^s_{(0,1)}(\Omega)$ for any $s > 0$.

Let $W^s(\Omega)$ and $W^s_{(p,q)}(\Omega)$ denote the L^2-Sobolev space on Ω of order s and the space of (p,q)-forms with coefficient functions in $W^s(\Omega)$, respectively. Also $\|.\|_{s,\Omega}$ denotes the norm on $W^s_{(p,q)}(\Omega)$. Let N_q denote the inverse of the complex Laplacian, $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, on square integrable $(0,q)$-forms. It is an observation of Kohn, as the following proposition says, that on a smooth bounded pseudoconvex domain the $\overline{\partial}$-Neumann problem is regular in the Sobolev scale for sufficiently small levels.

Proposition 1 (Kohn)**.** *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Then there exist positive ε and C (depending on Ω) such that*

$$\|N_q u\|_{\varepsilon,\Omega} \leq C\|u\|_{\varepsilon,\Omega}, \quad \|\overline{\partial} N_q u\|_{\varepsilon,\Omega} \leq C\|u\|_{\varepsilon,\Omega}, \quad \|\overline{\partial}^* N_q u\|_{\varepsilon.\Omega} \leq C\|u\|_{\varepsilon,\Omega}$$

for $u \in W^s_{(0,q)}(\Omega)$ and $1 \leq q \leq n$.

We show that if one drops the smoothness assumption, then the $\overline{\partial}$-Neumann operator, N_1, may not map any positive Sobolev space into itself continuously.

Theorem 1. *There exists a bounded pseudoconvex domain Ω in $\mathbb{C}^2$, smooth except at one point, such that the $\overline{\partial}$-Neumann operator on Ω is not bounded on $W^s_{(0,1)}(\Omega)$ for any $s > 0$.*

Proof. We will build the domain by attaching infinitely many worm domains (constructed by Diederich and Fornæss in [DF77]) with progressively larger winding. Let Ω_j be a worm domain, a smooth bounded pseudoconvex domain, in $\mathbb{C}^2$ that winds $2\pi j$ such that

$$\Omega_j \subset \{(z,w) \in \mathbb{C}^2 : |z| < 2^{-j}, 4^{-j} < |w| < 4^{-j}2\}$$

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for $j = 1, 2, \ldots$. Let γ_j be a straight line that connects an extreme point on the cap of Ω_j to a closest point on the cap of Ω_{j+1}. Then using the barbell lemma (see [FS77, HW68]) we get a bounded pseudoconvex domain Ω that is smooth except for one point $(0,0) \in b\Omega$. Notice that Ω is the union of $\Omega_j \subset \Omega$ for $j = 1, 2, \ldots$ and all connecting bands. In the rest of the proof we will show that if the $\overline{\partial}$-Neumann operator on Ω is continuous on $W^s_{(0,1)}(\Omega)$, then the $\overline{\partial}$-Neumann operator on Ω_j is continuous on $W^s_{(0,1)}(\Omega_j)$ for $j = 1, 2, \ldots$. However this is a contradiction with a theorem of Barrett ([Bar92]). Let us define $\Box^j = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $L^2_{(0,1)}(\Omega_j)$, and $\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $L^2_{(0,1)}(\Omega)$. Let us fix j and choose a defining function ρ for Ω_j such that $\|\nabla\rho\| = 1$ on $b\Omega_j$. Let $\nu = \mathrm{Re}\left(\sum_{j=1}^{2} \frac{\partial\rho}{\partial\bar{z}_j}\frac{\partial}{\partial z_j}\right)$ and J denote the complex structure of $\mathbb{C}^2$. Now we will construct a smooth cut-off function that fixes the domain of $\Box$ and $\Box^j$ under multiplication. We can choose open sets U_1, U_2, and U_3 and $\chi \in C^\infty_0(U_2)$ such that

i) $U_1 \Subset U_2 \Subset U_3$,
ii) U_1, U_2, and U_3 contain all boundary points of Ω_j that meet the (strongly pseudoconvex) band created using γ_j and γ_{j-1}, and they do not contain any weakly pseudoconvex boundary point of Ω_j,
iii) $0 \le \chi \le 1$, $\chi \equiv 1$ on U_1,
iv) there exists an open set U such that $b\Omega_j \cup U_2 \Subset U$ and the following two ordinary differential equations can be solved in U:

$$\nu(\widetilde{\psi}) = 0, \qquad \widetilde{\psi}|_{b\Omega_j} = \chi, \tag{1}$$

$$\nu(\widetilde{\phi}) = -J(\nu)(\chi), \quad \widetilde{\phi}|_{b\Omega_j} = 0. \tag{2}$$

Notice that $\widetilde{\psi} \equiv 1$ and $\widetilde{\phi} \equiv 0$ on U_1, and $\widetilde{\psi} = \widetilde{\phi} = 0$ in a neighborhood of the set of weakly pseudoconvex boundary points of Ω_j. We choose a neighborhood $V \Subset U$ of $b\Omega_j$ and $\widetilde{\chi} \in C^\infty_0(V)$ such that $\widetilde{\chi} \equiv 1$ in a neighborhood $\widetilde{V}$ of $b\Omega_j$. Let us define $\phi = \widetilde{\chi}\widetilde{\phi}, \psi = \widetilde{\chi}\widetilde{\psi}$, and $\xi = \psi + i\phi$. We would like to make some observations about ξ that will be useful later:

i) $\xi \equiv 1$ on $\widetilde{V} \cap U_1$,
ii) $(\nu + iJ(\nu))(\xi) \equiv 0$ on $b\Omega_j$,
iii) $\xi \equiv 0$ in a neighborhood of the weakly pseudoconvex boundary points of Ω_j.

Claim: If $f \in Dom(\Box^j)$, then $\xi f \in Dom(\Box^j)$ and $(1-\xi)f \in Dom(\Box)$.

Proof of Claim. First we will show that $\xi f \in Dom(\Box^j)$. Then we will talk about how one can show that $(1-\xi)f \in Dom(\Box)$.

One can easily show that $\xi f \in Dom(\overline{\partial}^*) \cap Dom(\overline{\partial})$ (on Ω_j). On the other hand, by the Kohn-Morrey-Hörmander formula [CS01] since the L^2-norm of any "bar" derivatives of any terms of f on Ω_j is dominated by $\|\overline{\partial}f\|_{\Omega_j} + \|\overline{\partial}^* f\|_{\Omega_j}$, we have $\overline{\partial}^*(\xi f) \in Dom(\overline{\partial})$. So we need to show that $\overline{\partial}(\xi f) = \overline{\partial}\xi \wedge f + \xi\overline{\partial}f \in Dom(\overline{\partial}^*)$. Since $\xi\overline{\partial}f \in Dom(\overline{\partial}^*)$ we only need to show that $\overline{\partial}\xi \wedge f \in Dom(\overline{\partial}^*)$. We will use the special boundary frames. Let

$$L_\tau = \frac{\partial\rho}{\partial z_1}\frac{\partial}{\partial z_2} - \frac{\partial\rho}{\partial z_2}\frac{\partial}{\partial z_1}, \quad L_\nu = \frac{\partial\rho}{\partial\bar{z}_1}\frac{\partial}{\partial z_1} + \frac{\partial\rho}{\partial\bar{z}_2}\frac{\partial}{\partial z_2}.$$

Also let w_τ and w_ν be the dual $(1,0)$-forms. We note that $L_\nu = \nu - iJ(\nu)$ and so $\overline{L}_\nu(\xi) \equiv 0$ on $b\Omega_j$. We can write $f = f_\tau \overline{w}_\tau + f_\nu \overline{w}_\nu$. Therefore, $\overline{\partial}\xi \wedge f = (\overline{L}_\tau(\xi) f_\nu - \overline{L}_\nu(\xi) f_\tau)\overline{w}_\tau \wedge \overline{w}_\nu$. Using the fact that $f_\nu \in W^1_0(\Omega_j)$ (it is easy to see this for $f \in C^1(\overline{\Omega}_j)$. For $f \in Dom(\overline{\partial}^*) \cap Dom(\overline{\partial})$ one can use the fact that $\Delta : W^1_0(\Omega_j) \to W^{-1}(\Omega_j)$ is an isomorphism and the density lemma [CS01, Lemma 4.3.2]) and $\overline{L}_\tau(\xi)$ is smooth, we may reduce the problem of showing $\overline{\partial}\xi \wedge f \in Dom(\overline{\partial}^*)$ to show the following:

$$\overline{L}_\nu(\xi) f_\tau \overline{w}_\tau \wedge \overline{w}_\nu \in Dom(\overline{\partial}^*).$$

Let $\{\phi_k\}_{k=1}^\infty$ be a sequence of smooth compactly supported functions converging to $\overline{L}_\nu(\xi)$ in the C^1-norm and u be a $(0,1)$-form with smooth compactly supported coefficient functions in Ω_j. Then

$$\langle \overline{L}_\nu(\xi) f_\tau \overline{w}_\tau \wedge \overline{w}_\nu, \overline{\partial} u\rangle_{\Omega_j} = \lim_{k\to\infty} \langle \phi_k f_\tau \overline{w}_\tau \wedge \overline{w}_\nu, \overline{\partial} u\rangle_{\Omega_j}$$

where $\langle , \rangle_{\Omega_j}$ is the inner product on forms on Ω_j. If we integrate by parts and use $\lim_{k\to\infty} \|L_l(\phi_k f_\tau)\|_{\Omega_j} = \|L_l(\overline{L}_\nu(\xi) f_\tau)\|_{\Omega_j}$ for $l = \tau, \nu$ we can reduce the problem of showing $\overline{\partial}\xi \wedge f \in Dom(\overline{\partial}^*)$ to showing that $\|\frac{\partial}{\partial z_1}(\overline{L}_\nu(\xi) f_\tau)\|_{\Omega_j}$ and $\|\frac{\partial}{\partial z_2}(\overline{L}_\nu(\xi) f_\tau)\|_{\Omega_j}$ are finite. One can show that

$$\left\|\frac{\partial}{\partial z_m}(\overline{L}_\nu(\xi) f_\tau)\right\|_{\Omega_j} = \lim_{k\to\infty} \left\|\frac{\partial}{\partial z_m}(\phi_k f_\tau)\right\|_{\Omega_j} = \lim_{k\to\infty} \left\|\frac{\partial}{\partial \bar{z}_m}(\phi_k f_\tau)\right\|_{\Omega_j}.$$

On the second equality we used integration by parts. On the other hand, we have

$$\begin{aligned} \lim_{k\to\infty} \left\|\frac{\partial}{\partial \bar{z}_m}(\phi_k f_\tau)\right\|_{\Omega_j} &= \left\|\frac{\partial}{\partial \bar{z}_m}(\overline{L}_\nu(\xi) f_\tau)\right\|_{\Omega_j} \\ &= \left\|\frac{\partial}{\partial \bar{z}_m}(\overline{L}_\nu(\xi)) f_\tau\right\|_{\Omega_j} + \left\|\overline{L}_\nu(\xi)\frac{\partial}{\partial \bar{z}_m}(f_\tau)\right\|_{\Omega_j} \\ &\leq C(\|\overline{\partial} f\|_{\Omega_j} + \|\overline{\partial}^* f\|_{\Omega_j}) < \infty \end{aligned}$$

for $m = 1, 2$ and a positive constant C that does not depend on f. In the last inequality we used the fact that L^2-norms of f and the "bar" derivatives of f_τ on Ω_j are bounded by $C(\|\overline{\partial} f\|_{\Omega_j} + \|\overline{\partial}^* f\|_{\Omega_j})$. We remark that it is essential that ξ is complex-valued and Ω is smooth in a neighborhood of $\overline{\Omega}_j$. Therefore, we showed that $\xi f \in Dom(\Box^j)$.

As for $(1-\xi)f$ being in $Dom(\Box)$, since $\xi \equiv 1$ in a neighborhood of the boundary points of Ω_j that meet the band created using γ_j and γ_{j-1}, we have $(1-\xi)f \equiv 0$ on $\Omega \setminus \Omega_j$. Also since $\overline{L}_\nu(1-\xi) = -\overline{L}_\nu(\xi)$ similar calculations as before show that $(1-\xi)f \in Dom(\Box)$. This completes the proof of the claim.

We will use generalized constants in the sense that $\|A\|_{s,\Omega_j} \lesssim \|B\|_{s,\Omega_j}$ means that there is a constant $C = C(s, \Omega_j) > 0$ that depends only on s and Ω_j but not on A or B such that $\|A\|_{s,\Omega_j} \leq C\|B\|_{s,\Omega_j}$. Assume that the $\overline{\partial}$-Neumann operator on Ω maps $W^s_{(0,1)}(\Omega)$ into itself continuously for some $s > 0$. That is, $\|N_1 h\|_{s,\Omega} \lesssim \|h\|_{s,\Omega}$ for $h \in W^s_{(0,1)}(\Omega)$. Then we have $\|g\|_{s,\Omega} \lesssim \|\Box g\|_{s,\Omega}$ for $g \in Dom(\Box)$ and $\Box g \in W^s_{(0,1)}(\Omega)$. Let $f \in Dom(\Box^j)$ and $\Box^j f \in W^s_{(0,1)}(\Omega_j)$. Then we have

$$\|f\|_{s,\Omega_j} \leq \|\xi f\|_{s,\Omega_j} + \|(1-\xi) f\|_{s,\Omega_j}.$$

Since $\xi \equiv 0$ in a neighborhood of the weakly pseudoconvex boundary points of Ω_j, we can use pseudolocal estimates on Ω_j (see [KN65]) to get

$$\|\xi f\|_{s,\Omega_j} \lesssim \|\Box^j f\|_{s-1,\Omega_j} + \|\Box^j f\|_{\Omega_j}. \tag{3}$$

Let us choose η to be a smooth compactly supported function that is constant 1 around the support of $\nabla\xi$ and zero in a neighborhood of the weakly pseudoconvex points of Ω_j. Therefore, we have

$$\begin{aligned}\|(1-\xi)f\|_{s,\Omega_j} = \|(1-\xi)f\|_{s,\Omega} &\lesssim \|\Box(1-\xi)f\|_{s,\Omega}\\ &\lesssim \|(\triangle\xi)f\|_{s,\Omega} + \|\nabla\xi\cdot\nabla f\|_{s,\Omega} + \|(1-\xi)\triangle f\|_{s,\Omega_j}\\ &\lesssim \|\eta f\|_{s,\Omega_j} + \|\eta f\|_{s+1,\Omega_j} + \|\Box^j f\|_{s,\Omega_j}\\ &\lesssim \|\Box^j f\|_{s,\Omega_j}.\end{aligned}$$

The first inequality comes from the assumption that the $\overline{\partial}$-Neumann operator on Ω is continuous on $W^s_{(0,1)}(\Omega)$. The second inequality comes from the fact that $\Box$ operates as a Laplacian componentwise on forms. In the last inequality we used the pseudolocal estimates as we did in (3). Therefore we showed that

$$\|f\|_{s,\Omega_j} \lesssim \|\xi f\|_{s,\Omega_j} + \|(1-\xi)f\|_{s,\Omega_j} \lesssim \|\Box^j f\|_{s,\Omega_j}$$

for $f \in Dom(\Box^j)$ and $\Box^j f \in W^s_{(0,1)}(\Omega_j)$. One can check that this is equivalent to the condition that the $\overline{\partial}$-Neumann operator on Ω_j is continuous on $W^s_{(0,1)}(\Omega_j)$. $\Box$

One can check that $\overline{\partial}^* N_1$ maps $W^s_{(0,1)}(\Omega)$ into $W^s(\Omega)$ continuously if and only if $\|\overline{\partial}^* f\|_{s,\Omega} \lesssim \|\Box f\|_{s,\Omega}$ for $f \in Dom(\Box)$ and $\Box f \in W^s_{(0,1)}(\Omega)$. Using this observation one can give a proof, similar to the proof of the theorem, for the following corollary.

Corollary 1. *There exists a bounded pseudoconvex domain Ω in $\mathbb{C}^2$, smooth except at one point, such that $\overline{\partial}^* N_1$ is not bounded from $W^s_{(0,1)}(\Omega)$ into $W^s(\Omega)$ for any $s > 0$.*

It is interesting that for a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^2$ the operator $\overline{\partial} N_1$ is bounded from $W^s_{(0,1)}(\Omega)$ into $W^s_{(0,2)}(\Omega)$ for any $s \geq 0$. (One can use (4) in [BS90] to see this.)

Remark 1. We would like to note the following additional property for the domain we constructed in the proof of Theorem 1. There is no open set U that contains the non-smooth boundary point of Ω such that $\overline{U \cap \Omega}$ has a Stein neighborhood basis. That is, non-smooth domains may not have a "local" Stein neighborhood basis. However, this is not the case for smooth domains (see for example [Ran86, Lemma 2.13]).

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