A REMARK ON IRREGULARITY OF THE \(\overline{\partial}\)-NEUMANN PROBLEM ON NON-SMOOTH DOMAINS

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Abstract. It is an observation due to J. J. Kohn that for a smooth bounded pseudoconvex domain \(\Omega\) in \(\mathbb{C}^n\) there exists \(s > 0\) such that the \(\overline{\partial}\)-Neumann operator on \(\Omega\) maps \(W^s_{0,1}(\Omega)\) (the space of \((0,1)\)-forms with coefficient functions in \(L^2\)-Sobolev space of order \(s\)) into itself continuously. We show that this conclusion does not hold without the smoothness assumption by constructing a bounded pseudoconvex domain \(\Omega\) in \(\mathbb{C}^2\), smooth except at one point, whose \(\overline{\partial}\)-Neumann operator is not bounded on \(W^s_{0,1}(\Omega)\) for any \(s > 0\).

Let \(W^s(\Omega)\) and \(W^s_{p,q}(\Omega)\) denote the \(L^2\)-Sobolev space on \(\Omega\) of order \(s\) and the space of \((p,q)\)-forms with coefficient functions in \(W^s(\Omega)\), respectively. Also \(\|\cdot\|_{s,\Omega}\) denotes the norm on \(W^s_{p,q}(\Omega)\). Let \(N_q\) denote the inverse of the complex Laplacian, \(\overline{\partial}\partial + \partial\overline{\partial}\), on square integrable \((0,q)\)-forms. It is an observation of Kohn, as the following proposition says, that on a smooth bounded pseudoconvex domain the \(\overline{\partial}\)-Neumann problem is regular in the Sobolev scale for sufficiently small levels.

**Proposition 1 (Kohn).** Let \(\Omega\) be a smooth bounded pseudoconvex domain in \(\mathbb{C}^n\). Then there exist positive \(\varepsilon\) and \(C\) (depending on \(\Omega\)) such that

\[
\|N_q u\|_{\varepsilon,\Omega} \leq C\|u\|_{\varepsilon,\Omega}, \quad \|\overline{\partial}N_q u\|_{\varepsilon,\Omega} \leq C\|u\|_{\varepsilon,\Omega}, \quad \|\partial N_q u\|_{\varepsilon,\Omega} \leq C\|u\|_{\varepsilon,\Omega}
\]

for \(u \in W^s_{0,q}(\Omega)\) and \(1 \leq q \leq n\).

We show that if one drops the smoothness assumption, then the \(\overline{\partial}\)-Neumann operator, \(N_1\), may not map any positive Sobolev space into itself continuously.

**Theorem 1.** There exists a bounded pseudoconvex domain \(\Omega\) in \(\mathbb{C}^2\), smooth except at one point, such that the \(\overline{\partial}\)-Neumann operator on \(\Omega\) is not bounded on \(W^s_{0,1}(\Omega)\) for any \(s > 0\).

**Proof.** We will build the domain by attaching infinitely many worm domains (constructed by Diederich and Fornæss in [DF77]) with progressively larger winding. Let \(\Omega_j\) be a worm domain, a smooth bounded pseudoconvex domain, in \(\mathbb{C}^2\) that winds \(2\pi j\) such that

\[
\Omega_j \subset \{(z,w) \in \mathbb{C}^2 : |z| < 2^{-j}, 4^{-j} < |w| < 4^{-j/2}\}
\]

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for \( j = 1, 2, \ldots \). Let \( \gamma_j \) be a straight line that connects an extremal point on the cap of \( \Omega_j \) to a closest point on the cap of \( \Omega_{j+1} \). Then using the barbell lemma (see \cite{FS77, HW08}) we get a bounded pseudoconvex domain \( \Omega \) that is smooth except for one point \((0, 0) \in b\Omega \). Notice that \( \Omega \) is the union of \( \Omega_j \subseteq \Omega \) for \( j = 1, 2, \ldots \) and all connecting bands. In the rest of the proof we will show that if the \( \overline{\partial} \)-Neumann operator on \( \Omega \) is continuous on \( W^s_{(0,1)}(\Omega) \), then the \( \overline{\partial} \)-Neumann operator on \( \Omega_j \) is continuous on \( W^s_{(0,1)}(\Omega_j) \) for \( j = 1, 2, \ldots \). However this is a contradiction with a theorem of Barrett \((\text{Bar}92)\). Let us define \( \square = \overline{\partial}\partial + \overline{\partial}\partial \) on \( L^2_{(0,1)}(\Omega) \), and \( \square = \overline{\partial}\partial + \overline{\partial}\partial \) on \( L^2_{(0,1)}(\Omega_j) \). Let us fix \( j \) and choose a defining function \( \rho \) for \( \Omega_j \) such that \( \|\nabla \rho\| = 1 \) on \( b\Omega_j \). Let \( \nu = \text{Re} \left( \sum_{j=1}^2 \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial \overline{z}_j} \right) \) and \( J \) denote the complex structure of \( \mathbb{C}^2 \). Now we will construct a smooth cut-off function that fixes the domain of \( \square \) and \( \overline{\square} \) under multiplication. We can choose open sets \( U_1, U_2, \) and \( U_3 \) and \( \chi \in C_0^\infty(U_2) \) such that

\begin{enumerate} 
  \item \( U_1 \subseteq U_2 \subseteq U_3 \),
  \item \( U_1, U_2, \), and \( U_3 \) contain all boundary points of \( \Omega_j \) that meet the (strongly pseudoconvex) band created using \( \gamma_j \) and \( \gamma_{j-1} \), and they do not contain any weakly pseudoconvex boundary point of \( \Omega_j \),
  \item \( 0 \leq \chi \leq 1 \), \( \chi \equiv 1 \) on \( U_1 \),
  \item there exists an open set \( U \) such that \( b\Omega_j \cup U_2 \subseteq U \) and the following two ordinary differential equations can be solved in \( U \):
\end{enumerate}

\begin{align*}
(1) & \quad \nu(\psi) = 0, \quad \psi|_{\partial\Omega_j} = \chi, \\
(2) & \quad \nu(\phi) = -J(\nu)(\chi), \quad \phi|_{\partial\Omega_j} = 0.
\end{align*}

Notice that \( \tilde{\psi} \equiv 1 \) and \( \tilde{\phi} \equiv 0 \) on \( U_1 \), and \( \psi = \phi = 0 \) in a neighborhood of the set of weakly pseudoconvex boundary points of \( \Omega_j \). We choose a neighborhood \( \bar{V} \subseteq U \) of \( b\Omega_j \) and \( \tilde{\chi} \in C_0^\infty(V) \) such that \( \tilde{\chi} \equiv 1 \) in a neighborhood \( \bar{V} \) of \( b\Omega_j \). Let us define \( \phi = \tilde{\chi}\phi, \psi = \tilde{\chi}\psi, \) and \( \xi = \psi + i\phi \). We would like to make some observations about \( \xi \) that will be useful later:

\begin{enumerate} 
  \item \( \xi \equiv 1 \) on \( \bar{V} \cap U_1 \),
  \item \( (\nu + iJ(\nu))(\xi) \equiv 0 \) on \( b\Omega_j \),
  \item \( \xi \equiv 0 \) in a neighborhood of the weak pseudconvex boundary points of \( \Omega_j \).
\end{enumerate}

**Claim:** If \( f \in \text{Dom}(\overline{\square}) \), then \( \xi f \in \text{Dom}(\overline{\square}) \) and \( (1 - \xi)f \in \text{Dom}(\square) \).

**Proof of Claim.** First we will show that \( \xi f \in \text{Dom}(\overline{\square}) \). Then we will talk about how one can show that \( (1 - \xi)f \in \text{Dom}(\square) \).

One can easily show that \( \xi f \in \text{Dom}(\overline{\square}) \cap \text{Dom}(\square) \) (on \( \Omega_j \)). On the other hand, by the Kohn-Morrey-Hörmander formula \cite{CS01} since the \( L^2 \)-norm of any “bar” derivatives of any terms of \( f \) on \( \Omega_j \) is dominated by \( \|\overline{\partial}f\|_{\Omega_j} + \|\overline{\partial}f\|_{\Omega_j} \), we have \( \overline{\partial}(\xi f) \in \text{Dom}(\overline{\square}) \). So we need to show that \( \overline{\partial}(\xi f) = \overline{\partial}\xi f + \xi \overline{\partial}f \in \text{Dom}(\overline{\square}) \). Since \( \xi f \in \text{Dom}(\overline{\square}) \) we only need to show that \( \overline{\partial}\xi f + \overline{\partial}f \in \text{Dom}(\overline{\square}) \). We will use the special boundary frames. Let

\[ L_\tau = \frac{\partial \rho}{\partial z_1} \frac{\partial}{\partial z_2} - \frac{\partial \rho}{\partial z_2} \frac{\partial}{\partial z_1}, \quad L_\nu = \frac{\partial \rho}{\partial z_1} \frac{\partial}{\partial z_1} + \frac{\partial \rho}{\partial z_2} \frac{\partial}{\partial z_2}. \]
Also let \( w_\tau \) and \( w_\nu \) be the dual \((1,0)\)-forms. We note that \( L_\nu = \nu - iJ(\nu) \) and so \( \mathcal{L}_\nu(\xi) \equiv 0 \) on \( \partial \Omega_j \). We can write \( f = f_\tau w_\tau + f_\nu w_\nu \). Therefore, \( \partial \xi \wedge f = (\mathcal{L}_\nu(\xi) f_\nu - \mathcal{L}_\nu(\xi) f_\tau)\overline{w}_\tau \wedge \overline{w}_\nu \). Using the fact that \( f_\nu \in W^1_0(\Omega_j) \) (it is easy to see this for \( f \in C^1(\overline{\Omega}_j) \)). For \( f \in \text{Dom}(\overline{\nu}) \cap \text{Dom}(\overline{\tau}) \) one can use the fact that \( \Delta : W^1_0(\Omega_j) \rightarrow W^{-1}(\Omega_j) \) is an isomorphism and the density lemma [CS01, Lemma 4.3.2] and \( \mathcal{L}_\tau(\xi) \) is smooth, we may reduce the problem of showing \( \partial \xi \wedge f \in \text{Dom}(\overline{\tau}) \) to show the following:

\[
\mathcal{L}_\nu(\xi) f_\tau \overline{w}_\tau \wedge \overline{w}_\nu \in \text{Dom}(\overline{\tau}).
\]

Let \( \{\phi_k\}_{k=1}^\infty \) be a sequence of smooth compactly supported functions converging to \( \mathcal{L}_\nu(\xi) \) in the \( C^1 \)-norm and \( u \) be a \((0,1)\)-form with smooth compactly supported coefficient functions in \( \Omega_j \). Then

\[
\langle \mathcal{L}_\nu(\xi) f_\tau \overline{w}_\tau \wedge \overline{w}_\nu, \overline{\nu} u \rangle_{\Omega_j} = \lim_{k \to \infty} \langle \phi_k f_\tau \overline{w}_\tau \wedge \overline{w}_\nu, \overline{\nu} u \rangle_{\Omega_j}
\]

where \( \langle \cdot, \cdot \rangle_{\Omega_j} \) is the inner product on forms on \( \Omega_j \). If we integrate by parts and use

\[
\lim_{k \to \infty} \| L_\nu(\phi_k f_\tau) \|_{\Omega_j} = \| L_\nu(\mathcal{L}_\nu(\xi) f_\tau) \|_{\Omega_j},
\]

for \( l = \tau, \nu \) we can reduce the problem of showing \( \partial \xi \wedge f \in \text{Dom}(\overline{\tau}) \) to showing that

\[
\| \frac{\partial}{\partial \nu}(\mathcal{L}_\nu(\xi) f_\tau) \|_{\Omega_j} \text{ and } \| \frac{\partial}{\partial \nu}(\mathcal{L}_\nu(\xi) f_\tau) \|_{\Omega_j}
\]

are finite. One can show that

\[
\left| \frac{\partial}{\partial \nu}(\mathcal{L}_\nu(\xi) f_\tau) \right|_{\Omega_j} = \lim_{k \to \infty} \left| \frac{\partial}{\partial \nu}(\mathcal{L}_\nu(\xi) f_\tau) \right|_{\Omega_j} = \lim_{k \to \infty} \left| \frac{\partial}{\partial \nu}(\mathcal{L}_\nu(\xi) f_\tau) \right|_{\Omega_j}.
\]

On the second equality we used integration by parts. On the other hand, we have

\[
\lim_{k \to \infty} \left| \frac{\partial}{\partial \nu}(\mathcal{L}_\nu(\xi) f_\tau) \right|_{\Omega_j} = \left| \frac{\partial}{\partial \nu}(\mathcal{L}_\nu(\xi) f_\tau) \right|_{\Omega_j} = \left| \frac{\partial}{\partial \nu}(\mathcal{L}_\nu(\xi) f_\tau) \right|_{\Omega_j} \leq C(\| \overline{\nu} f \|_{\Omega_j} + \| \overline{\nu} f \|_{\Omega_j}) < \infty
\]

for \( m = 1, 2 \) and a positive constant \( C \) that does not depend on \( f \). In the last inequality we used the fact that \( L^2 \)-norms of \( f \) and the “bar” derivatives of \( f_\nu \) on \( \Omega_j \) are bounded by \( C(\| \overline{\nu} f \|_{\Omega_j} + \| \overline{\nu} f \|_{\Omega_j}) \). We remark that it is essential that \( \xi \) is complex-valued and \( \Omega \) is smooth in a neighborhood of \( \overline{\Omega}_j \). Therefore, we showed that \( \xi f \in \text{Dom}(\overline{\nu}) \).

As for \((1 - \xi) f \) being in \( \text{Dom}(\Box) \), since \( \xi \equiv 1 \) in a neighborhood of the boundary points of \( \Omega_j \) that meet the band created using \( \gamma_j \) and \( \gamma_{j-1} \), we have \((1 - \xi) f \equiv 0 \) on \( \Omega \setminus \Omega_j \). Also since \( \mathcal{L}_\nu(1 - \xi) = -\mathcal{L}_\nu(\xi) \) similar calculations as before show that \((1 - \xi) f \in \text{Dom}(\Box) \). This completes the proof of the claim.

We will use generalized constants in the sense that \( \| A \|_{s, \Omega_j} \lesssim \| B \|_{s, \Omega_j} \) means that there is a constant \( C = C(s, \Omega_j) > 0 \) that depends only on \( s \) and \( \Omega_j \) but not on \( A \) or \( B \) such that \( \| A \|_{s, \Omega_j} \leq C \| B \|_{s, \Omega_j} \). Assume that the \( \overline{\nu} \)-Neumann operator on \( \Omega \) maps \( W^s_{(0,1)}(\Omega) \) into itself continuously for some \( s > 0 \). That is, \( \| N f \|_{s, \Omega_j} \lesssim \| h \|_{s, \Omega_j} \) for \( h \in W^s_{(0,1)}(\Omega) \). Then we have \( \| g \|_{s, \Omega_j} \lesssim \| \Box g \|_{s, \Omega_j} \) for \( g \in \text{Dom}(\Box) \) and \( \Box g \in W^s_{(0,1)}(\Omega) \). Let \( f \in \text{Dom}(\Box) \) and \( \Box f \in W^s_{(0,1)}(\Omega_j) \). Then we have

\[
\| f \|_{s, \Omega_j} \leq \| \xi f \|_{s, \Omega_j} + \| (1 - \xi) f \|_{s, \Omega_j}.
\]
Since $\xi \equiv 0$ in a neighborhood of the weakly pseudoconvex boundary points of $\Omega_j$, we can use pseudolocal estimates on $\Omega_j$ (see [KN65]) to get

$$\|\xi f\|_{s, \Omega_j} \lesssim \|\Box^j f\|_{s-1, \Omega_j} + \|\Box^j f\|_{\Omega_j}.$$  \hfill (3)  

Let us choose $\eta$ to be a smooth compactly supported function that is constant 1 around the support of $\nabla \xi$ and zero in a neighborhood of the weakly pseudoconvex points of $\Omega_j$. Therefore, we have

$$\|(1 - \xi) f\|_{s, \Omega_j} = \|(1 - \xi) f\|_{s, \Omega} \lesssim \|\Box (1 - \xi) f\|_{s, \Omega}$$

$$\lesssim \|(\Delta \xi) f\|_{s, \Omega} + \|\nabla \xi \cdot \nabla f\|_{s, \Omega} + \|(1 - \xi) \Delta f\|_{s, \Omega_j}$$

$$\lesssim \|\eta f\|_{s, \Omega_j} + \|\eta f\|_{s+1, \Omega_j} + \|\Box^j f\|_{s, \Omega_j}$$

$$\lesssim \|\Box^j f\|_{s, \Omega_j}.$$  

The first inequality comes from the assumption that the $\overline{\partial}$-Neumann operator on $\Omega$ is continuous on $W^s_{(0,1)}(\Omega)$. The second inequality comes from the fact that $\Box$ operates as a Laplacian componentwise on forms. In the last inequality we used the pseudolocal estimates as we did in (3). Therefore we showed that

$$\|f\|_{s, \Omega_j} \lesssim \|\xi f\|_{s, \Omega_j} + \|(1 - \xi) f\|_{s, \Omega_j} \lesssim \|\Box^j f\|_{s, \Omega_j}$$

for $f \in \text{Dom}(\Box^j)$ and $\Box^j f \in W^s_{(0,1)}(\Omega_j)$. One can check that this is equivalent to the condition that the $\overline{\partial}$-Neumann operator on $\Omega_j$ is continuous on $W^s_{(0,1)}(\Omega_j)$.  

One can check that $\overline{\partial} N_1$ maps $W^s_{(0,1)}(\Omega)$ into $W^s(\Omega)$ continuously if and only if $\|\overline{\partial} f\|_{s, \Omega} \lesssim \|\Box f\|_{s, \Omega}$ for $f \in \text{Dom}(\Box)$ and $\Box f \in W^s_{(0,1)}(\Omega)$. Using this observation one can give a proof, similar to the proof of the theorem, for the following corollary.

**Corollary 1.** There exists a bounded pseudoconvex domain $\Omega$ in $\mathbb{C}^2$, smooth except at one point, such that $\overline{\partial} N_1$ is not bounded from $W^s_{(0,1)}(\Omega)$ into $W^s(\Omega)$ for any $s > 0$.

It is interesting that for a smooth bounded pseudoconvex domain $\Omega$ in $\mathbb{C}^2$ the operator $\overline{\partial} N_1$ is bounded from $W^s_{(0,1)}(\Omega)$ into $W^s_{(0,2)}(\Omega)$ for any $s \geq 0$. (One can use (4) in [BS90] to see this.)

**Remark 1.** We would like to note the following additional property for the domain we constructed in the proof of Theorem 1. There is no open set $U$ that contains the non-smooth boundary point of $\Omega$ such that $U \cap \overline{\Omega}$ has a Stein neighborhood basis. That is, non-smooth domains may not have a “local” Stein neighborhood basis. However, this is not the case for smooth domains (see for example [Ran86, Lemma 2.13]).

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IRREGULARITY OF THE $\bar{\partial}$-NEUMANN PROBLEM

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