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DYNAMICS OF THE w FUNCTION AND THE GREEN-TAO THEOREM ON ARITHMETIC PROGRESSIONS IN THE PRIMES

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ABSTRACT. Let A_3 be the set of all positive integers pqr, where p, q, r are primes and possibly two, but not all three of them are equal. For any $n = pqr \in A_3$, define a function w by $w(n) = P(p+q)P(p+r)P(q+r)$, where $P(m)$ is the largest prime factor of m. It is clear that if $n = pqr \in A_3$, then $w(n) \in A_3$. For any $n \in A_3$, define $w^0(n) = n$, $w^i(n) = w(w^{i-1}(n))$ for $i = 1, 2, \ldots$. An element $n \in A_3$ is semi-periodic if there exists a nonnegative integer s and a positive integer t such that $w^{s+t}(n) = w^s(n)$. We use $\operatorname{ind}(n)$ to denote the least such nonnegative integer s. Wushi Goldring [*Dynamics of the w function and primes*, J. Number Theory 119(2006), 86-98] proved that any element $n \in A_3$ is semi-periodic. He showed that there exists i such that $w^i(n) \in \{20, 98, 63, 75\}$, $\operatorname{ind}(n) \leqslant 4(\pi(P(n)) - 3)$, and conjectured that $\operatorname{ind}(n)$ can be arbitrarily large.

In this paper, it is proved that for any $n \in A_3$ we have $\operatorname{ind}(n) = O((\log P(n))^2)$, and the Green-Tao Theorem on arithmetic progressions in the primes is employed to confirm Goldring's above conjecture.

1. Introduction

Let

$$A_3 = \{n = pqr \mid p,\ q,\ r \text{ are all primes}\} \setminus \{n = p^3 \mid p \text{ is prime}\}.$$

For any $n \in A_3$, define a function w by

$$w(n) = P(p+q)P(p+r)P(q+r),$$

where $P(m)$ is the largest prime factor of m. It is clear that $w(A_3) \subseteq A_3$ (see [1, Lemma 2.1]). For any $n \in A_3$, define $w^0(n) = n$, $w^i(n) = w(w^{i-1}(n))$ for $i = 1, 2, \ldots$, and define the w-orbit of n to be the sequence $W(n) = [n, w(n), \ldots, w^i(n), \ldots]$.

Wushi Goldring [1] proved that for any $n \in A_3$, there exists i such that $W(n) = [\ n,\ w(n), \ldots, w^{i-1}(n),\ \overline{20, 98, 63, 75}\ ]$. For $n \in A_3$, the periodicity index of n is defined to be the least nonnegative integer i such that $w^i(n) \in \{20, 98, 63, 75\}$. Denote the periodicity index of n by $\operatorname{ind}(n)$. Wushi Goldring[1] proved that $\operatorname{ind}(n) \leqslant 4(\pi(P(n)) - 3)$, and posed several conjectures related to $w(n)$. Two of them are

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Conjecture 2.9. *ind*$(n) = O(\log \pi(P(n)))$.

Conjecture 2.10. *There are sets in* A_3 *of arbitrarily large periodicity index.*

In this paper we improve the upper bound of ind(n) and employ the Green-Tao Theorem on arithmetic progressions in the primes to confirm Goldring's Conjecture 2.10.

Theorem 1. *For any* $n \in A_3$*, we have ind*$(n) = O((\log P(n))^2)$.

Theorem 2. *ind*(n) *can be arbitrarily large.*

Remark. By the Prime Number Theorem, it is clear that $\log(\pi(P(n))) \sim \log P(n)$. By the proofs of Theorems 1 and 2, we believe that Conjecture 2.9 is not true.

2. Proofs of theorems

We will make repeated use of the following trivial facts.

Lemma 1. *Let* p, q *be odd primes. Then*

(a) $P(p+q) \leq \frac{1}{2}(p+q)$*;*

(b) if $p+2$ *is composite, then* $P(p+2) \leq \frac{1}{3}(p+2)$*;*

(c) if $p+2$ *is prime and* $p > 3$*, then* $p+4$ *must be composite and* $P(p+4) \leq \frac{1}{3}(p+4)$.

Lemma 2. *Let* X *be an integer with* $X \geqslant 3$ *and* α *be a real number with* $0 < \alpha < 1$. *For any* $n = pqr \in A_3$*, where* p*,* q*,* r *are all primes with* $p \leqslant X$*,* $q \leqslant \alpha X$ *and* $r \leqslant \alpha X$*, there exists* $1 \leqslant i \leqslant 3$ *such that*

$$P(w^i(n)) \leqslant \frac{1}{4}(\alpha+3)X + 4.$$

Proof. By the definition of w we have

$$w(n) = P(p+q)P(p+r)P(q+r).$$

If $p \leqslant \alpha X$, then by Lemma 1,

$$P(w(n)) \leqslant \alpha X + 2 \leqslant \frac{1}{4}(\alpha+3)X + 4.$$

Now we may assume that $p > \alpha X$. If $p = 3$, then by $q \leqslant \alpha X$ and $r \leqslant \alpha X$ we have $q = r = 2$. Then

$$P(w(n)) = 5 \leqslant \frac{1}{4}(\alpha+3)X + 4.$$

Thus we may assume that $p \geqslant 5$ and $p > \alpha X$.

Case 1. $p+2$ is composite.

By Lemma 1 we have

$$P(p+q) \leqslant \begin{cases} \frac{p+2}{3}, & \text{if } q = 2, \\ \frac{1+\alpha}{2}X, & \text{if } q \geqslant 3, \end{cases} \quad P(p+r) \leqslant \begin{cases} \frac{p+2}{3}, & \text{if } r = 2\ , \\ \frac{1+\alpha}{2}X, & \text{if } \ r \geqslant 3, \end{cases}$$

and

$$P(q+r) \leqslant \begin{cases} \alpha X, & \text{if } q \geqslant 3 \text{ and } r \geqslant 3, \\ \alpha X + 2, & \text{if } q = 2 \text{ or } r = 2. \end{cases}$$

Hence

$$P(w(n)) \leqslant \frac{1}{4}(\alpha+3)X + 4.$$

Case 2. $p+2$ is prime. Since $p>3$ and $p,\ p+2$ are both primes, we have $3 \mid p+4$.

Subcase 2.1. $q,\ r$ are both odd primes. By Lemma 1 we have

$$P(w(n)) \leqslant \max\{\frac{1+\alpha}{2}X,\ \alpha X\} \leqslant \frac{1}{4}(\alpha+3)X+4.$$

Subcase 2.2. $q>r=2$. Then $w(n)=(p+2)P(p+q)P(q+2)$. Let $p_1=p+2,\ q_1=P(p+q),\ r_1=P(q+2)$. Since $p_1 \leqslant p+2,\ q_1 \leqslant \frac{1}{2}(1+\alpha)X,\ r_1 \leqslant \alpha X+2$, and r_1 is odd, by Lemma 1 we have

$$P(p_1+q_1) \leqslant \begin{cases} \frac{p+4}{3}, & \text{if } q_1=2, \\ \frac{3+\alpha}{4}X+1, & \text{if } q_1 \text{ is odd,} \end{cases}$$

$$P(q_1+r_1) \leqslant \begin{cases} \alpha X+4, & \text{if } q_1=2, \\ \frac{1+3\alpha}{4}X+1, & \text{if } q_1 \text{ is odd,} \end{cases}$$

and

$$P(p_1+r_1) \leqslant \frac{1+\alpha}{2}X+2.$$

Hence

$$P(w^2(n)) \leqslant \frac{1}{4}(\alpha+3)X+4.$$

Similarly, when $r>q=2$, we can get the same conclusion.

Subcase 2.3. $q=r=2$. Then

$$w^3(n)=P(p+4)\cdot P^2(P(p+4)+p+2).$$

By Lemma 1 we have

$$P(p+4) \leqslant \frac{p+4}{3},\ P(P(p+4)+p+2) \leqslant \frac{p+2+(p+4)/3}{2}=\frac{2p+5}{3}.$$

Hence

$$P(w^3(n)) \leqslant \frac{1}{4}(\alpha+3)X+4.$$

This completes the proof of Lemma 2. □

Corollary 3. *Let X be an integer with $X \geqslant 3$ and α be a real number with $0<\alpha<1$. For $n=pqr \in A_3$ with $p \geqslant q \geqslant r \geqslant 3,\ p \leqslant X$ and $r \leqslant \alpha X$, there exists $2 \leqslant i \leqslant 4$ such that*

$$P(w^i(n)) \leqslant \frac{1}{8}(\alpha+7)X+4.$$

Proof. Let

$$p_1=P(p+q),\ q_1=P(p+r),\ r_1=P(q+r).$$

By Lemma 1 we have

$$p_1 \leqslant X,\ q_1 \leqslant \frac{1+\alpha}{2}X,\ r_1 \leqslant \frac{1+\alpha}{2}X.$$

Since $w(n)=P(p+q)P(p+r)P(q+r)$, by Lemma 2, there exists $1 \leqslant j \leqslant 3$ such that

$$P(w^j(w(n))) \leqslant \frac{3+(1+\alpha)/2}{4}X+4.$$

Therefore, there exists $2 \leqslant i \leqslant 4$ such that

$$P(w^i(n)) \leqslant \frac{1}{8}(\alpha+7)X+4.$$

This completes the proof of Corollary 3. □

Lemma 4. *For any* $n \in A_3$ *there exists* $1 \leqslant i \leqslant 2\log(P(n)+2)+6$ *such that*

$$P(w^i(n)) \leqslant \frac{15}{16}P(n)+6.$$

Proof. For any $n = pqr \in A_3$ with $p \geqslant q \geqslant r \geqslant 2$, we have

$$w(n) = P(p+q)P(p+r)P(q+r).$$

Case 1. $p,\ q,\ r$ are all odd primes.

Subcase 1.1. At most one of $\frac{1}{2}(p+q),\ \frac{1}{2}(p+r)$, and $\frac{1}{2}(q+r)$ is an odd prime.

If there is an odd prime, then this odd prime is not larger than p, but the other two prime factors of $w(n)$ are not larger than $\frac{1}{2}p$. Otherwise all of $P(p+q),\ P(p+r)$ and $P(q+r)$ are not larger than $\frac{1}{2}p$. Hence, by Lemma 2, there exists $1 \leqslant i \leqslant 4$ such that

$$P(w^i(n)) \leqslant \frac{1}{4}(3+\frac{1}{2})p+4 = \frac{7}{8}p+4.$$

Subcase 1.2. Precisely two of $\frac{1}{2}(p+q),\ \frac{1}{2}(p+r)$, and $\frac{1}{2}(q+r)$ are odd primes.

Firstly, we suppose that $\frac{1}{2}(p+q)$ and $\frac{1}{2}(p+r)$ are odd primes. Let

$$p_1 = P(p+q) = \frac{p+q}{2}, \quad q_1 = P(p+r) = \frac{p+r}{2}, \quad r_1 = P(q+r).$$

Since $p_1+q_1 = p+\frac{q+r}{2}$ is even, $1 \neq \frac{q+r}{2}$ is odd. Thus $3 \leqslant r_1 = P(q+r) \leqslant \frac{1}{2}p$. By Corollary 3, there exists $1 \leqslant i \leqslant 5$ such that

$$P(w^i(n)) \leqslant \frac{1}{8}(7+\frac{1}{2})p+4 = \frac{15}{16}p+4.$$

Similarly, we have the same conclusion for the other two cases: (a) $\frac{1}{2}(p+q)$ and $\frac{1}{2}(q+r)$ are odd primes; (b) $\frac{1}{2}(p+r)$ and $\frac{1}{2}(q+r)$ are odd primes.

Subcase 1.3. All of $\frac{1}{2}(p+q),\ \frac{1}{2}(p+r)$, and $\frac{1}{2}(q+r)$ are odd primes.

Let

$$p_1 = \frac{p+q}{2},\ q_1 = \frac{p+r}{2},\ r_1 = \frac{q+r}{2},$$
$$p_i = \frac{p_{i-1}+q_{i-1}}{2},\ q_i = \frac{p_{i-1}+r_{i-1}}{2},\ r_i = \frac{q_{i-1}+r_{i-1}}{2}\ (i = 2,3,...).$$

Assume that for $i = 1,2,...,k$, all of $p_i,\ q_i,\ r_i$ are odd primes, and at least one of $p_{k+1},\ q_{k+1},\ r_{k+1}$ is not an odd prime. Since

$$1 \leqslant p_k - r_k = \frac{p_{k-1}-r_{k-1}}{2} = \cdots = \frac{p-r}{2^k},$$

we have $k \leqslant 2\log p$. By Subcases 1.1 and 1.2, there exists $1 \leqslant j \leqslant 5$ such that

$$P(w^j(p_kq_kr_k)) \leqslant \frac{15}{16}p_k+4 \leqslant \frac{15}{16}p+4.$$

Thus

$$P(w^{k+j}(n)) \leqslant \frac{15}{16}p+4.$$

Hence, if $p,\ q,\ r$ are all odd primes, there exists $1 \leqslant i \leqslant 2\log p+5$ such that

$$P(w^i(n)) \leqslant \frac{15}{16}p+4.$$

Case 2. $p \geqslant q > r = 2$.

Now we note that $P(p+2)$ and $P(q+2)$ are odd primes.

Subcase 2.1. $P(p+q)$ is an odd prime.
As in Case 1, there exists $1 \leqslant i \leqslant 2\log(p+2)+6$ such that

$$P(w^i(n)) \leqslant \frac{15}{16}(p+2)+4 \leqslant \frac{15}{16}p+6.$$

Subcase 2.2. $P(p+q) = 2$.

- If at least one of $p+2$ and $q+2$ is not prime, then by Lemma 1 we have
$$P(p+2) \leqslant \frac{p+2}{3} \quad \text{or} \quad P(q+2) \leqslant \frac{p+2}{3}.$$
Noting that $2 \leqslant \frac{1}{2}(p+2)$, by Lemma 2 there exists $1 \leqslant i \leqslant 4$ such that
$$P(w^i(n)) \leqslant \frac{1}{4}(3+\frac{1}{2})(p+2)+4 = \frac{7}{8}(p+2)+4.$$
- If $p+2$, $q+2$ are both primes, then $p+4$ is odd composite and $q+4$ is odd composite or 7, and
$$w^2(n) = P(p+4)P(q+4)P(p+q+4).$$
Since $P(p+q)=2$, we have $4 \mid p+q+4$. So
$$P(p+q+4) \leqslant \frac{p+q+4}{4}.$$
By Lemma 1 we have
$$P(p+4) \leqslant \frac{p+4}{3},\ P(q+4) \leqslant \max\{\ 7,\ \frac{p+4}{3}\}.$$
Hence
$$P(w^2(n)) \leqslant \frac{15}{16}p+6.$$

Case 3. $q = r = 2$. Then $n = 2^2p$. By Lemma 2 with $\alpha = \frac{2}{3}$ and $X = p$, there exists $1 \leqslant i \leqslant 3$ such that

$$P(w^i(n)) \leqslant \frac{11}{12}p+4.$$

This completes the proof of Lemma 4. □

Proof of Theorem 1. For any $n = pqr \in A_3$ let $i_0 = 0$ and $P(n) = p$. By Lemma 4 there exist positive integers $i_1 < i_2 < \cdots$ such that

$$P(w^{i_k}(n)) \leqslant \frac{15}{16}P(w^{i_{k-1}}(n))+6, \quad k=1,2,\ldots \tag{1}$$

and

$$i_k - i_{k-1} \leqslant 2\log(P(w^{i_{k-1}}(n))+2)+6, \quad k=1,2,\ldots. \tag{2}$$

By (1) we have

$$\begin{aligned} P(w^{i_k}(n)) &\leqslant \frac{15}{16}P(w^{i_{k-1}}(n))+6 \leqslant (\frac{15}{16})^2P(w^{i_{k-2}}(n))+6(\frac{15}{16})+6 \leqslant \cdots \\ &\leqslant (\frac{15}{16})^kP(w^{i_0}(n))+6(\frac{15}{16})^{k-1}+6(\frac{15}{16})^{k-2}+\cdots+6(\frac{15}{16})+6 \\ &< (\frac{15}{16})^kp+96. \end{aligned}$$

If

$$k \geqslant \frac{\log p}{\log 16 - \log 15},$$

then

$$P(w^{i_k}(n)) \leqslant (\frac{15}{16})^k p + 96 \leqslant 97.$$

For each $m \in A_3$ with $P(m) \leqslant 97$, by [1, Theorem 1.1] there exists j_m such that

$$w^{j_m}(n) \in \{20, 98, 63, 75\}.$$

Let $c = \max\{j_m \mid m \in A_3,\ P(m) \leq 97\}$. Then there exists j with $1 \leqslant j \leqslant c$ such that

$$w^{i_k+j}(n) = w^j(w^{i_k}(n)) \in \{20, 98, 63, 75\}.$$

By [1, Corollary 2.6] we have

$$P(w^{i_k}(n)) \leqslant p + 4, \quad k = 1, 2, \ldots. \tag{3}$$

By (2) and (3) we have

$$i_k - i_{k-1} \leqslant 2\log(p+6) + 6, \quad k = 1, 2, \ldots.$$

Thus

$$i_k \leqslant 2k\log(p+6) + 6k, \quad k = 1, 2, \ldots.$$

Take an integer k with

$$k \geqslant \frac{\log p}{\log 16 - \log 15} > k - 1.$$

Then

$$\mathrm{ind}(n) \leqslant i_k + j \leqslant 2k\log(p+6) + 6k + c \ll (\log p)^2.$$

That is,

$$\mathrm{ind}(n) = O((\log p)^2).$$

This completes the proof of Theorem 1. □

Finally, we employ the Green-Tao Theorem on arithmetic progressions in the primes to confirm Goldring's Conjecture 2.10 [1]; i.e., $\mathrm{ind}(n)$ can be arbitrarily large.

Proof of Theorem 2. By the Green-Tao Theorem on arithmetic progressions in the primes (see [2]), for any positive integer $k \geqslant 7$, there exist two positive integers $a,\ d$ such that $a + id$ $(0 \leqslant i \leqslant 2^k)$ are all primes. Let

$$p_0 = a + 2^k d,\ q_0 = a + 2^{k-1} d,\ r_0 = a,$$

and

$$p_i = \frac{p_{i-1} + q_{i-1}}{2},\ q_i = \frac{p_{i-1} + r_{i-1}}{2},\ r_i = \frac{q_{i-1} + r_{i-1}}{2}\ ,\ i = 1, 2, \ldots, k-1.$$

Then $p_i,\ q_i$ and r_i ($i = 0$, 1, 2, ..., $k-1$) are in the arithmetic progression $a+id$ $(0 \leqslant i \leqslant 2^k)$. Hence $p_i,\ q_i$ and r_i ($i = 0$, 1, 2, ..., $k-1$) are all primes. Let $n = p_0\, q_0\, r_0$. Then $w^i(n) = p_i\, q_i\, r_i$ ($i = 1$, 2, ..., $k-1$). Noting that

$$p_i + q_i + r_i = p_0 + q_0 + r_0 > 2^k,$$

we have $w^i(n) \notin \{20,\ 63,\ 75,\ 98\}$ ($i = 1$, 2, ..., $k-1$). So $\mathrm{ind}(n) \geqslant k$. This completes the proof of Theorem 2. □

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