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A SHARP ROGERS AND SHEPHARD INEQUALITY FOR THE p-DIFFERENCE BODY OF PLANAR CONVEX BODIES

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Abstract. We prove a sharp Rogers and Shephard type inequality for the p-difference body of a convex body in the two-dimensional case, for every $p \geq 1$.

1. Introduction

A *convex body* is a non-empty convex compact subset of $\mathbb{R}^n$; let us indicate the set of convex bodies in $\mathbb{R}^n$ with $\mathscr{K}^n$. To each convex body K we can associate in a biunique way its *support function* h_K:

$$h_K(u) = \sup\{\langle x, u\rangle \mid x \in K\}, \qquad \text{for all } u \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product. The support function is a fundamental tool since the main properties of the body can be deduced from it.

One of the most interesting aspects of convex geometry, i.e. the theory of convex bodies, are geometric inequalities. An important family of inequalities are those leading to estimate the volume of a special body associated with a convex body (for example the difference body or the reflection body) in terms of the volume of the body itself. A remarkable inequality of this type is the classical Rogers and Shephard inequality (see [15]) which asserts that for all $K \in \mathscr{K}^n$

$$V_n\Big(K + (-K)\Big) \leq \binom{2n}{n} V_n(K), \tag{1.1}$$

and equality holds if and only if K is a simplex. Here $V_n(K)$ denotes the n-dimensional volume of K (i.e. the n-dimensional Lebesgue measure). The body $K + (-K)$ is called the difference body of K and it is the Minkowski sum of K and its reflected body with respect to the origin, $-K$. We recall more generally that the Minkowski sum of K and $L \in \mathscr{K}^n$ is

$$K + L = \{z \in \mathbb{R}^n \mid z = k + l, \quad k \in K, l \in L\}.$$

Another inequality due to Rogers and Shephard ([16]) concerns the convex hull (here denoted by conv) of K and $-K$, under the assumption that the origin o belongs to K:

$$V_n\Big(\operatorname{conv}\big(K \cup (-K)\big)\Big) \leq 2^n V_n(K), \tag{1.2}$$

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where equality holds if and only if K is a simplex with one vertex at the origin.

In [6] Firey introduced a new operation for convex bodies, called p-sum, which depends on the parameter $p \geq 1$ and extends the Minkowski sum. This notion originated the so-called Brunn-Minkowski-Firey theory, or L_p Brunn-Minkowski theory of convex bodies. The L_p Brunn-Minkowski theory can be seen as an interpolation of the classical Brunn-Minkowski theory (L_1 theory) that arises from Minkowski sums of convex bodies and the study of convex hulls of sets (L_∞ theory). Although it does not have the same direct geometric interpretation as the L_1 and L_∞ cases, the L_p theory has been attracting a lot of attention from both geometers and analysts because it provides analytic tools for geometric problems and new connections between convex geometry and analysis. An account on the L_p theory of convex bodies can be found in the papers [9], [10]; recent important developments of this theory are contained in [11], [12], [2], [3], [13], [7], [4].

Let us fix $K, L \in \mathscr{K}^n$ both containing the origin; the p-sum of K and L, $K +_p L$, is defined by its support function in the following way:

$$h_{K+_pL}(u) = \Big(h_K^p(u) + h_L^p(u)\Big)^{\frac{1}{p}}, \qquad u \in \mathbb{R}^n.$$

This definition admits a natural extension to the case $p = \infty$:

$$h_{K+_\infty L}(u) = \lim_{p\to\infty} h_{K+_pL}(u) = \max\{h_K(u), h_L(u)\},\ u \in \mathbb{R}^n.$$

Note that the extremal values $p = 1$ and $p = \infty$ correspond to the Minkowski sum and the convex hull of the union, respectively. Indeed one has

$$h_{K+_1L}(u) = h_K(u) + h_L(u) = h_{K+L}(u)$$

and

$$h_{K+_\infty L}(u) = \max\{h_K(u), h_L(u)\} = h_{\mathrm{conv}(K\cup L)}.$$

As proved by Firey [6], the p-sum is monotone with respect to the parameter p: for all $K, L \in \mathscr{K}^n$ such that $o \in K, L$, if $p \leq q$, then

$$K +_q L \subseteq K +_p L.$$

This implies that for all $p \geq 1$,

$$\mathrm{conv}(K \cup L) \subseteq K +_p L \subseteq K + L.$$

Another simple inclusion is

$$K +_p L \subseteq 2^{\frac{1}{p}} \mathrm{conv}(K \cup L).$$

In particular, choosing $L = -K$ and using inequalities (1.1) and (1.2), we have

$$V_n\Big(K +_p (-K)\Big) \leq \min\left\{ \binom{2n}{n},\ 2^{n\frac{(1+p)}{p}} \right\} V_n(K).$$

A natural problem is then to find the best constant $c = c_{n,p}$, depending on n and p, such that

$$V_n\Big(K +_p (-K)\Big) \leq\ c_{n,p}\, V_n(K), \qquad \text{for all } K \in \mathscr{K}^n,\ o \in K. \tag{1.3}$$

In this paper we solve this problem in the planar case $n = 2$, for every $p \geq 1$.

Theorem 1.1. *For every $p \geq 1$ there exists a constant c_p such that*

$$V_2\Big(K +_p (-K)\Big) \leq c_p\, V_2(K), \tag{1.4}$$

for all $K \in \mathscr{K}^2$. In particular if K is a triangle with one vertex at the origin, then equality holds.

An explicit expression of c_p will be presented in Section 3.

We will show the p-Rogers and Shephard inequality (1.4) as a consequence of a theorem about the p-sum of the so-called parallel chord movements of convex bodies.

A parallel chord movement is a special one-parameter family of convex bodies which can be seen as continuous deformations of a fixed convex body. More precisely, fix $K \in \mathscr{K}^n$ and a direction $v \in \mathbb{R}^n$, which is the direction of the movement. We move each chord of K parallel to v in that direction with a certain speed, and we consider the union of these chords as the time parameter varies. If the speed function is suitably chosen, namely if the union of the chords is convex for all values of the parameter, then the family of the resulting convex bodies is a parallel chord movement.

Parallel chord movements are special cases of a wider class of movements of convex bodies introduced by Rogers and Shephard in [17], which have recently been applied in the proof of several inequalities in convex geometry (see, for example, [1], [2]–[5], [14]).

The importance of these movements is due principally to the behaviour of several geometric functionals with respect to the parameter of the movement. Indeed many of them, and the volume is the main example, are convex functions of the time parameter of the movement.

In particular in this paper we prove that if K_t is a parallel chord movement, then the volume of its p-difference body $V_n\Big(K_t +_p (-K_t)\Big)$ is a convex function of t, for all $p \geq 1$. This result, together with a technique used in [1], leads to the proof of Theorem 1.1. As noted in [1] this technique is successful only in the planar case, so our method cannot be used to prove inequality (1.3) in the general case $n \geq 2$.

The paper is organized as follows. In Section 2 we introduce several kinds of movements of convex bodies and we show some of their properties. Next to basic results we present a theorem about the p-sum of a particular type of movement. In section 3 we prove Theorem 1.1 as an application of the results concerning movements of convex bodies.

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2. Shadow systems and linear parameter systems

A *shadow system* is a family of n-dimensional convex bodies $\{K(u)\}$ obtained as the projection of a fixed convex body $\tilde{K} \subseteq \mathbb{R}^{n+1}$ onto the hyperplane $\{e_{n+1}^{\perp}\}$, which we identify with $\mathbb{R}^n$, along the direction $e_{n+1} + u$. Here u varies in $\{e_{n+1}^{\perp}\}$. The shadow system is said to be originated from the $(n+1)$-dimensional body $\tilde{K}$.

A *linear parameter system* is a family of convex bodies $\{K_t\}$ that can be written in the form

$$K_t = \operatorname{conv}\{x_i + \lambda_i t v \;:\; i \in I\}, \qquad t \in \mathscr{I}, \tag{2.1}$$

where I is an arbitrary index set, $\{x_i\}_{i\in I}$ and $\{\lambda_i\}_{i\in I}$ are bounded subsets of $\mathbb{R}^n$ and of $\mathbb{R}$ respectively, $\mathscr{I}$ is an interval of $\mathbb{R}$ and $v \in \mathbb{R}^n$ is the direction of the linear parameter system.

Linear parameter systems are shadow systems in which u lies on a line; indeed we have the following result.

Proposition 2.1. *$\{K_t\}_{t\in\mathscr{I}}$ is a linear parameter system in $\mathbb{R}^n$ if and only if there exist a convex body $\tilde{K}$ in $\mathbb{R}^{n+1}$ and $v \in \{e_{n+1}^{\perp}\}$ such that for every $t \in \mathscr{I}$, K_t is the projection of $\tilde{K}$ onto the hyperplane $\{e_{n+1}^{\perp}\}$ along the direction $e_{n+1} - tv$.*

The idea to view linear parameter systems as projections of higher-dimensional convex bodies is contained in the original papers by Rogers and Shephard ([17], [19]) and was largely used by Campi and Gronchi ([2]–[5]). For the sake of completeness here we present the proof of Proposition 2.1.

Proof. Let K_t be of the form (2.1) and let us define the body $\tilde{K}$ as follows:

$$\tilde{K} = \operatorname{conv}\Big(\{x_i + \lambda_i e_{n+1} \ : \ i \in I\ \}\Big).$$

For all $t \in \mathscr{I}$ let us call L_t the projection of $\tilde{K}$ onto $\{e_{n+1}^{\perp}\}$ along $e_{n+1} - tv$. For all $y \in L_t$ there exists $z \in \tilde{K}$ such that $y = z - \langle z, e_{n+1}\rangle\big(e_{n+1} - tv\big)$. Furthermore there exist $a_i \in e_{n+1}^{\perp}$, $\lambda_i \in \mathbb{R}$, and $\sigma_i \geq 0$, $i = 1, ..., n+1$, such that $\sum_{i=1}^{n+1}\sigma_i = 1$ and

$$z = \sum_{i=1}^{n+1}\sigma_i(a_i + \lambda_i e_{n+1}).$$

Therefore

$$y = \sum_{i=1}^{n+1}\sigma_i(a_i + \lambda_i tv).$$

This implies that L_t is contained in K_t. To prove the reverse inclusion one can observe that the previous implications are true in both directions.

Conversely, let $\tilde{K}$ be any $(n+1)$-dimensional convex body and fix $t \in \mathscr{I}$; its projection onto $\{e_{n+1}^{\perp}\}$ along $e_{n+1} - tv$ is the set

$$L_t = \{e_{n+1}^{\perp}\} \cap \{x \in \mathbb{R}^{n+1} \mid x = z + s(e_{n+1} - tv),\ z \in \tilde{K},\ s \in \mathbb{R}\}.$$

This is equivalent to

$$L_t = \big\{z - \langle z, e_{n+1}\rangle e_{n+1} + t\langle -z, e_{n+1}\rangle v \ : \ z \in \tilde{K}\big\},$$

and, by the convexity of $\tilde{K}$, $\{L_t\}_{t\in\mathscr{I}}$ is a linear parameter system as defined in (2.1). □

From the previous proof it follows that the body $\tilde{K}$ which generates a linear parameter system of the form (2.1) can be explicitly written as

$$\tilde{K} = \operatorname{conv}\{x_i + \lambda_i e_{n+1} \ : \ i \in I\}. \tag{2.2}$$

Campi and Gronchi showed in [4] the following formula, which relates the support functions of K_t and $\tilde{K}$:

$$h_{K_t}(u) = h_{\tilde{K}}(u + t\langle u, v\rangle e_{n+1}), \qquad u \in \mathbb{R}^n,\ t \in \mathscr{I}. \tag{2.3}$$

We can give a cinematic interpretation of a linear parameter system viewing the numbers λ_i as the speeds of the points x_i along the direction v and t as the time parameter.

If the index set I is a convex body $K \in \mathscr{K}^n$ and the speed is a function of the point, then the linear parameter system is called *continuous movement*:

$$K_t = \operatorname{conv}\{x + \alpha(x)tv \ : \ x \in K\}, \qquad t \in \mathscr{I},$$

where $\alpha(\cdot)$ is a bounded function on K.

Assume that the speed function is constant on each chord parallel to v, i.e. $\alpha(x) = \beta(x|v^\perp)$, where $x|v^\perp$ is the projection of x onto $\{v^\perp\}$ and β is a function defined on the ortoghonal projection of K onto $\{v^\perp\}$. Moreover, if β is such that convexity is preserved for any t, namely

$$\{x + \beta(x|v^\perp)tv \ : \ x \in K\} = \operatorname{conv}\{x + \beta(x|v^\perp)tv \ : \ x \in K\},$$

then the continuous movement is called *parallel chord movement*.

In other words a parallel chord movement is obtained by assigning to each chord parallel to the direction v a speed vector $\beta(x|v^\perp)v$ and considering for each fixed time t the union of these chords; such a union has to be convex. If for some value of $t \in \mathscr{I}$ it is not convex, then its convex hull is just a continuous movement but not a parallel chord movement (usually in this case one considers only the subset $\mathscr{J}$ of $\mathscr{I}$ such that K_t is convex for $t \in \mathscr{J}$). The reason why we consider only convex unions of chords is due to the behaviour of the volume. Indeed if $\{K_t\}_{t\in\mathscr{I}}$ is a parallel chord movement, then the volume of K_t is independent of t.

The following theorem is due to Rogers and Shephard (see [17]), and it is one of the main motivations for the use of linear parameter systems in the theory of convex bodies.

Theorem 2.2. *The volume $V_n(K_t)$ of a linear parameter system is a convex function of the parameter t.*

In [4] it is proved that the Minkowski sum of linear parameter systems is a linear parameter system. Here we extend this result to the p-sum. This fact is one of the main ingredients in the proof of the p-Rogers and Shephard inequality.

Theorem 2.3. *Let $\{K_t\}_{t\in\mathscr{I}}$ and $\{L_t\}_{t\in\mathscr{I}}$ be linear parameter systems along the direction v and let $p \geq 1$. Then $\{K_t +_p L_t\}_{t\in\mathscr{I}}$ is also a linear parameter system along the direction v.*

The proof is a straightforward consequence of Proposition 2.1 and the following lemma.

Lemma 2.4. *Let $\{K_t\}_{t\in\mathscr{I}}$ and $\{L_t\}_{t\in\mathscr{I}}$ be linear parameter systems along the same direction v and let $\tilde{K}$ and $\tilde{L}$ be the $(n+1)$-dimensional convex bodies which generate K_t and L_t, respectively, defined as in* (2.2). *Hence for all $t \in \mathscr{I}$, $K_t +_p L_t$ is the projection of $\tilde{K} +_p \tilde{L}$ onto the hyperplane $\{e_{n+1}^\perp\}$ along the direction $e_{n+1} - tv$.*

Proof. Using (2.3) one has

$$\begin{aligned} h^p_{\tilde{K}+_p\tilde{L}}\big(u + t\langle u, v\rangle e_{n+1}\big) &= h^p_{\tilde{K}}\big(u + t\langle u, v\rangle e_{n+1}\big) + h^p_{\tilde{L}}\big(u + t\langle u, v\rangle e_{n+1}\big) \\ &= h^p_{K_t}(u) + h^p_{L_t}(u) = h^p_{K_t+_pL_t}(u). \end{aligned}$$

This implies that $K_t +_p L_t$ is the projection of the body $\tilde{K} +_p \tilde{L}$ onto the hyperplane $\{e_{n+1}^{\perp}\}$ along the direction $e_{n+1} - tv$, which means, by Proposition 2.1, that $K_t +_p L_t$ is a linear parameter system along v. □

3. The proof of the p-Rogers and Shephard inequality

Let us call $\mathscr{K}_0^n$ the set of convex bodies with non-empty interior and containing the origin, and let us consider the functional F_p defined on $\mathscr{K}_0^n$:

$$F_p(K) = \frac{V_n\Big(K +_p (-K)\Big)}{V_n(K)}.$$

It is clear that the best constant $c_{n,p}$ such that (1.3) holds is the supremum of F_p in $\mathscr{K}_0^n$.

We will use linear parameter systems to find a maximum for the functional F_p in the planar case. The starting point is the next proposition which follows from Theorem 2.2 and Theorem 2.3 and the fact that the volume is constant on each parallel chord movement.

Proposition 3.1. *If K_t is any parallel chord movement such that $K_t \in \mathscr{K}_0^n$ for all $t \in \mathscr{I}$, then $F_p(K_t)$ is a convex function of the parameter t.*

In [1] the following fact is proved: if P is a planar convex polygon with m vertices, $m > 3$, then there exists a parallel chord movement $\{P_t\}_{t \in [t_0, t_1]}$, with $t_0 < 0 < t_1$, such that $P = P_0$ and P_{t_0} and P_{t_1} have at most $(m-1)$ vertices. By Proposition 3.1 it follows that

$$F_p(P) \leq \max\{F_p(P_{t_0}), F_p(P_{t_1})\}.$$

Using this fact recursively we deduce that

$$\sup_{\mathscr{P}} F_p = \sup_{\mathscr{T}} F_p,$$

where $\mathscr{P} = \{K \in \mathscr{K}_0^2 \mid K \text{ is a polygon}\}$ and $\mathscr{T} = \{K \in \mathscr{K}_0^2 \mid K \text{ is a triangle}\}$. Moreover, by the continuity of $F_p(\cdot)$ and a standard density argument, one has

$$\sup_{\mathscr{K}_0^2} F_p = \sup_{\mathscr{T}} F_p.$$

In particular we are going to show that triangles with one vertex at the origin are maximizers for F_p. In order to do this, let $T \in \mathscr{T}$ and assume $o \in \mathrm{int}(T)$ (int denotes the interior). Then there exists a parallel chord movement (whose elements are translates of T), $\{T_t\}_{t \in [t_0, t_1]}$ with $t_0 < 0 < t_1$, such that $T_0 = T$ and $o \in \mathrm{bd}(T_{t_0})$, $o \in \mathrm{bd}(T_{t_1})$ (bd denotes the boundary). Similarly, if $o \in \mathrm{bd}(T)$, then there exists a parallel chord movement containing T, whose endpoints are triangles with one vertex at o. Again using Proposition 3.1, we have proved that

$$\sup_{\mathscr{K}_0^2} F_p = \sup_{\mathscr{T}_0} F_p,$$

where $\mathscr{T}_0$ is the set of triangles with one vertex at the origin.

Note that F_p is invariant under non-singular linear transformations. This implies that F_p is constant on $\mathscr{T}_0$.

This argument proves the following result.

Theorem 3.2. *If T is a triangle in $\mathscr{K}_0^2$ with one vertex at the origin, then T is a maximizer for F_p.*

To compute the best constant $c_{2,p}$, we can choose as a maximizer the triangle with vertices at the origin, at $(1,0)$ and $(0,1)$; let us indicate it with $\overline{K}$. Namely

$$c_{2,p} = \frac{V_2\Big(\overline{K} +_p (-\overline{K})\Big)}{V_2(\overline{K})}.$$

Then to express the value of $c_{2,p}$ it is necessary to know how the p-difference body $\overline{K} +_p (-\overline{K})$ looks. Here we use the parametrization of the boundary of a convex body in terms of its support function (see [18, Corollary 1.7.3]).

The support function of $\overline{K} +_p (-\overline{K})$ is

$$h_{\overline{K}+_p(-\overline{K})}(w) = \begin{cases} \cos\theta & \text{if } 0 \le \theta < \frac{\pi}{4}, \\ \sin\theta & \text{if } \frac{\pi}{4} \le \theta < \frac{\pi}{2}, \\ \big(\sin^p\theta + (-\cos\theta)^p\big)^{\frac{1}{p}} & \text{if } \frac{\pi}{2} \le \theta \le \pi, \end{cases}$$

where $w = \mathbf{e}^{i\theta} \in S^1$. Furthemore, by the symmetry of $\overline{K} +_p (-\overline{K})$,

$$h_{\overline{K}+_p(-\overline{K})}(\mathbf{e}^{i\theta}) = h_{\overline{K}+_p(-\overline{K})}(\mathbf{e}^{i(\theta-\pi)}),$$

for all $\pi \le \theta \le 2\pi$. Then a parametrization for the boundary of $\overline{K} +_p (-\overline{K})$, for $1 < p < +\infty$, is $\zeta(\theta) = \Big(x(\theta), y(\theta)\Big)$, where

$$x(\theta) = \begin{cases} 1 - \frac{2}{\pi}\theta & \text{for } \theta \in [0, \frac{\pi}{2}], \\ -\big(\sin^p\theta + (-\cos\theta)^p\big)^{\frac{1-p}{p}}(-\cos\theta)^{p-1} & \text{for } \theta \in (\frac{\pi}{2}, \pi), \end{cases}$$

$$y(\theta) = \begin{cases} \frac{2}{\pi}\theta & \text{for } \theta \in [0, \frac{\pi}{2}], \\ \big(\sin^p\theta + (-\cos\theta)^p\big)^{\frac{1-p}{p}}\sin^{p-1}\theta & \text{for } \theta \in (\frac{\pi}{2}, \pi), \end{cases}$$

and the remaining part of the boundary can be found using the symmetry of the body.

A picture can perhaps better show the geometry of the body. In Figure 1, $\overline{K} +_p (-\overline{K})$ is represented for the the values 1, 1.5, 2, 15, ∞ of the parameter p.

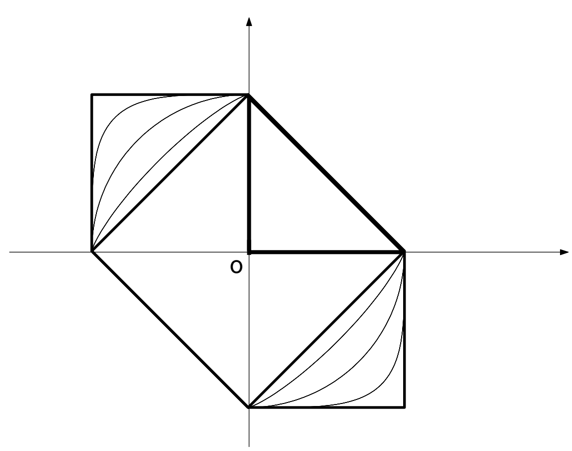

FIGURE 1

Using the above parametrization and Gauss-Green's formulas we can express the area of $\overline{K} +_p (-\overline{K})$ and then the value of the best constant $c_{2,p}$:

$$c_{2,p} = 2\left(1 + (p-1)\int_0^{\frac{\pi}{2}} \frac{\sin^{p-2} t \cos^{p-2} t}{\left(\sin^p t + \cos^p t\right)^{2\frac{(p-1)}{p}}}\, dt\right), \quad 1 < p < +\infty.$$

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