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ERRATUM TO "A FINITELY PRESENTED GROUP WITH UNBOUNDED DEAD-END DEPTH"

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Abstract. In our earlier work we exhibited a finitely presented group G that we claimed enjoyed a geometric property called *unbounded dead-end depth*. We described a model for understanding the word metric on G, which regrettably was incorrect. Here, we describe the corrected model and the revised proof that G does indeed have this property.

For an infinite group G with finite generating set $\mathcal{A}$, the (*dead-end*) *depth* of $g \in G$ is the distance in the word metric $d_{\mathcal{A}}$ between g and the complement in G of the closed ball B_g of radius $d_{\mathcal{A}}(1, g)$ centered at 1.

In [1] we claimed that for $(G, \mathcal{A})$, defined by

$$\begin{aligned} G &= \langle\, a, s, t \mid a^2 = 1,\ [a, a^t] = 1,\ [s, t] = 1,\ a^s = aa^t \,\rangle \\ \mathcal{A} &= \{a, s, t, at, ta, ata, as, sa, asa\}, \end{aligned}$$

depth admits no bound. (Notation: $[a, b] = a^{-1}b^{-1}ab$ and $a^b = b^{-1}ab$.)

We thank Jörg Lehnert for pointing out our mistake in [1], where our model did not behave as claimed for lamplighter locations below the t-axis. A revision of [1] incorporating the corrections described here can be found on the ArXiv: `math.GR/0406443`.

1. The corrected lamplighter grid model for G

The subgroup $\langle a, t\rangle$ of G is the *lamplighter group* $\mathbb{Z}_2 \wr \mathbb{Z}$ — it has presentation $\langle a, t \mid a^2, [a, a^{t^i}],\ \forall i \in \mathbb{Z}\rangle$. We will describe a lamplighter model for G, building on Cannon's *lamplighter model* for $\mathbb{Z}_2 \wr \mathbb{Z}$. More precisely, we will give a faithful, transitive, left action of G on $\mathcal{P}_{fin}(\mathbb{L}) \times \mathbb{Z}^2$, where $\mathcal{P}_{fin}(\mathbb{L})$ denotes the set of finite subsets of a countable set $\mathbb{L}$ that we define below.

In our model a *lamplighter* moves among the lattice points of the infinite rhombic grid illustrated in Figure 1. We will refer to the union of the t-axis and the portion of the s-axis that is below the t-axis as the *lampstand*. Let $\mathbb{L}$ be the set of lattice points on the lampstand; these are the locations of the lamps in our model. An element of $\mathcal{P}_{fin}(\mathbb{L})$ denotes a finite configuration of illuminated lamps.

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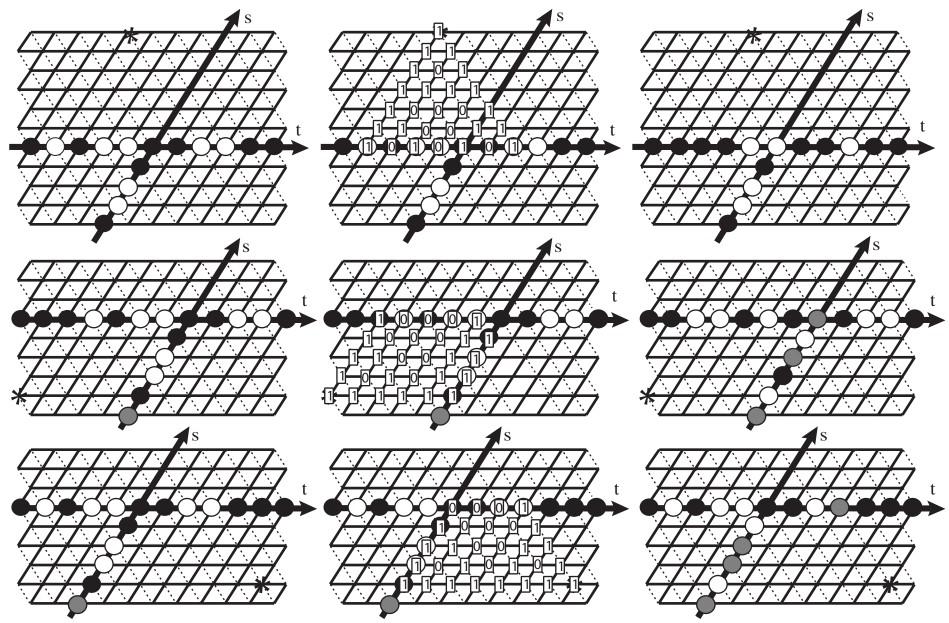

FIGURE 1. Examples of a acting. The first column of diagrams shows $hg(\emptyset, (0,0))$, where $g = s^3as^{-1}as^{-2}t^4at^{-2}at^{-1}at^{-3}at^{-1}at^{-1}at^3$ and h is s^6t^{-4}, $s^{-4}t^{-4}$, and $s^{-4}t^7$, respectively. The third column show the corresponding $ahg(\emptyset, (0,0))$. (Off and on lamps are represented by circles with black and white interiors, respectively.)

The actions of s and t are to move the lamplighter one unit in the s- and t-directions, respectively, in the rhombic grid. The rhombic grid is subdivided into a triangular grid by inserting a negatively sloped diagonal into each rhombus (the dashed lines in Figure 1). At every lattice point there is a *button*; the action of a is to *press* the button at the location of the lamplighter, and this has the following effect. If the button is on the lampstand, then it toggles the lamp at its location. If the button is off the lampstand, a *signal is set off* that propagates in the triangular grid toward the lampstand and toggles finitely many lamps as follows.

When the button is at a lattice point above the t-axis, the signal propagates downward in the triangular grid along the sloped grid lines. At each vertex *en route* it splits into two signals, one advancing along the positively sloped diagonal below and one along the negatively sloped diagonal below. The signals stop when they hit the t-axis, and each lamp on the t-axis switches between on and off once for every signal it receives. The manner in which these signals split as they propagate toward the lampstand leads to a connection with Pascal's triangle modulo 2, which is illustrated in the example shown in the top row of diagrams of Figure 1.

When the button is at a lattice point below the t-axis and to the left of the s-axis, the signals propagate similarly through the triangular grid toward the lampstand, but move in the horizontal direction and in the s-direction. Again there is a connection with Pascal's triangle modulo 2 but this time it is rotated by $2\pi/3$ and the pattern is interrupted at the s- and t-axes. The middle row of diagrams in Figure 1 shows an example.

When the button is at a lattice point below the t-axis and to the right of the s-axis, the signals propagate in the horizontal direction and in the negatively sloped diagonal direction toward the lampstand, where they again stop. Pascal's triangle modulo 2 rotated by $4\pi/3$ from its standard orientation (and interrupted on the t-axis) can be used in describing the action, as illustrated in the bottom row of diagrams of Figure 1.

To verify that we have a well-defined action of G on $\mathcal{P}_{fin}(\mathbb{L}) \times \mathbb{Z}^2$, it suffices to check that a^2, $[a, a^t]$, and $[s,t]$ all act trivially and that the actions of aa^t and a^s agree. This and the proof that the action is transitive are left to the reader. The fact that the action is faithful follows from converting a word $g \in G$ which satisfies $g(\emptyset, (0,0)\,) = (\emptyset, (0,0)\,)$ first into a word which does not leave the lampstand and then by using commutation relations to show that it represents the identity in G.

2. Proof that depth in $(G, \mathcal{A})$ is unbounded

As in [1], we define maps $\mathcal{I} : G \to \mathcal{P}_{fin}(\mathbb{L})$ and $\mathcal{L} : G \to \mathbb{Z}^2$ by $(\,\mathcal{I}(g), \mathcal{L}(g)\,) = g(\emptyset, (0,0)\,)$, giving the lamps illuminated and the location of the lamplighter respectively, after the action of g on the configuration in which no lamps are lit and the lamplighter is at the origin. We define H_n to be the subset of $\mathbb{Z}^2$ of lattice points in (and on the boundary of) the hexagonal region of the grid with corners at $(\pm n, 0), (0, \pm n), (n, -n), (-n, n)$. The shaded region in Figure 2 is H_4.

The crucial feature of $\mathcal{A}$ is that the button at a vertex at which the lamplighter is leaving or arriving can be pressed with no additional cost to word length. So if $g \in G \smallsetminus \{a\}$, then $d_{\mathcal{A}}(g, 1)$ is the length of the shortest path in the rhombic grid that starts at $(0,0)$, finishes at $\mathcal{L}(g)$, and such that pressing some of the buttons at the vertices visited produces the configuration $\mathcal{I}(g)$ of illuminated bulbs.

Proposition 2.1. *If $g \in G$ is such that $\mathcal{I}(g) \subseteq H_n$ and $\mathcal{L}(g) \in H_n$, then $d_{\mathcal{A}}(1, g) \leq 6n$.*

Proof. Define $(p,q) := \mathcal{L}(g)$. So p and q are the s- and t-coordinates, respectively, of the position of the lamplighter. We will describe a path from $(0,0)$ to (p,q) and specify buttons to push *en route* that will illuminate $\mathcal{I}(g)$.

Case $p \geq 0$. We let the lamplighter begin by travelling distance $2n$ first along the s-axis to $(-n, 0)$ and then back to $(0,0)$, illuminating all bulbs in $\mathcal{I}(g)$ that are not on the t-axis as it goes. Then we illuminate the lamps of $\mathcal{I}(g)$ on the t-axis by following a path of length at most $4n$ from the origin to $\mathcal{L}(g)$ as described in [1].

Case $p < 0$ and $q \leq 0$. The lamplighter begins by travelling along the t-axis to $(0, n)$ and then back to $(0,0)$, illuminating all the bulbs in $\mathcal{I}(g)$ that it traverses *en route*. Next it moves along the t-axis to $(0, -n)$ and then parallel to the s-axis to $(-n, -n)$. As in the previous case, between $(0, -n)$ and $(-n, -n)$ the lamplighter can press a combination of buttons which put the lights on the s-axis into the configuration of $\mathcal{I}(g)$. This will have some effect on the lights between $(-1, 0)$ and $(-n, 0)$. But, when travelling from $(0,0)$ to $(0, -n)$, the lamplighter can toggle lights both to counteract this effect and to implement the required configuration for those lights. By this stage the lamplighter has travelled distance $4n$, is located at $(-n, -n)$, and $\mathcal{I}(g)$ has been achieved. All that remains is for the lamplighter to move to (p,q), which is within a further distance of $2n$.

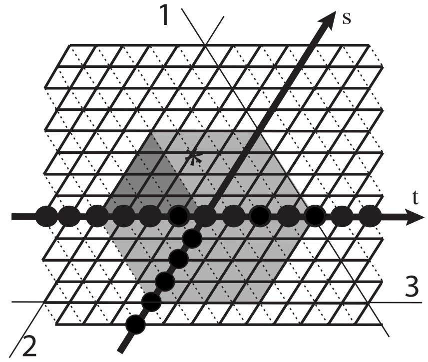

FIGURE 2. Regions and lines in the lamplighting grid.

Case $p < 0$ and $q > 0$. First the lamplighter travels along the t-axis to $(0, -n)$ and then back to $(0, 0)$, illuminating all the bulbs as per $\mathcal{I}(g)$ *en route*. Then the lamplighter moves along the t-axis to and then parallel to the t-axis to $(-n, n)$. Between $(0, n)$ and $(-n, n)$ buttons are pressed to illuminate the lamps on the s-axis as per $\mathcal{I}(g)$, and between $(0, 0)$ and $(0, n)$ buttons are pressed to counteract the effect on the bulbs there and to achieve the configuration $\mathcal{I}(g)$. The lamplighter then moves to (p, q), which is within distance $2n$ from $(-n, n)$. $\square$

Proposition 2.2. *If $g \in G$ satisfies $\mathcal{L}(g) = (0, 0)$ and $\mathcal{I}(g) = \{(0, n), (0, -n), (-n, 0)\}$, then $d_{\mathcal{A}}(1, g) \geq 6n$.*

Proof. In order to light the lamps at $(0, n)$, $(0, -n)$ and $(-n, 0)$, the lamplighter must visit (or cross) the straight lines L_1 through $(0, n)$ and $(n, 0)$, L_2 through $(0, -n)$ and $(n, -n)$, and L_3 through $(-n, 0)$ and $(-n, n)$, respectively (see Figure 2). Suppose the lamplighter follows a minimal length path that makes these visits. We will show that this path has length at least $6n$.

Drawing L_1, L_2, L_3 in standard $\mathbb{R}^2$ with the ℓ_1-metric, a symmetry becomes apparent on account of which we may as well assume that the lamplighter first visits L_2. But then because we are using the ℓ_1-metric, the path can be altered without increasing its length so that it first travels from the origin to L_2 along the t-axis. If the path visits L_3 before L_1, then it can be altered without increasing its length so that it does so by next travelling distance n in the (negative) s-direction. One then easily checks that visiting L_1 and returning to the origin costs at least $4n$ in additional length. Suppose, on the other hand, the lamplighter proceeds next to L_1 after L_2. Then the final portion of the lamplighters journey will be between L_3 and the origin and the path can be altered, with no change in length, so that it follows the s-axis from $(-n, 0)$ to $(0, 0)$. One checks that a shortest path from $(0, -n)$ to $(-n, 0)$ via L_1 has length at least $4n$.

So in each case the length of the path is at least $6n$ and so $d_{\mathcal{A}}(1, g) \geq 6n$. $\square$

It follows, for reasons in [1], that

$$g_n := s^n a s^{-n} t^n a t^{-2n} a t^n = s^{n-1}(sa)s^{-n}t^{n-1}(ta)t^{-2n}(at)t^{n-1}$$

has depth at least n in $(G, \mathcal{A})$. So there is no bound on depth in $(G, \mathcal{A})$.

References

[1] S. Cleary and T. R. Riley. A finitely presented group with unbounded dead-end depth. *Proc. Amer. Math. Soc.*, 134(2):343–349, 2006. MR2176000 (2006g:20066)

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