PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 7, July 2008, Pages 2445–2451
S 0002-9939(08)09220-4
Article electronically published on February 29, 2008

EXTREME POINTS, EXPOSED POINTS, DIFFERENTIABILITY POINTS IN CL-SPACES

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(Communicated by N. Tomczak-Jaegermann)

Abstract. This paper presents a property of geometric and topological nature of Gateaux differentiability points and Fréchet differentiability points of almost CL-spaces. More precisely, if we denote by M a maximal convex set of the unit sphere of a CL-space X, and by C_M the cone generated by M, then all Gateaux differentiability points of X are just $\bigcup$n-s(C_M), and all Fréchet differentiability points of X are $\bigcup \text{int}(C_M)$ (where n-s(C_M) denotes the non-support points set of C_M).

1. Introduction

Speaking of the classification of Banach spaces by convexity, we should mention here that there are two extreme classes: One is the well-known class of uniformly convex spaces, the other is that of "flat spaces" (CL-spaces). The theoretical research of uniformly convex spaces has continued for over 70 years since Clarkson [3] introduced the notion of uniformly convex Banach spaces (see, for instance, [2, 4, 5, 6, 7, 9, 12, 13]). The study of various properties of CL-spaces has also brought mathematicians great attention (see, for instance, [1, 10, 15, 16, 17, 18, 22]).

The notion of a CL-space was first introduced by R. Fullerton [8] in 1960, and a generalized notion of a CL-space (i.e., almost CL-space) was introduced by A. Lima in 1978 [14]. A Banach space is called a (an almost) CL-space provided its closed unit ball is the (closed) absolutely convex hull of each maximal convex set of the unit sphere.

In this note, the letter X will always be a real Banach space and X^* its dual. We denote by B_X and S_X, the closed unit ball of X and the unit sphere of X, respectively. For a convex set $K \subset X$, C_K stands for the cone generated by K, that is, $C_K = \bigcup_{\lambda>0} \lambda K$. n-s$(K)$ and int(K) will represent the set of all non-support points of K and the interior of K, respectively. The aim of this note is to study the differentiability property of the norms of almost CL-spaces. As a result, it mainly shows the following theorems.

Theorem 1.1. *Suppose that X is a real almost CL-space, and that $\Im$ is the set of all maximal convex sets of S_X. Then*

Received by the editors December 18, 2006.
2000 *Mathematics Subject Classification.* Primary 46B20, 46G05.
Key words and phrases. Extreme point, exposed point, differentiability point, CL-space and Banach space.

i) The set of all Gateaux differentiability points of the norm is precisely $\bigcup_{M\in\Im} n\text{-}s(C_M)$*;*

ii) The set of all Fréchet differentiability points of the norm is precisely $\bigcup_{M\in\Im} \mathrm{int}(C_M)$*.*

Corollary. *The set of all Fréchet differentiability points of a real almost CL-space is open.*

Theorem 1.2. *Suppose that X is a real almost CL-space. Then every extreme point of B_X is an extreme point of $B_{X^{**}}$.*

Theorem 1.3. *Suppose that X is a separable almost CL-space. Then*

i) $x^* \in S_{X^*}$ is a w^*-exposed point of B_X if and only if $M \equiv \{x \in B_X : \langle x^*, x\rangle = 1\}$ is a maximal convex set of S_X. If, in addition, X is a CL-space and B_X is the closed convex hull of its extreme points, then

ii) every extreme point of B_{X^*} is a w^*-exposed point of B_{X^*}.

2. Notions and preliminaries

To begin this section, we recall a sequence of definitions which will be used in what follows.

Definition 2.1 ([20])**.** Suppose that X is a Banach space and C is a non-empty convex set of X.

i) A point $x \in X$ is said to be a support point of C if there exists $x^* \in X^*$ with $x^* \neq 0$ such that

$$\langle x^*, x\rangle = \sup_{y\in C}\langle x^*, y\rangle \equiv \sup_C x^*.$$

In this case, x^* is called a support functional of C. We denote by n-s(C) the set of all non-support points of C.

ii) $x \in C$ is called an extreme point of C if $C\backslash\{x\}$ is again a convex set.

Definition 2.2 ([20])**.** Suppose that X is a Banach space and C is a non-empty convex set of X.

i) $x_0 \in C$ is said to be an exposed point of C provided that there exists $x^* \in X^*\backslash\{0\}$ such that $\langle x^*, x_0\rangle > \langle x^*, y\rangle$ for all $y \neq x_0$ in C. In this case, the functional x^* is called an exposing functional of C and exposing C at x_0.

ii) We say that $x_0 \in C$ is a strongly exposed point of C if there is an $x^* \in X^*\backslash\{0\}$ such that $\langle x^*, x_n\rangle \to \sup_C x^*$ implies $x_n \to x_0$ whenever $\{x_n\}$ is a sequence in C. In this case, x^* is said to be a strongly exposing functional of C and strongly exposing C at x_0.

iii) In particular, if $C \subset X^*$, we can analogously define a w^*-exposed point and a w^*-strongly exposed point of C, respectively, with the functional x^* coming from X rather than X^{**} in i) and ii), respectively.

Definition 2.3 ([20])**.** Suppose that f is a continuous convex function on a non-empty convex open set D of a Banach space X.

i) The subdifferential mapping $\partial f : D \to 2^{X^*}$ is defined by $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle$, for all $y \in D\}$.

ii) We say that f is Gateaux differentiable at x if ∂f is single-valued at x, i.e., $\partial f \equiv x^*$ is a singleton. In this case, we call the functional x^* the Gateaux derivative of f at x, and we denote it by $d_G f(x) = x^*$.

iii) We say that f is Fréchet differentiable at x if ∂f is single-valued and norm-to-norm upper semi-continuous at x. In this case, we call $\partial f \equiv x^*$ the Fréchet derivative of f at x and denote it by $d_F f(x) = x^*$.

The following properties are classical (see, for instance, [20]).

Proposition 2.4. *Suppose that f is a continuous convex function defined on a non-empty open subset of a Banach space X. Then the following statements are equivalent.*

i) f is Fréchet differentiable at $x \in D$.

ii) There exists a selection φ for ∂f which is norm-to-norm continuous at x.

iii) Every selection of ∂f is norm-to-norm continuous at x.

Proposition 2.5. *Suppose that X is a Banach space and $x \in X$. Then*

i) $\partial \| x \| = \{x^* \in B_{X^*} : \langle x^*, x\rangle = \| x \|\}$.

ii) The norm $\| \,.\, \|$ is Gateaux differentiable at x with $d_G f(x) = x^*$ if and only if x^* is a w^*-exposed point of B_{X^*} and is exposed by x.

iii) The norm $\| \,.\, \|$ is Fréchet differentiable at x with $d_F f(x) = x^*$ if and only if x^* is a w^*-strongly exposed point of B_{X^*} and is strongly exposed by x.

Proposition 2.6 ([24])**.** *Suppose that C is a closed convex set with n-s(C) $\neq \emptyset$ of a Bananch space X. Then for every continuous convex function f defined on an open convex set D with $D \supset C$,*

i) $\partial f(x) = \partial(f_C)(x)$ for every $x \in$n-s(C);

ii) f_C is Gateaux differentiable at $x \in$n-s(C) if and only if there exists a selection φ of ∂f_C which is norm-to-$*$ continuous at x, where f_C denotes the function f restricted to C.

Proposition 2.7 ([11])**.** *Suppose that X is a separable space, and D is a non-empty closed convex set of X. Then n-s(D) $\neq \emptyset$ if and only if D is not contained in a closed hyperplane.*

Proposition 2.8 ([21])**.** *Let X be a Banach space and let $x_0 \in B_X$ be such that $|x^*(x_0)| = 1$ for all $x^* \in \text{ext}B_{X^*}$. The $|\tau(x_0)| = 1$ for all $\tau \in \text{ext}B_{X^{**}}$.*

3. Proof of the theorems

Now, we are ready to prove the theorems presented in section 1, and we also restate and renumber them as follows.

Theorem 3.1. *Suppose that X is a real almost CL-space, and that $\Im$ is the set of all maximal convex sets of S_X. Then*

i) The set of all Gateaux differentiability points of the norm is precisely $\bigcup_{M\in\Im}$n-s(C_M).

ii)The set of all Fréchet differentiability points of the norm is precisely $\bigcup_{M\in\Im} \text{int}(C_M)$.

Proof. Note that every point $x \in X$ with $x \neq 0$ is lying in a cone C_M generated by some maximal convex set M of S_X. Then, it suffices to characterize the Gateaux (Fréchet) differentiability points contained in C_M for every maximal convex set M of S_X. So, let us fix a maximal convex set M of S_X. We observe that by the Hahn-Banach and Krein-Milman theorems, there is an extreme point $x_0^* \in B_{X^*}$ such that $M = \{x \in B_X : \langle x_0^*, x\rangle = 1\}$, and thus, $\langle x_0^*, x\rangle = \|x\|$ for every $x \in C_M$.

This implies that $\varphi : C_M \longrightarrow X^*$ defined by $\varphi(x) = x_0^*$ for every $x \in C_M$, is a selection for $\partial\|\cdot\|$ restricted to C_M, which is clearly norm-to-norm continuous.

i) Suppose that $x_0 \in$ n-s(C_M). Then, it follows from Proposition 2.6 that x_0 is a Gateaux differentiability point of the norm.

Conversely, suppose that $x_0 \in C_M$ is a Gateaux differentiability point. We can assume $\| x_0 \| = 1$. Let $x_0^* = d_G \| x_0 \|$. Then it is a w^*-exposed point of B_{X^*} and exposed by x_0. Now we assert that $M = \{x \in B_X : \langle x_0^*, x\rangle = 1\}$. Otherwise, there is an extreme point e^* of B_{X^*} with $e^* \neq x_0^*$ such that $\langle e^*, x\rangle = 1$ for all $x \in M$. Therefore $e^* \in \partial\| x_0 \| = x_0^*$. If $x_0 \notin$ n-s(C_M), then there exists $x^* \in S_{X^*}$ such that $\langle x^*, x_0\rangle = \sup_{C_M} x^*$. Note that C_M is a cone and note both 0 and $2x_0$ are in C_M. We obtain that $\langle x^*, x_0\rangle = 0$ and $\langle x^*, x\rangle \le 0$ for all $x \in C_M$. Thus $\| x_0^* + \lambda x^* \| > 1$ for all $\lambda > 0$. (Otherwise,

$$1 \ge \| x_0^* + \lambda x^* \| \ge \langle x_0^* + \lambda x^*, x_0\rangle = 1,$$

which in turn tells us $x_0^* + \lambda x^* \in \partial\| x_0 \| = x_0^*$.) Now, fix $0 < \lambda < 1$. The density of $co(M \cup -M)$ in B_X further says that there exists $z \in M \cup -M$ such that $\langle x_0^* + \lambda x^*, z\rangle > 1$. But this is impossible since

$$\langle x_0^* + \lambda x^*, z\rangle \le \langle x_0, z\rangle \le 1 \quad \text{if} \quad z \in M$$

and

$$\langle x_0^* + \lambda x^*, z\rangle \le -1 + \lambda < 0 \quad \text{if} \quad z \in -M.$$

ii) By Proposition 2.5, it is easy to show that all points in int(C_M) are Fréchet differentiability points of the norm since φ is norm-to-norm continuous.

Conversely, if $x_0 \notin C_M \backslash \text{int}(C_M)$, we want to show x_0 is not a Fréchet differentiability point of the norm. We can again assume $\| x_0 \| = 1$. For every $r > 0$, we can find $x_r \in B(x_0, r) \cap S_X$ with $x_r \notin M$. Let $x_r \in M_r$ for some maximal convex set M_r of S_X and x_r^* be an extreme point of B_{X^*} such that $M_r = \{x \in S_X : \langle x_r^*, x\rangle = 1\}$. Then we have $\| x_0^* - x_r^* \| = 2$ for all $r > 0$. Let φ be a selection of $\partial\| \, . \, \|$ such that $\varphi(x_r) = x_r^*$ for all $r > 0$. It is clear that φ is not norm-to-norm continuous at x_0, and hence x_0 is not a Fréchet differentiability point. □

Corollary. *The set of all Fréchet differentiability points of a real almost CL-space is open.*

Theorem 3.2. *Suppose that X is a real CL-space. Then every extreme point of B_X is an extreme point of $B_{X^{**}}$.*

Proof. Let x_0 be an extreme point of B_X. Let E be the set of those extreme points $x^* \in B_{X^*}$ satisfying that the convex set $\{x \in B_X : \langle x^*, x\rangle = 1\}$ is maximal in S_X. By the definition of a CL-space, it is clear that $x_0 \in M \cup -M$ for every maximal convex subset M of S_X and, therefore, $|\langle x^*, x_0\rangle| = 1$ for every $x^* \in E$. On the other hand, B_{X^*} is the w^*-closed convex hull of E, and the reversed Krein-Milman theorem gives us that the set of all extreme points of B_{X^*} is contained in the w^*-closed hull of E. Therefore, one has $|\langle x^*, x_0\rangle| = 1$ for every extreme point x^* of B_{X^*}. Now, Proposition 2.8 gives us that $|\langle x^{***}, x_0\rangle| = 1$ for every extreme point x^{***} of $B_{X^{***}}$. It clearly follows that x_0 is an extreme point of $B_{X^{**}}$. □

Theorem 3.3. *Suppose that X is a separable almost CL-space. Then*

i) $x^* \in X^*$ is a w^*-exposed point of B_{X^*} if and only if $M \equiv \{x \in B_X : \langle x^*, x\rangle = 1\}$ is a maximal convex set of S_X. If, in addition, X is a CL-space and B_X is the closed convex hull of its extreme points, then

ii) every extreme point of B_{X^*} is a w^*-exposed point.

Proof. i) Suppose that M is a maximal convex set of S_X. Since $co(M \cup -M)$ is dense in B_X, the closed convex set $D \equiv \overline{co}(M \cup \{0\}) = co(M \cup \{0\})$ cannot be contained in a closed hyperplane. Since X is separable, n-s$(D) \neq \emptyset$. Theorem 3.1 explains that $x^* \in B_{X^*}$ satisfying $M = \{x \in B_X : \langle x^*, x\rangle = 1\}$ is a w^*-exposed point of B_{X^*}.

Conversely, suppose that x_0^* is a w^*-exposed point of B_{X^*} and exposed by $x_0 \in S_X$. Then by Proposition 2.6, $d_G\| x_0 \| = x_0^*$. Let $M = \{x \in B_X : \langle x^*, x\rangle = 1\}$. We extend M to be a maximal convex set $\widetilde{M}$ of S_X and let $x^* \in B_{X^*}$ such that $\widetilde{M} = \{x \in B_X : \langle x^*, x\rangle = 1\}$. Then $x^* \in \partial\| x_0 \| = \{d_G\| x_0 \|\}$, i.e., $x^* = x_0^*$ and $\widetilde{M} = M$.

ii) By Theorem 3.1 i), it suffices to show that for every extreme point x^* of B_{X^*}, $C \equiv \bigcup_{\lambda>0} \lambda M$ has at least one non-support point , where $M = \{x \in B_X : \langle x^*, x\rangle = 1\}$. By Theorem 3.2, every extreme point of B_X is an extreme point of $B_{X^{**}}$. Thus, $|\ \langle x^*, x\rangle\ | = 1$ for every extreme point of B_X. Let $E^{\pm} = \{x$ is an extreme of point B_X: $\langle x^*, x\rangle = \pm 1\}$. Then $co(E^+ \cup E^-)$ is dense in B_X. Therefore, $co(M \cup -M)$ is dense in B_X. Thus the closed cone C_M cannot be contained in a closed hyperplane. Separability of X implies n-s$(C_M) \neq \emptyset$. □

4. Final remarks

Remark 4.1. Without a separability assumption on X, Theorem 3.3 no longer holds. For example, for any uncountable set Γ, $\ell^1(\Gamma)$ is a CL-space and with the *RNP*(hence the *KMP*), but for every maximal convex set M of $S_{\ell^1(\Gamma)}$, n-s$(C_M) = \emptyset$, since there are no w^*-exposed points of $B_{\ell^\infty(\Gamma)}$.

Remark 4.2. Assume X is a (an almost) CL-space. Applying the results of this note, maximal convex sets of S_X and differentiability points of X can be easily obtained. Now we give some examples as follows.

Example 4.2.1. It is well known that $\{\pm\delta_t : t \in [a, b]\}$ is just the set of all w^*-exposed points of X^* for $X = C[a, b]$. Thus every maximal convex set M of S_X has the following form:

$$M \equiv M_t = \{x \in B_X : x(t) = 1\}$$

or

$$M \equiv M_t = \{x \in B_X : x(t) = -1\}$$

for some $t \in [a, b]$. Therefore

$$\text{n-s}(C_{M_t}) = \bigcup_{\lambda>0} \lambda\{x \in S_X : 1 = x(t) > x(s), \text{for all } s \neq t\}$$

or

$$\text{n-s}(C_{M_t}) = \bigcup_{\lambda>0} \lambda\{x \in S_X : 1 = x(t) < x(s), \text{for all } s \neq t\}.$$

Since $\text{int}(C_{M_t}) = \emptyset$, $C_{[a,b]}$ does not admit a Fréchet differentiability point.

Example 4.2.2. Assume $X = \ell^1$. Then all w^*-exposed points of B_{X^*} are just $\{(\sigma_i)_{i=1}^{\infty} : \sigma_i = \pm 1\}$. Thus, for every maximal convex set M of S_X,

$$\text{n-s}(C_M) = \bigcup_{\lambda>0} \lambda\{x \in M : x(i) \neq 0, \text{for all } i \in N\}$$

Example 4.2.3. Assume $X = c_0$. Then $\{\pm e_n\}_{n=1}^{\infty}$ is the set of all w^*-exposed points of B_{X^*}. Note c_0 is an Asplund space. There exists a w^*-strongly exposed point of B_{X^*} (of course, contained in $\{\pm e_n\}$), which implies that $\{\pm e_n\}$ are w^*-strongly exposed points of B_{X^*}. Therefore, every Gateaux differentiability point is a Fréchet differentiability point in c_0, and they form a dense open set.

Remark 4.3. Some questions are have arisen naturally. Theorem 3.2 explains that every extreme point of B_X is an extreme point of $B_{X^{**}}$ if X is a CL-space.

Problem 4.4.1. Is this true for every almost CL-space?

Theorem 3.3 tells us that if X is a separable CL-space and B_X is the closed convex hull of its extreme points, then every extreme point of B_{X^*} is a w^*-exposed point.

Problem 4.4.2. Whether Theorem 3.3 still holds without the assumption that B_X is the closed convex hull of its extreme points.

Problem 4.4.3. Whether every w^*-exposed point of B_X is a w^*-strongly exposed point in an Asplund CL-space.

Problem 4.4.4. Assume X is an almost CL-space or a CL-space. If X is an Asplund space, whether every extreme point of B_{X^*} is an exposed point, a w^*-exposed point, a strongly exposed point or a w^*-strongly exposed point.

Problem 4.4.5. Assume X is a (an almost) CL-space.Whether every extreme point of B_X is an exposed point or a strongly exposed point if X has the RNP.

Acknowledgments

The authors would like to express their thanks to the referee for his (or her) helpful corrections and suggestions.

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