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SMOOTH APPROXIMATION OF DEFINABLE CONTINUOUS FUNCTIONS

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ABSTRACT. Let $\mathcal{M}$ be an *o*-minimal expansion of the real exponential field which possesses smooth cell decomposition. We prove that for every definable open set, the definable indefinitely continuously differentiable functions are a dense subset of the definable continuous function with respect to the *o*-minimal Whitney topology.

1. INTRODUCTION

In [3] and [6] it was shown that semialgebraic continuous functions can be approximated by Nash functions with respect to the semialgebraic Whitney topology. Similar approximations were studied for *o*-minimal structures.

We assume the reader to be familiar with the basic concepts of *o*-minimal structures, as they are presented in [1] or [2]. In the sequel, "definable" always means "definable with parameters in $\mathcal{M}$" for a given *o*-minimal structure $\mathcal{M}$. Let $\mathbb{R}^n$ be endowed with the Euclidean topology. For a definable open set U and $p \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{C}^p(U, \mathbb{R}^k)$ the definable p times continuously differentiable functions from U to $\mathbb{R}^k$, and by $\mathcal{C}(U, \mathbb{R})$ the definable continuous functions from U to $\mathbb{R}$.

The *o*-minimal Whitney topology on $\mathcal{C}(U, \mathbb{R})$ is the coarsest topology for which the sets

$$(1.1) \quad \{g \in \mathcal{C}(U, \mathbb{R}) : \ |f(u) - g(u)| < \varepsilon(u), \ u \in U\}, \ f \in \mathcal{C}(U, \mathbb{R}), \ \varepsilon \in \mathcal{C}(U, (0, \infty)),$$

form a basis. In all *o*-minimal expansions of the reals, $\mathcal{C}^p(U, \mathbb{R}) \subset \mathcal{C}(U, \mathbb{R})$ is dense with respect to the *o*-minimal Whitney topology if $p < \infty$; cf. [4].

Here we consider *o*-minimal expansions $\mathcal{M}$ of the real exponential field. In [5, Chap. 8] it was shown that for every definable open set $U \subset \mathbb{R}^n$, $\mathcal{C}^\infty(U, \mathbb{R}) \subset \mathcal{C}(U, \mathbb{R})$ is dense with respect to the *o*-minimal Whitney topology if (a) $\mathcal{M}$ is locally polynomially bounded, and (b) $\mathcal{M}$ possesses $\mathcal{C}^\infty$ cell decomposition.

By proving the following theorem, we show that assumption (a) can be omitted.

Theorem 1.1. *Let $\mathcal{M}$ be an o-minimal expansion of the real exponential field which possesses $\mathcal{C}^\infty$ cell decomposition. Let $U \subset \mathbb{R}^n$ be open and let $f \in \mathcal{C}(U, \mathbb{R})$. Then, for every $\varepsilon \in \mathcal{C}(U, (0, \infty))$, there is a $g \in \mathcal{C}^\infty(U, \mathbb{R})$ with $|g(u) - f(u)| < \varepsilon(u)$, $u \in U$.*

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2. Preliminaries

A definable function $f : A \to \mathbb{R}^k$ is called a $\mathcal{C}^\infty$ function, if there are an open definable neighborhood U of A and an $F \in \mathcal{C}^\infty(U, \mathbb{R}^k)$ such that $F|_A = f$.

Definition 2.1. A $\mathcal{C}^\infty$ cell in $\mathbb{R}$ is either a single point or an open interval. Supposing all cells of $\mathbb{R}^n$ are known, then a $\mathcal{C}^\infty$ cell in $\mathbb{R}^{n+1}$ is a set of either the form $(h)_X := \{(x,r) : x \in X, r = h(x)\}$ where $X \subset \mathbb{R}^n$ is a $\mathcal{C}^\infty$ cell and $h \in \mathcal{C}^\infty(X, \mathbb{R})$, or of the form $(f,g)_X := \{(x,r) : x \in X, f(x) < r < g(x)\}$ where $X \subset \mathbb{R}^n$ is a $\mathcal{C}^\infty$ cell and $f, g \in \mathcal{C}^\infty(X,\mathbb{R}) \cup \{\pm\infty\}$ such that $f(x) < g(x)$, $x \in X$.

The functions used to describe a $\mathcal{C}^\infty$ cell Z we call the *defining functions* of Z.

Note that a $\mathcal{C}^\infty$ cell decomposition of $\mathbb{R}$ is a finite partition of $\mathbb{R}$ into $\mathcal{C}^\infty$ cells. A finite partition of $\mathbb{R}^{n+1}$ into $\mathcal{C}^\infty$ cells $Z_1, ..., Z_r$ is called a $\mathcal{C}^\infty$ cell decomposition, if the set of projections $\pi(Z_i)$, $i = 1, ..., r$, is a $\mathcal{C}^\infty$ cell decomposition of $\mathbb{R}^n$, where π is the projection onto the first n coordinates.

From now on we assume that $\mathcal{M}$ possesses $\mathcal{C}^\infty$ cell decomposition. That is, *for any finite collection of definable sets $A_1, ..., A_k \subset \mathbb{R}^n$ there exists a $\mathcal{C}^\infty$ cell decomposition of $\mathbb{R}^n$ partitioning each A_i, $i = 1, ..., k$.*

So far, all known *o*-minimal structures have this property. For a set Z, $\mathrm{cl}(Z)$ denotes its topological closure, and $\partial Z := \mathrm{cl}(Z) \setminus Z$ its frontier.

Lemma 2.2. *Let $Z \subset \mathbb{R}^n$ be a bounded open $\mathcal{C}^\infty$ cell and let $f \in \mathcal{C}(\mathrm{cl}(Z), [0, \infty))$ satisfy $f > 0$ on Z. Then there is a $g \in \mathcal{C}^\infty(Z, \mathbb{R})$ such that*

$$0 < g(z) < f(z), \ z \in Z. \tag{2.1}$$

Proof. Let f_i, g_i, $i = 1, ..., n$, be the defining functions of Z. Then $\rho : Z \to \mathbb{R}$,

$$\rho(x_1, ..., x_n) := \prod_{i=1}^{n} (x_i - f_i(x_1, ..., x_{i-1}))(g_i(x_1, ..., x_{i-1}) - x_i), \ (x_1, ..., x_n) \in Z, \tag{2.2}$$

is a $\mathcal{C}^\infty$ function, which is positive on Z and extends to a definable continuous function $\overline{\rho} : \mathrm{cl}(Z) \to \mathbb{R}$ with $\overline{\rho}(z) = 0$, $z \in \partial Z$.

We apply a generalized Łojasiewicz inequality (cf. [2, C.14]) to $\overline{\rho}$, f and $\mathrm{cl}(Z)$, and obtain a definable continuous strictly monotone function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that $0 < \phi(\overline{\rho}(z)) \le f(z)$, $z \in Z$.

By $\mathcal{C}^\infty$ cell decomposition, ϕ restricted to $(0, \delta)$ is a $\mathcal{C}^\infty$ function for some $\delta \in (0, 1)$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined by $\psi(t) := \phi(\delta t^2/(1+t^2))$. Then $g := \psi \circ \rho : Z \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function and satisfies inequality (2.1), as $\delta < 1$. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. By applying the function $\tau : \mathbb{R}^n \to \mathbb{R}^n$ which is defined by $\tau(x_1, ..., x_n) = (x_1/\sqrt{1 + x_1^2}, ..., x_n/\sqrt{1 + x_n^2})$ we can reduce our consideration to bounded open sets.

By $\mathcal{C}^\infty$ cell decomposition there are finitely many disjoint $\mathcal{C}^\infty$ cells $Z_1, ..., Z_s$, which cover U such that $f|_{Z_i}$ is a $\mathcal{C}^\infty$ function, $i = 1, ..., s$. This partition can be refined in such a way that ∂Z_i is the union of some of the $\mathcal{C}^\infty$ cells; cf. [1, Chap. 4, Prop. 1.13]. Therefore, each $\mathcal{C}^\infty$ cell Z_i has a definable open neighborhood U_i, which is disjoint to all cells Z_j, for $j \ne i$ and $\dim(Z_j) \le \dim(Z_i)$. Let $Z_1, ..., Z_q$ be the cells of dimension less than n. We order these cells, such that $\dim(Z_{i+1}) \ge \dim(Z_i)$, $i = 1, ..., q-1$.

We prove the following statement, which implies the conclusion of Theorem 1.1, by induction on r.

For all $\tilde{\varepsilon} \in \mathcal{C}(U,(0,\infty))$ *and* $F \in \mathcal{C}(U,\mathbb{R})$, *which satisfy the conditions*

(a) $F|_{Z_i}$ *is* $\mathcal{C}^\infty$ *smooth,* $i = 1,...,r$, *and*

(b) F *is* $\mathcal{C}^\infty$ *smooth in* $U \setminus \bigcup_{i=1}^r Z_i$,

there is a $g \in \mathcal{C}^\infty(U,\mathbb{R})$ *such that* $|g(u) - F(u)| < \tilde{\varepsilon}(u)$, $u \in U$.

The case $r = 0$ is evident. We assume that the statement holds for $r \geq 0$.

Let $Z := Z_{r+1}$. After some permutation of the coordinates, Z is the graph of a $\mathcal{C}^\infty$ function $h = (h_{d+1},...,h_n) : X \to \mathbb{R}^{n-d}$ where $X \subset \mathbb{R}^d$ is some open $\mathcal{C}^\infty$ cell. There also exists an open definable neighborhood U' of Z and a function $e \in \mathcal{C}^\infty(U',\mathbb{R})$ with $e|_Z = F|_Z$. In addition, U' may be chosen so small that $U' \cap Z_i = \emptyset$, $i = 1,...,r$, and $U' \subset X \times \mathbb{R}^{n-d}$.

We take a definable open neighborhood V of Z, which is contained in U' such that $|e(u) - F(u)| < \tilde{\varepsilon}(u)/2$, $u \in V$. Let $\Delta : X \to \mathbb{R}$ be the definable continuous function, which maps $x \in X$ to

(3.1)
$$\Delta(x) := \min\left(\operatorname{dist}((x,h(x)),\bigcup_{i\leq r} Z_i),\operatorname{dist}((x,h(x)),\partial Z),\operatorname{dist}((x,h(x)),\partial V)\right).$$

By Lemma 2.2 there is a $\varphi \in \mathcal{C}^\infty(X,\mathbb{R})$ such that $0 < \varphi(x) < \Delta(x)/n$, $x \in X$. For $0 < s \leq 1$, let W_s be given by

$$W_s := \{(x,y) : \; x \in X, \; |y_i - h_i(x)| < s\varphi(x), \; i = d+1,...,n\} \subset V. \tag{3.2}$$

The W_s contain Z, and they are $\mathcal{C}^\infty$ cells with defining $\mathcal{C}^\infty$ functions f_i, g_i of X, $i = 1,...,d$, and $f_i = h_i - s\varphi$, $g_i = h_i + s\varphi$, $i = d+1,...,n$.

We let $\sigma : \mathbb{R} \to \mathbb{R}$ be the $\mathcal{C}^\infty$ function given by $\sigma(t) = \exp(-1/t)$ if $t > 0$, and $\sigma(t) = 0$ if $t \leq 0$. Let the functions $\psi_1, \psi_2 : X \times \mathbb{R}^{n-d} \to \mathbb{R}$ be defined by

$$\psi_1(x,y) := \prod_{i=d+1}^{n} \sigma(y_i - h_i(x) + \varphi(x))\sigma(h_i(x) + \varphi(x) - y_i), \text{ respectively,} \tag{3.3}$$
$$\psi_2(x,y) := \sum_{i=d+1}^{n} \left(\sigma\left(h_i(x) - \frac{1}{2}\varphi(x) - y_i\right) + \sigma\left(y_i - h_i(x) - \frac{1}{2}\varphi(x)\right)\right).$$

The functions ψ_1 and ψ_2 are definable $\mathcal{C}^\infty$ functions, and their sum is positive in $X \times \mathbb{R}^{n-d}$. Finally, we define the function $G : U \to \mathbb{R}$ by

$$G(u) := \begin{cases} \dfrac{\psi_1(u)e(u) + \psi_2(u)F(u)}{\psi_1(u) + \psi_2(u)}, & \text{if } u \in V, \\ F(u), & \text{otherwise.} \end{cases} \tag{3.4}$$

The function ψ_1 vanishes outside of W_1, and ψ_2 equals zero in $W_{1/2}$. Hence, $G = F$ outside W_1, and $G = e$ in $W_{1/2}$. Moreover, $G \in \mathcal{C}(U,\mathbb{R})$, and G satisfies

$$|G(u) - F(u)| < \frac{1}{2}\tilde{\varepsilon}(u), \; u \in U. \tag{3.5}$$

In addition, the set of points at which G is not $\mathcal{C}^\infty$ smooth is contained in $\bigcup_{i=1}^r Z_i$, and $G|_{Z_i}$ is a $\mathcal{C}^\infty$ function, $i = 1,...,r$, and G restricted to $U \setminus \bigcup_{i=1}^r Z_i$ is $\mathcal{C}^\infty$ smooth. We apply the induction hypothesis to $\tilde{\varepsilon}/2$ and G in place of $\tilde{\varepsilon}$ and F so that we obtain a $g \in \mathcal{C}^\infty(U,\mathbb{R})$ which satisfies $|g(u) - G(u)| < \tilde{\varepsilon}(u)/2$, $u \in U$. Therefore,

$$|g(u) - F(u)| < |g(u) - G(u)| + |G(u) - F(u)| < \tilde{\varepsilon}(u), \; u \in U. \tag{3.6}$$ □

Remark 3.1. In the situation described in Theorem 1.1, let $S \subset U$ be a definable set that is closed in U and contains the set of non-smooth points of f. If W is any definable open neighborhood of S, we may assume that $g = f$ outside of W. This is due to the fact that in the proof of Theorem 1.1 we can select a $\mathcal{C}^\infty$ cell decomposition partitioning S, and for the induction we only have to consider the cells contained in S. The corresponding neighborhoods of the cells may be chosen as subsets of W.

4. Consequences

Theorem 1.1 implies a more general statement.

Corollary 4.1. *Let $A \subset \mathbb{R}^n$ be a locally closed definable set, and let $f \in \mathcal{C}(A, \mathbb{R})$. Then for every $\varepsilon \in \mathcal{C}(A, (0, \infty))$ there is a definable open neighborhood V of A and a $g \in \mathcal{C}^\infty(V, \mathbb{R})$ such that $|g(a) - f(a)| < \varepsilon(a)$, $a \in A$.*

Proof. Since A is locally closed, there are definable sets B and V, where the former is closed and the latter is open, such that $A = B \cap V$. Take an $F \in \mathcal{C}(V, \mathbb{R})$ with $F|_A = f$ and $\tilde{\varepsilon} \in \mathcal{C}(V, (0, \infty))$ with $\tilde{\varepsilon}|_A = \varepsilon$; cf. [1, p. 138, Cor. 3.10]. By Theorem 1.1 there is a $g \in \mathcal{C}^\infty(V, \mathbb{R})$ such that $|g(v) - F(v)| < \tilde{\varepsilon}(v)$, $v \in V$. Hence, g has the desired properties. $\square$

The next consequence is $\mathcal{C}^\infty$ separation of sets.

Corollary 4.2. *Let $A_0, ..., A_k \subset \mathbb{R}^n$ be disjoint closed definable sets. Then there is a $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that $A_i \subset \phi^{-1}(\{i\})$, $i = 0, ..., k$.*

Proof. Let U_i be a definable open neighborhood of A_i, $i = 0, ..., k$, such that $\mathrm{cl}(U_i) \cap \mathrm{cl}(U_j) = \emptyset$, $i \neq j$. Select for each $i = 1, ..., k$ a definable continuous function $g_i : \mathbb{R}^n \to [0, 1]$ that vanishes in $\mathrm{cl}(U_j)$, $j \neq i$, and that equals 1 in $\mathrm{cl}(U_i)$; cf. [1, p. 102, Lem. 3.8]. Then $g : \mathbb{R}^n \to \mathbb{R}$,

$$g(u) = \sum_{i=1}^{k} i g_i(u), \; u \in U, \tag{4.1}$$

is definable and continuous such that $\mathrm{cl}(U_i) \subset \{x : \; g(x) = i\}$, $i = 0, ..., k$. Then $\mathbb{R}^n \setminus (\bigcup_{i=0}^k A_i)$ is a definable open neighborhood of the points at which g is not $\mathcal{C}^\infty$ smooth. By Theorem 1.1 in connection with Remark 3.1 there is an $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ with $f = g$ on $\bigcup_{i=0}^k A_i$. $\square$

Finally, we note the following corollary.

Corollary 4.3. *Let $A \subset \mathbb{R}^n$ be a closed definable set, let $U \supset A$ be definable and open, and let $f \in \mathcal{C}^\infty(U, \mathbb{R})$. Then there is an $F \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that $F|_A = f|_A$.*

Proof. Let V be an open definable neighborhood of A such that $\mathrm{cl}(V) \subset U$. Set $A_0 := \mathbb{R}^n \setminus V$ and $A_1 := A$. By Corollary 4.2 there is a $g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that $A_i \subset \{x : \; g(x) = i\}$. Set $F(x) := f(x)g(x)$ if $x \in V$, and $F(x) := 0$ otherwise. $\square$

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