

## THE SUB-SUPERSOLUTION METHOD FOR WEAK SOLUTIONS

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ABSTRACT. We extend the method of sub and supersolutions in order to prove existence of  $L^1$ -solutions of the equation  $-\Delta u = f(x, u)$  in  $\Omega$ , where  $f$  is a Carathéodory function. The proof is based on Schauder's fixed point theorem.

### 1. INTRODUCTION

We consider the following semilinear problem:

$$(1.1) \quad \begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  is a smooth bounded domain and  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function.

The classical method of sub and supersolutions (see, e.g., [4, 9]) asserts that if  $f$  is smooth and if one can find smooth sub and supersolutions  $v_1 \leq v_2$  of (1.1), then there exists a classical solution  $u$  of (1.1) such that  $v_1 \leq u \leq v_2$ .

The usual proof is based on a monotone iteration scheme; this requires  $f$  to be Lipschitz (or locally Lipschitz). The argument also shows that there exist a smallest and a largest solution  $u_1 \leq u_2$  in the interval  $[v_1, v_2]$ . Another proof, based on Schauder's fixed point theorem, can be found in Akô [1]. In this case, the existence of a smallest and a largest solution is proved separately, via a Perron-type approach. Based on Akô's strategy, Clément and Sweers [6] were able to implement the method of sub-supersolutions when  $v_1, v_2 \in C(\overline{\Omega})$  and  $f$  is only assumed to be continuous. Other versions can also be found for instance in Deuel-Hess [8] (see also Dancer-Sweers [7]) for  $H^1$ -solutions and in Brezis-Marcus-Ponce [3, Theorem 4] for  $L^1$ -solutions and  $f$  continuous, nondecreasing. However, none of these results is contained in the other. We shall compare them in Section 5 below.

In this paper, we extend the method of sub-supersolutions in order to establish existence of  $L^1$ -solutions of (1.1) in the sense of Definition 1.1 below. We follow the strategy of [1, 6, 7] based on Schauder's fixed point theorem. However, some of the details had to be substantially modified.

We assume throughout that

$$(1.2) \quad \Omega \text{ is a smooth bounded domain}$$

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and

$$(1.3) \quad f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \text{ is a Carathéodory function.}$$

The notion of solution we will consider is the following:

**Definition 1.1.** We say that  $u$  is an  $L^1$ -solution of (1.1) if

- (i)  $u \in L^1(\Omega)$ ;
- (ii)  $f(\cdot, u)\rho_0 \in L^1(\Omega)$ ;
- (iii)

$$(1.4) \quad - \int_{\Omega} u \Delta \zeta = \int_{\Omega} f(x, u) \zeta \, dx \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

Here,  $\rho_0(x) = d(x, \partial\Omega)$ ,  $\forall x \in \Omega$ , and  $C_0^2(\overline{\Omega}) = \{\zeta \in C^2(\overline{\Omega}); \zeta = 0 \text{ on } \partial\Omega\}$ . Note that (1.4) makes sense in view of (i) and (ii).

We also consider  $L^1$ -sub and  $L^1$ -supersolutions in analogy with this definition. For instance,  $u$  is an  $L^1$ -subsolution of (1.1) if  $u$  satisfies (i)–(iii) with “ $\leq$ ” instead of “ $=$ ” in (1.4). We will systematically omit the term “ $L^1$ ” and simply say that  $u$  is a solution of (1.1), meaning (1.4); a similar convention for sub and supersolutions.

Our main result is

**Theorem 1.1.** *Let  $v_1$  and  $v_2$  be a sub and a supersolution of (1.1), respectively. Assume that  $v_1 \leq v_2$  a.e. and*

$$(1.5) \quad f(\cdot, v)\rho_0 \in L^1(\Omega) \quad \text{for every } v \in L^1(\Omega) \text{ such that } v_1 \leq v \leq v_2 \text{ a.e.}$$

*Then, there exist solutions  $u_1 \leq u_2$  of (1.1) in  $[v_1, v_2]$  such that any solution  $u$  of (1.1) in the interval  $[v_1, v_2]$  satisfies*

$$v_1 \leq u_1 \leq u \leq u_2 \leq v_2 \quad \text{a.e.}$$

In general, (1.1) need not have a solution if (1.5) fails (see [13]). However, Orsina-Ponce [13] were recently able to prove existence of solutions of (1.1) (and (5.2) below) for *some* nonlinearities  $f$  which need not satisfy (1.5).

The paper is organized as follows. In Section 2, we develop some tools used in the proof of Theorem 1.1. In Section 3, we recall some existence, compactness and comparison results related to the linear equation  $-\Delta w = h$  when  $h\rho_0 \in L^1(\Omega)$ . We then establish Theorem 1.1 in Section 4. In the last section, we recover some known results; we also apply Theorem 1.1 in order to study semilinear problems involving measures.

ADDED NOTE: After this paper was completed, the authors were informed of other related works of M. Marcus [12] (assuming  $f$  monotone) and M. C. Palmeri [14] (with (1.5) replaced by a stronger assumption on  $f$  and without proving the existence of a smallest and a largest solution).

## 2. BOUNDEDNESS AND EQUI-INTEGRABILITY IN $L^1_{\rho_0}$

We begin with the following well-known result. We shall present a proof below for the convenience of the reader.

**Theorem 2.1.** *Let  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that*

$$(2.1) \quad g(\cdot, v)\rho_0 \in L^1(\Omega) \quad \text{for every } v \in L^1(\Omega).$$

Then, the Nemytskii operator

$$(2.2) \quad \begin{aligned} G : L^1(\Omega) &\longrightarrow L^1(\Omega; \rho_0 dx), \\ v &\longmapsto g(\cdot, v) \end{aligned}$$

is continuous.

We first prove

**Lemma 2.1.** *Let  $(w_n) \subset L^1(\Omega)$  and let  $(E_n)$  be a sequence of measurable subsets of  $\Omega$  such that*

$$(2.3) \quad |E_n| \rightarrow 0 \quad \text{and} \quad \int_{E_n} |w_n| \geq 1 \quad \forall n \geq 1.$$

*Then, there exist a subsequence  $(w_{n_k})$  and a sequence of disjoint measurable sets  $(F_k)$  such that*

$$(2.4) \quad F_k \subset E_{n_k} \quad \text{and} \quad \int_{F_k} |w_{n_k}| \geq \frac{1}{2} \quad \forall k \geq 1.$$

*Proof.* Let  $n_1 := 1$  and  $A_1 := E_1$ . By induction, we construct an increasing sequence of integers  $(n_k)$  and measurable sets  $(A_k)$  as follows.

Let  $k \geq 2$ . Assume we are given integers  $n_1 < \dots < n_{k-1}$  and sets  $A_1, \dots, A_{k-1}$  (not necessarily disjoint) such that  $A_j \subset E_j$  and

$$\int_{A_{j+1} \cup \dots \cup A_{k-1}} |w_{n_j}| \leq \frac{1}{2} - \frac{1}{2^{k-1}} \quad \forall j = 1, \dots, k-2.$$

(This condition is vacuous when  $k = 2$ .)

Since  $|E_n| \rightarrow 0$ , then for  $n_k > n_{k-1}$  sufficiently large we have

$$\int_{E_{n_k}} |w_{n_j}| \leq \frac{1}{2^k} \quad \forall j = 1, \dots, k-1.$$

Let  $A_k := E_{n_k}$ . Then,

$$\int_{A_{j+1} \cup \dots \cup A_k} |w_{n_j}| \leq \frac{1}{2} - \frac{1}{2^k} \quad \forall j = 1, \dots, k-1.$$

Proceeding with this construction, one gets sequences  $(n_k)$  and  $(A_k)$ . We now set

$$F_k := A_k \setminus \bigcup_{i=k+1}^{\infty} A_i.$$

Then, the sets  $F_k$  are disjoint and

$$\int_{F_k} |w_{n_k}| = \lim_{i \rightarrow \infty} \int_{A_k \setminus (A_{k+1} \cup \dots \cup A_i)} |w_{n_k}| \geq \frac{1}{2}.$$

The proof of the lemma is complete. □

As a consequence of Lemma 2.1 we have

**Proposition 2.1.** *Let  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function and  $v_1, v_2 \in L^1(\Omega)$  be such that  $v_1 \leq v_2$  a.e. Assume that*

$$(2.5) \quad g(\cdot, v)\rho_0 \in L^1(\Omega) \quad \text{for every } v \in L^1(\Omega) \text{ such that } v_1 \leq v \leq v_2 \text{ a.e.}$$

Then, the set

$$(2.6) \quad \mathcal{B} = \left\{ g(\cdot, v) \in L^1(\Omega; \rho_0 dx); v \in L^1(\Omega) \text{ and } v_1 \leq v \leq v_2 \text{ a.e.} \right\}$$

is bounded and equi-integrable in  $L^1(\Omega; \rho_0 dx)$ .

We recall that a set  $\mathcal{B} \subset L^1(\Omega; \rho_0 dx)$  is equi-integrable if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$E \subset \Omega \quad \text{and} \quad |E| < \delta \quad \implies \quad \int_E |g|\rho_0 < \varepsilon \quad \forall g \in \mathcal{B}.$$

Here,  $|E|$  denotes the Lebesgue measure of  $E$ .

*Proof of Proposition 2.1.* Since  $\Omega$  is bounded, it suffices to show that  $\mathcal{B}$  is equi-integrable. Assume by contradiction that  $\mathcal{B}$  is not equi-integrable. Then, there exist  $\varepsilon > 0$ ,  $(u_n) \subset L^1(\Omega)$  with  $v_1 \leq u_n \leq v_2$  a.e., and a sequence of measurable sets  $(E_n)$  in  $\Omega$  such that

$$|E_n| \rightarrow 0 \quad \text{and} \quad \int_{E_n} g(x, u_n)\rho_0 dx \geq \varepsilon \quad \forall n \geq 1.$$

Applying Lemma 2.1 with  $w_n = g(\cdot, u_n)\rho_0/\varepsilon$ , we can extract a subsequence  $(u_{n_k})$  and a sequence of disjoint measurable sets  $(F_k)$  in  $\Omega$  such that

$$(2.7) \quad \int_{F_k} |g(x, u_{n_k})|\rho_0 dx \geq \frac{\varepsilon}{2} \quad \forall k \geq 1.$$

Let

$$v(x) = \begin{cases} u_{n_k}(x) & \text{if } x \in F_k \text{ for some } k \geq 1, \\ v_1(x) & \text{otherwise.} \end{cases}$$

Then,  $v_1 \leq v \leq v_2$  a.e.; hence,  $v \in L^1(\Omega)$ . Moreover,

$$\int_{\Omega} |g(x, v)|\rho_0 dx \geq \sum_{k=1}^{\infty} \int_{F_k} |g(x, u_{n_k})|\rho_0 dx = \infty.$$

This contradicts (2.5). Therefore,  $\mathcal{B}$  is equi-integrable in  $L^1(\Omega; \rho_0 dx)$ . □

We now present the

*Proof of Theorem 2.1.* Assume  $v_n \rightarrow v$  in  $L^1(\Omega)$ . Let  $(v_{n_k})$  be a subsequence such that  $v_{n_k} \rightarrow v$  a.e. and  $|v_{n_k}| \leq V$  a.e. for some function  $V \in L^1(\Omega)$ . In particular,

$$g(\cdot, v_{n_k}) \rightarrow g(\cdot, v) \quad \text{a.e.}$$

Moreover, by Proposition 2.1 above (applied to  $v_1 = -V$  and  $v_2 = V$ ), the sequence  $(g(\cdot, v_{n_k}))$  is equi-integrable in  $L^1(\Omega; \rho_0 dx)$ . It then follows from Egorov's theorem that

$$g(\cdot, v_{n_k}) \rightarrow g(\cdot, v) \quad \text{in } L^1(\Omega; \rho_0 dx).$$

Since the limit is independent of the subsequence  $(v_{n_k})$ , we deduce that

$$G(v_n) \rightarrow G(v) \quad \text{in } L^1(\Omega; \rho_0 dx). \quad \square$$

3. EXISTENCE, COMPACTNESS AND COMPARISON RESULTS

In this section, we recall some well-known results related to weak solutions of the linear problem:

$$(3.1) \quad \begin{cases} -\Delta w = h & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

We begin with the existence and compactness of weak solutions of (3.1):

**Theorem 3.1.** *Given  $h \in L^1(\Omega; \rho_0 dx)$ , there exists a unique  $w \in L^1(\Omega)$  such that*

$$(3.2) \quad - \int_{\Omega} w \Delta \zeta = \int_{\Omega} h \zeta \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

Moreover,

(i) For every  $1 \leq p < \frac{N}{N-1}$ ,  $w \in L^p(\Omega)$  and

$$(3.3) \quad \|w\|_{L^p} \leq C_p \|h\rho_0\|_{L^1}.$$

(ii) Given  $h_n \in L^1(\Omega; \rho_0 dx)$ ,  $n \geq 1$ , let  $w_n$  be the solution of (3.2) associated to  $h_n$ . If  $(h_n)$  is bounded in  $L^1(\Omega; \rho_0 dx)$ , then  $(w_n)$  is relatively compact in  $L^p(\Omega)$  for every  $1 \leq p < \frac{N}{N-1}$ .

*Proof.* We refer the reader to [2] for the existence and uniqueness of  $w$ . We split the proof of (i)–(ii) into two steps:

*Step 1.* Proof of (i).

Note that  $w$  satisfies

$$(3.4) \quad \left| \int_{\Omega} w \Delta \zeta \right| \leq \|h\rho_0\|_{L^1} \|\zeta/\rho_0\|_{L^\infty} \quad \forall \zeta \in C_0^2(\overline{\Omega}).$$

Given  $f \in C^\infty(\overline{\Omega})$ , let  $\zeta \in C_0^2(\overline{\Omega})$  be the solution of

$$\begin{cases} -\Delta \zeta = f & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

By Calderón-Zygmund estimates (see [10]),

$$(3.5) \quad \|\zeta\|_{W^{2,p'}} \leq C \|f\|_{L^{p'}}.$$

Since  $p' > N$ , it follows from Morrey’s imbedding that

$$(3.6) \quad \|\zeta/\rho_0\|_{L^\infty} \leq C (\|\zeta\|_{L^\infty} + \|\nabla \zeta\|_{L^\infty}) \leq C \|\zeta\|_{W^{2,p'}}.$$

Combining (3.4)–(3.6), we get

$$(3.7) \quad \left| \int_{\Omega} w f \right| \leq C_p \|h\rho_0\|_{L^1} \|f\|_{L^{p'}} \quad \forall f \in C^\infty(\overline{\Omega}).$$

By duality, one deduces that  $w \in L^p(\Omega)$  and (3.3) holds.

*Step 2.* Proof of (ii).

Given a smooth domain  $U \subset\subset \Omega$ , let  $v_n \in L^1(U)$  be the solution of

$$\begin{cases} -\Delta v_n = h_n & \text{in } U, \\ v_n = 0 & \text{on } \partial U. \end{cases}$$

By elliptic estimates (see [15]),

$$(3.8) \quad \|v_n\|_{W^{1,p}(U)} \leq C_p \|h_n\|_{L^1(U)} \leq C_p$$

for every  $1 \leq p < \frac{N}{N-1}$ . On the other hand, since  $w_n - v_n$  is harmonic in  $U$ , for every  $\omega \subset\subset U$  we have

$$(3.9) \quad \|w_n - v_n\|_{C^1(\bar{\omega})} \leq C_\omega \|w_n - v_n\|_{L^1(U)} \leq C_\omega \|h_n \rho_0\|_{L^1} \leq C_\omega.$$

Thus, by a diagonalization argument there exists a subsequence  $(w_{n_k})$  such that  $w_{n_k} \rightarrow w$  a.e. in  $\Omega$ . On the other hand, by (i) the sequence  $(w_n)$  is bounded in  $L^p(\Omega)$  for every  $1 \leq p < \frac{N}{N-1}$ . The conclusion then follows from Egorov’s theorem.  $\square$

We also need the following version of Kato’s inequality (see [3, Proposition B.5]):

**Proposition 3.1.** *Let  $w \in L^1(\Omega)$  and  $h \in L^1(\Omega; \rho_0 dx)$  be such that*

$$-\int_\Omega w \Delta \zeta \leq \int_\Omega h \zeta \quad \forall \zeta \in C_0^2(\bar{\Omega}), \zeta \geq 0 \text{ in } \Omega.$$

Then,

$$-\int_\Omega w^+ \Delta \zeta \leq \int_{[w \geq 0]} h \zeta \quad \forall \zeta \in C_0^2(\bar{\Omega}), \zeta \geq 0 \text{ in } \Omega.$$

**Corollary 3.1.** *If  $u, v$  are solutions of (1.1), then  $\max\{u, v\}$  is a subsolution.*

*Proof.* Apply Proposition 3.1 with  $w = v - u$  and  $h = f(\cdot, v) - f(\cdot, u)$ . Then,

$$-\int_\Omega (v - u)^+ \Delta \zeta \leq \int_{[v \geq u]} [f(x, v) - f(x, u)] \zeta dx \quad \forall \zeta \in C_0^2(\bar{\Omega}), \zeta \geq 0 \text{ in } \Omega.$$

Since  $\max\{u, v\} = u + (v - u)^+$ , the result follows.  $\square$

#### 4. PROOF OF THEOREM 1.1

We split the proof into two steps:

*Step 1.* Problem (1.1) has a solution  $u$  such that  $v_1 \leq u \leq v_2$  a.e.

Given  $(x, t) \in \Omega \times \mathbb{R}$ , let

$$g(x, t) := \begin{cases} v_1(x) & \text{if } t < v_1(x), \\ t & \text{if } v_1(x) \leq t \leq v_2(x), \\ v_2(x) & \text{if } v_2(x) < t. \end{cases}$$

Then,  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function and, by (1.5), we have

$$g(\cdot, v) \rho_0 \in L^1(\Omega) \quad \text{for every } v \in L^1(\Omega).$$

We now consider

$$G : L^1(\Omega) \longrightarrow L^1(\Omega; \rho_0 dx), \\ v \longmapsto g(\cdot, v)$$

and

$$K : L^1(\Omega; \rho_0 dx) \longrightarrow L^1(\Omega), \\ h \longmapsto w$$

where  $w$  is the unique solution of

$$(4.1) \quad \begin{cases} -\Delta w = h & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorems 2.1 and 3.1,  $KG : L^1(\Omega) \rightarrow L^1(\Omega)$  is continuous. Moreover, by Proposition 2.1 (applied to  $f$ ),  $G(L^1(\Omega))$  is a bounded subset of  $L^1(\Omega; \rho_0 dx)$ . Hence, by Theorem 3.1,  $KG$  is compact and there exists  $C > 0$  such that

$$\|K(G(v))\|_{L^1} \leq C_1 \|G(v)\rho_0\|_{L^1} \leq C \quad \forall v \in L^1(\Omega).$$

It then follows from Schauder's fixed point theorem that  $KG$  has a fixed point  $u \in L^1(\Omega)$ . In other words,  $u$  satisfies

$$(4.2) \quad \begin{cases} -\Delta u = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We claim that  $u$  is a solution of (1.1) and

$$(4.3) \quad v_1 \leq u \leq v_2 \quad \text{a.e.}$$

It suffices to prove (4.3). We show that  $u \leq v_2$  a.e.; the proof of the inequality  $v_1 \leq u$  a.e. is similar. Note that

$$g(\cdot, u) = g(\cdot, v_2) \quad \text{a.e. on the set } [u \geq v_2].$$

Thus, applying Proposition 3.1 to  $w = u - v_2$ , we get

$$-\int_{\Omega} w^+ \Delta \zeta \leq \int_{[u \geq v_2]} [g(x, u) - g(x, v_2)] \zeta dx = 0 \quad \forall \zeta \in C_0^2(\bar{\Omega}), \zeta \geq 0 \text{ in } \Omega.$$

We easily deduce that  $w^+ \leq 0$  a.e.; hence,  $w^+ = 0$  a.e. This implies  $u \leq v_2$  a.e. The proof of Step 1 is complete.

*Step 2.* There exist a smallest and a largest solution  $u_1 \leq u_2$  of (1.1) in the interval  $[v_1, v_2]$ .

(In [7], the proof of this step is based on Zorn's lemma. We could have followed their approach, but we present a different reasoning.)

We prove the existence of the largest solution  $u_2$ ; the existence of  $u_1$  is similar. Let

$$A = \sup \left\{ \int_{\Omega} w; v_1 \leq w \leq v_2 \text{ a.e. and } w \text{ is a solution of (1.1)} \right\}.$$

Clearly,  $A < \infty$ . Before we proceed, let us prove the following

*Claim.* If  $w_1, w_2$  are two solutions of (1.1) such that  $v_1 \leq w_1, w_2 \leq v_2$  a.e., then (1.1) has a solution  $w$  such that

$$(4.4) \quad v_1 \leq \max\{w_1, w_2\} \leq w \leq v_2 \quad \text{a.e.}$$

Indeed, by Corollary 3.1,  $\max\{w_1, w_2\}$  is a subsolution of (1.1). Applying Step 1 above with  $\max\{w_1, w_2\}$  and  $v_2$ , one finds a solution  $w$  of (1.1) satisfying (4.4). This establishes the claim.

It follows from the claim above that one can find a *nondecreasing* sequence of solutions  $(w_n)$  of (1.1) such that

$$(4.5) \quad v_1 \leq w_n \leq v_2 \text{ a.e. and } \int_{\Omega} w_n \rightarrow A.$$

By monotone convergence, there exists  $w_0 \in L^1(\Omega)$  such that  $w_n \rightarrow w_0$  a.e.,

$$v_1 \leq w_0 \leq v_2 \text{ a.e. and } \int_{\Omega} w_n \rightarrow \int_{\Omega} w_0 = A.$$

On the other hand, by Proposition 2.1 the sequence  $(f(\cdot, w_n))$  is equi-integrable in  $L^1(\Omega; \rho_0 dx)$ . It then follows from Egorov’s theorem that

$$f(\cdot, w_n) \rightarrow f(\cdot, w_0) \quad \text{in } L^1(\Omega; \rho_0 dx).$$

Thus,  $w_0$  is a solution of (1.1) and

$$\int_{\Omega} w_0 = A.$$

By the claim above,  $w_0$  is the largest solution of (1.1) in the interval  $[v_1, v_2]$ . The proof of Theorem 1.1 is complete.  $\square$

### 5. SOME CONSEQUENCES AND FURTHER RESULTS

In this section, we discuss some consequences of Theorem 1.1. In what follows, we denote by  $v_1, v_2$  sub and supersolutions of (1.1) satisfying  $v_1 \leq v_2$  a.e. Using regularity theory, in each case one shows that the solution provided in Theorem 1.1 lies in a better space.

**Corollary 5.1.** *If  $v_1, v_2 \in C^2(\overline{\Omega})$  and  $f \in C^1(\overline{\Omega} \times \mathbb{R})$ , then (1.1) has a classical solution  $u \in C^2(\overline{\Omega})$  such that  $v_1 \leq u \leq v_2$  in  $\Omega$ .*

**Corollary 5.2.** *If  $v_1, v_2 \in L^\infty(\Omega)$  and  $f \in C(\overline{\Omega} \times \mathbb{R})$ , then (1.1) has a solution  $u \in C^{1,\alpha}(\overline{\Omega})$ ,  $\forall \alpha \in (0, 1)$ , such that  $v_1 \leq u \leq v_2$  in  $\Omega$ .*

Corollary 5.2 is established in [6] for functions  $v_1, v_2 \in C(\overline{\Omega})$ .

**Corollary 5.3.** *If  $v_1, v_2 \in L^1(\Omega)$  and  $f$  is a Carathéodory function satisfying*

$$(5.1) \quad f(\cdot, v) \in L^{\frac{2N}{N+2}}(\Omega) \quad \text{for every } v \in L^1(\Omega) \text{ such that } v_1 \leq v \leq v_2 \text{ a.e.,}$$

*then (1.1) has a solution  $u \in H_0^1(\Omega)$  such that  $v_1 \leq u \leq v_2$  a.e.*

Corollary 5.3 is stated in [7] under the assumption that  $f(\cdot, v) \in L^p$  for some  $p > \frac{2N}{N+2}$ .

We now apply Theorem 1.1 to study semilinear problems with measure data, namely

$$(5.2) \quad \begin{cases} -\Delta u = f(x, u) + \mu & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega, \end{cases}$$

where  $\mu$  is a bounded measure in  $\Omega$  and  $\nu$  is a bounded measure on  $\partial\Omega$ . We say that  $u$  is a solution of (5.2) if  $u \in L^1(\Omega)$ ,  $f(\cdot, u)\rho_0 \in L^1(\Omega)$  and

$$-\int_{\Omega} u \Delta \zeta = \int_{\Omega} f(x, u) \zeta + \int_{\Omega} \zeta d\mu - \int_{\partial\Omega} \frac{\partial \zeta}{\partial n} d\nu \quad \forall \zeta \in C_0^2(\overline{\Omega}),$$

where  $n$  denotes the outward unit norm on  $\partial\Omega$ . This type of problem has been extensively studied in [3, 5] when  $f$  is of the form  $f(x, t) = -g(t)$ , where  $g$  is continuous, nondecreasing and  $g(0) = 0$ .

Applying Theorem 1.1 we deduce the following corollary which extends results in [3, 5]:

**Corollary 5.4.** *If  $v_1 \leq v_2$  are sub and supersolutions of (5.2) and  $f$  is a Carathéodory function satisfying (1.5), then (5.2) has a solution  $u \in L^1(\Omega)$  such that  $v_1 \leq u \leq v_2$  a.e.*



*Proof.* Let  $w \in L^1(\Omega)$  be the unique solution of (see [15, 11])

$$\begin{cases} -\Delta w = \mu & \text{in } \Omega, \\ w = \nu & \text{on } \partial\Omega. \end{cases}$$

Let  $\tilde{u} := u - w$ . The resulting equation in terms of  $\tilde{u}$  satisfies the assumptions of Theorem 1.1. Hence, (5.2) has a solution.  $\square$

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