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SPLICING AND THE $SL_2(\mathbb{C})$ CASSON INVARIANT

HANS U. BODEN AND CYNTHIA L. CURTIS

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Abstract. We establish a formula for the $SL_2(\mathbb{C})$ Casson invariant of spliced sums of homology spheres along knots. Along the way, we show that the $SL_2(\mathbb{C})$ Casson invariant vanishes for spliced sums along knots in S^3.

1. Introduction

In [13], Kronheimer and Mrowka prove that all nontrivial knots in S^3 have Property P. Their proof is based on strong existence results for irreducible $SU(2)$ representations of 3-manifolds obtained by Dehn surgery. It remains an interesting and important problem to determine whether a given 3-manifold admits irreducible $SU(2)$ representations. For example, for homology spheres Σ, nontriviality of the Casson invariant or Floer homology implies the existence of an irreducible $SU(2)$ representation. Since every irreducible $SU(2)$ representation is also irreducible as an $SL_2(\mathbb{C})$ representation, one expects stronger results for $SL_2(\mathbb{C})$. For example, Boyer and Zhang [5], and independently Dunfield and Garoufalidis [9], show that any nontrivial knot K in S^3 has nontrivial A-polynomial by using [14] to establish the existence of an arc of irreducible $SL_2(\mathbb{C})$ characters on $\pi_1(S^3 \setminus K)$.

Given a closed 3-manifold Σ, the $SL_2(\mathbb{C})$ Casson invariant $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ is defined (roughly) as the sum of isolated points of irreducible characters in the $SL_2(\mathbb{C})$ character variety $X(\Sigma)$. Thus, nontriviality of $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ guarantees the existence of an irreducible representation $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$, and this gives motivation for studying the $SL_2(\mathbb{C})$ Casson invariant.

In this paper, we use the spliced sum construction to present a family of homology spheres with $\lambda_{SL_2(\mathbb{C})}(\Sigma) = 0$. Since every isolated irreducible character contributes positively to the $SL_2(\mathbb{C})$ invariant, homology spheres Σ with $\lambda_{SL_2(\mathbb{C})}(\Sigma) = 0$ appear to be comparatively rare. We prove that, for any homology sphere Σ obtained by a spliced sums along two knots in S^3, every irreducible representation $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$ lies on a component X_i of the $SL_2(\mathbb{C})$ character variety $X(\Sigma)$ with $\dim X_i > 0$, and this implies $\lambda_{SL_2(\mathbb{C})}(\Sigma) = 0$.

More generally, we investigate the behavior of the invariant $\lambda_{SL_2(\mathbb{C})}$ under spliced sums along knots in arbitrary homology spheres. Using Casson's surgery formula, Fukuhara and Maruyama, and independently Boyer and Nicas, proved that the

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$SU(2)$ Casson invariant is additive under spliced sums [10, 2]. Unfortunately the same is not always true for the $SL_2(\mathbb{C})$ Casson invariant. Counterexamples are provided by Seifert fibered homology spheres. Recall that $\Sigma(p,q,r,s)$ is the spliced sum of $\Sigma(p,q,rs)$ and $\Sigma(pq,r,s)$ along the rs-singular fiber in the first case and the pq-singular fiber in the second case. However,

$$\lambda_{SL_2(\mathbb{C})}(\Sigma(p,q,r,s)) \neq \lambda_{SL_2(\mathbb{C})}(\Sigma(p,q,rs)) + \lambda_{SL_2(\mathbb{C})}(\Sigma(pq,r,s)).$$

For example, Theorem 2.7 of [1] shows that $\lambda_{SL_2(\mathbb{C})}(\Sigma(2,3,5,7)) = 20$, whereas $\lambda_{SL_2(\mathbb{C})}(\Sigma(2,3,35)) + \lambda_{SL_2(\mathbb{C})}(\Sigma(6,5,7)) = 17 + 30 = 47$.

In Theorem 3.4, our main result, we develop sufficient conditions, phrased in terms of the knots, under which the Casson $SL_2(\mathbb{C})$ invariant is additive under spliced sums.

For the remainder of the paper we will use the following notation: Given a finitely generated group π, denote by $R(\pi)$ the space of representations $\rho\colon \pi \to SL_2(\mathbb{C})$ and by $R^*(\pi)$ the subspace of irreducible representations. The character of a representation ρ will be denoted by χ_ρ. The variety of characters of $SL_2(\mathbb{C})$ representations is denoted by $X(\pi)$. Recall that there is a canonical projection $R(\pi) \to X(\pi)$ defined by $\rho \mapsto \chi_\rho$ that is surjective. Let $X^*(\pi)$ be the subspace of characters of irreducible characters. Given a manifold Σ, we denote by $R(\Sigma)$ the space of $SL_2(\mathbb{C})$ representations of $\pi_1\Sigma$ and by $X(\Sigma)$ the associated character variety.

For the definition of $\lambda_{SL_2(\mathbb{C})}$, see [7].

In section 2 we study homology spheres resulting from $1/q$ Dehn surgery on small knots in S^3 and show that the $SL_2(\mathbb{C})$ Casson invariants of such homology spheres are almost always nontrivial. In section 3, we introduce splicing and describe the behavior of the $SL_2(\mathbb{C})$ Casson invariant under a spliced sum.

We thank the referee for suggestions for improving Theorem 2.1.

2. Nonvanishing theorems

In this section, we show that the $SL_2(\mathbb{C})$ Casson invariant is nonzero for many homology spheres. Given a knot K in S^3 and slope $p/q \in \mathbb{Q} \cup \{1/0\}$, we denote by $S^3_{p/q}(K)$ the 3-manifold obtained by performing p/q Dehn surgery along K. Recall that $S^3_{1/q}(K)$ is always a homology sphere.

Theorem 2.1. *Let K be a small nontrivial knot in S^3, and let q be an integer with $|q| > 1$. Then $\lambda_{SL_2(\mathbb{C})}(S^3_{1/q}(K)) > 0$.*

Proof. By [13], there is an irreducible $SU(2)$ representation of $\pi_1(S^3_{1/q}(K))$, so the variety of characters of irreducible $SL_2(\mathbb{C})$ representations is nonempty. We must show that it contains a component of dimension 0. In fact we show that every component has dimension 0.

Suppose q is an integer such that the character variety $X(S^3_{1/q}(K))$ contains a component Y of dimension at least 1. We may view Y as a subset of the character variety $X(N)$ of the complement N of K in S^3 since $X(S^3_{1/q}(K)) \subset X(N)$. Since K is small, Y is one-dimensional, and there is a well-defined Culler-Shalen seminorm $\|\cdot\|_Y$ on Y given by

$$\|\alpha\|_Y = \deg(\tilde{I}^Y_{e(\alpha)} - 2)$$

where $\tilde{Y}$ is a smooth projective curve birationally equivalent to Y, $\tilde{I}^Y_\gamma$ is the function on $\tilde{Y}$ induced by the regular function $Y \to \mathbb{C}$ taking a character ξ to $\xi(\gamma)$, and $e : H_1(\partial N; \mathbb{Z}) \to \pi_1(\partial N)$ is the inverse of the Hurewicz isomorphism. But $\tilde{I}^Y_{e(1/q)} - 2$ vanishes on Y since Y lies in $X(S^3_{1/q}(K))$, so Y is an r-curve as defined in [3] with $r = 1/q$. Then by Corollary 6.7 of [3], we see that $1/q$ is an integer, contradicting the assumption that $|q| > 1$.

It follows that every component of $X(S^3_{1/q}(K))$ has dimension 0, whence the theorem. □

In particular, we have the following:

Theorem 2.2. *If K is a 2-bridge knot or a torus knot, then $\lambda_{SL_2(\mathbb{C})}(S^3_{1/q}(K)) > 0$ for all nonzero integers q.*

Proof. If $|q| > 1$, the claim follows from the previous theorem. By [13] we know that $X(S^3_{\pm 1}(K))$ contains an irreducible character. We show that every component of $X(S^3_{\pm 1}(K))$ is 0-dimensional, so $X(S^3_{\pm 1}(K))$ contains an isolated irreducible character.

Now as above we know that if Y is a component of dimension greater than 1 in $X(S^3_{\pm 1}(K))$, then Y has dimension 1, and the Culler-Shalen seminorm associated with Y is indefinite with $\| \pm 1\| = 0$.

It follows by Proposition 5.4 of [5] that there is a positive integer k and an integral boundary slope α for K such that the Culler-Shalen seminorm for the curve Y is given by

$$4\|p\mathscr{M} + q\mathscr{L}\|_Y = k|p - q\alpha|$$

for any slope p/q. If $\|1\| = 0$, we see that $\alpha = 1$, and if $\| - 1\| = 0$, then $\alpha = -1$. But the boundary slopes of 2-bridge knots are all even integers, and the boundary slopes of the (r, s)-torus knot are 0 and rs. Thus in neither case is either 1 or -1 a boundary slope, so no such curve Y exists.

Thus, $X(S^3_{\pm 1}(K))$ contains an irreducible character and contains only 0-dimensional components. Hence $\lambda_{SL_2(\mathbb{C})}(S^3_{1/q}(K)) > 0$. □

3. Splicing

The goal of this section is to investigate the behavior of the $SL_2(\mathbb{C})$ Casson invariant under the operation of spliced sum. Suppose K_1 and K_2 are knots in closed 3-manifolds Σ_1 and Σ_2, respectively, and let $M_1 = \Sigma_1 \setminus K_1$ and $M_2 = \Sigma_2 \setminus K_2$ denote their complements. Both M_1 and M_2 are manifolds with boundary $\partial M_1 = \partial M_2 = T$ a torus, and we denote by $\mathscr{M}_i$ and $\mathscr{L}_i$ the meridian and longitude of K_i for $i = 1, 2$. The *spliced sum* of K_1 and K_2 is the 3-manifold $\Sigma = M_1 \cup_T M_2$, with ∂M_1 glued to ∂M_2 by a diffeomorphism identifying $\mathscr{M}_1$ to $\mathscr{L}_2$ and $\mathscr{L}_1$ to $\mathscr{M}_2$. (See Figure 1.) If Σ_1 and Σ_2 are both homology spheres, then an elementary exercise shows that Σ is also a homology sphere.

Given a representation $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$, then by restriction we obtain representations $\rho_1 = \rho|_{\pi_1(M_1)}$ and $\rho_2 = \rho|_{\pi_1(M_2)}$. The next theorem shows that any irreducible character χ_ρ for which both of the induced characters χ_{ρ_1} and χ_{ρ_2} are irreducible must lie on a curve of characters. Therefore such representations do not contribute to $\lambda_{SL_2(\mathbb{C})}(\Sigma)$.

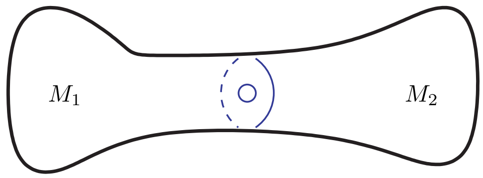

FIGURE 1. The spliced sum Σ along two knots K_1 and K_2

Theorem 3.1. *Suppose Σ is the spliced sum of two 3-manifolds Σ_1 and Σ_2 and $\chi_\rho \in X(\Sigma)$ is the character of an irreducible representation $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$ for which the induced characters χ_{ρ_1} and χ_{ρ_2} are irreducible. Then χ_ρ lies on a component X_i of $X(\Sigma)$ with $\dim X_i > 0$.*

Proof. By the Seifert-Van Kampen Theorem, any two irreducible representations $\rho_1\colon \pi_1(M_1) \to SL_2(\mathbb{C})$ and $\rho_2\colon \pi_1(M_2) \to SL_2(\mathbb{C})$ determine an irreducible representation $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$ provided the splicing relations hold, i.e. provided that $\rho_1(\mathscr{M}_1) = \rho_2(\mathscr{L}_2)$ and $\rho_2(\mathscr{M}_2) = \rho_1(\mathscr{L}_1)$. We use this fact to construct a curve of characters in the character variety containing χ_ρ.

Since ρ_1 and ρ_2 are irreducible, the stabilizer subgroup of each under the conjugation action is the group of central matrices $\{\left(\begin{smallmatrix}\pm1 & 0\\ 0 & \pm1\end{smallmatrix}\right)\}$. On the other hand, the restriction $\rho|_{\pi_1 T}$ of ρ to the spliced torus is abelian. Hence its stabilizer subgroup $\Gamma = \operatorname{Stab}(\rho|_{\pi_1 T})$ is either the subgroup

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle|\, a \in \mathbb{C}^* \right\}$$

of diagonal matrices or the subgroup

$$\left\{ \begin{pmatrix} \pm1 & a \\ 0 & \pm1 \end{pmatrix} \middle|\, a \in \mathbb{C} \right\}$$

of upper triangular univalent matrices. In either case, $\dim \Gamma = 1$. For any element $\gamma \in \Gamma$, the pair $(\rho_1, \gamma\rho_2\gamma^{-1})$ is a pair of irreducible representations of $\pi_1(M_1)$ and $\pi_1(M_2)$ that satisfy the splicing relations. The association $\gamma \in \Gamma \to \rho_\gamma$ gives a one-parameter family ρ_γ of $SL_2(\mathbb{C})$ representations of $\pi_1\Sigma$, and it is not hard to check that ρ_γ is conjugate to ρ if and only if $\gamma = \pm I$. Since distinct conjugacy classes of irreducible representations determine distinct characters, this shows that χ_ρ lies on a component X_i of irreducible characters with $\dim X_i > 0$. $\square$

The next several results rely on Proposition 6.1 of [6] regarding the complement M_1 of a knot K_1 in a homology sphere Σ_1 . This result asserts that the fundamental group of M_1 has a nonabelian reducible representation into $SL_2(\mathbb{C})$ with eigenvalue μ if and only if μ^2 is a root of the Alexander polynomial. Note that in this case the representation has the same character as an abelian representation, so such characters are the points of intersection of the curve of reducible characters with $X^*(\Sigma_1)$, as is noted in Proposition 6.2 of the same paper. For any knot K in a homology sphere, let $\Delta_{K_i}(t)$ denote the Alexander polynomial.

Proposition 3.2. *Given knots $K_1 \subset \Sigma_1$ and $K_2 \subset \Sigma_2$ in homology spheres, denote their complements by $M_1 = \Sigma_1 \setminus K_1$ and $M_2 = \Sigma_2 \setminus K_2$. If $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$ is an irreducible representation of the spliced sum $\Sigma = M_1 \cup_{T^2} M_2$, then at least one of $\rho_1 = \rho|_{\pi_1 M_1}$ or $\rho_2 = \rho|_{\pi_1 M_2}$ is irreducible.*

Proof. We will prove that if ρ_1 and ρ_2 are both reducible, then ρ is trivial. Since $\mathscr{L}_1$ lies in the second derived subgroup of $\pi_1(M_1)$, the reducibility of ρ_1 gives that $\rho_1(\mathscr{L}_1) = I$. Similarly, if ρ_2 is reducible, then $\rho_2(\mathscr{L}_2) = I$. Combined with the splicing relations, these facts imply that $\rho_1(\mathscr{M}_1) = I = \rho_2(\mathscr{M}_2)$. Now $\Delta_{K_1}(1) = \pm 1 \neq 0$, so Proposition 6.1 of [6] shows that ρ_1 is abelian. Since ρ_1 is abelian, it factors through $H_1(M_1)$, and hence $\rho_1(\mathscr{M}_1) = I$. This implies ρ_1 is trivial. A similar argument shows that ρ_2 is trivial; hence ρ is trivial. □

A direct consequence is that, for spliced sums along two knots K_1 and K_2 in S^3, the $SL_2(\mathbb{C})$ Casson invariant vanishes.

Corollary 3.3. *If Σ is a spliced sum along two knots K_1 and K_2 in S^3, then $\lambda_{SL_2(\mathbb{C})}(\Sigma) = 0$.*

Proof. Suppose ρ is an irreducible representation of $\pi_1(\Sigma)$ in $SL_2(\mathbb{C})$ with restrictions ρ_1 and ρ_2 as before. If ρ_1 and ρ_2 are both irreducible, then Theorem 3.1 shows that χ_ρ is not isolated and hence does not contribute to $\lambda_{SL_2(\mathbb{C})}(\Sigma)$. Otherwise, by Proposition 3.2, exactly one of ρ_1 and ρ_2 is irreducible.

Suppose that ρ_1 is irreducible and ρ_2 is reducible. The reducibility of ρ_2 implies $\rho_2(\mathscr{L}_2) = I$, so $\rho_1(\mathscr{M}_1) = I$ by the splicing relation. However, since K_1 is a knot in S^3, we know that the meridian $\mathscr{M}_1$ normally generates $\pi_1(M_1)$. Thus $\rho_1(\mathscr{M}_1) = I$ implies that ρ_1 is trivial, contradicting the irreducibility of ρ_1. A similar argument with the roles of ρ_1 and ρ_2 reversed reveals that $X^*(\Sigma)$ does not contain any components of dimension zero. Therefore $\lambda_{SL_2(\mathbb{C})}(\Sigma) = 0$. □

The next theorem is our main result, asserting additivity of the $SL_2(\mathbb{C})$ Casson invariant for spliced sums under certain restrictions. The restrictions we impose are necessary to rule out the types of counterexamples that were presented in the introduction. Specifically, the conditions given below use Proposition 6.1 of [6] to rule out unwanted interplay between the reducible and irreducible characters of M_1 and M_2.

Before stating the theorem, we find it convenient to define

$$X^\bullet(\Sigma) = \{\chi_\rho \in X^*(\Sigma) \mid \chi_\rho \text{ is isolated}\}$$

to be the subset of *isolated* irreducible characters of $\pi_1(\Sigma)$.

Theorem 3.4. *Assume $K_1 \subset \Sigma_1$ and $K_2 \subset \Sigma_2$ are knots in homology spheres, and consider the following conditions:*

(i) *For $\chi_\rho \in X^*(\Sigma_1)$, if μ is an eigenvalue of $\rho(\mathscr{L}_1)$, then $\Delta_{K_2}(\mu^2) \neq 0$.*
(ii) *For $\chi_\rho \in X^*(\Sigma_2)$, if μ is an eigenvalue of $\rho(\mathscr{L}_2)$, then $\Delta_{K_1}(\mu^2) \neq 0$.*

If condition (i) is satisfied for all $\chi \in X^\bullet(\Sigma_1)$ and condition (ii) is satisfied for all $\chi \in X^\bullet(\Sigma_2)$, then for the spliced sum, we have

$$\lambda_{SL_2(\mathbb{C})}(\Sigma) = \lambda_{SL_2(\mathbb{C})}(\Sigma_1) + \lambda_{SL_2(\mathbb{C})}(\Sigma_2).$$

Proof. If $\rho\colon \pi_1\Sigma \to SL_2(\mathbb{C})$ is an irreducible representation, then Proposition 3.2 implies that one of ρ_1 or ρ_2 is irreducible. If in addition $\chi_\rho \in X^\bullet(\Sigma)$ is isolated, then Theorem 3.1 shows that exactly one of ρ_1 and ρ_2 is irreducible. Hence we can partition $X^\bullet(\Sigma) = X_1^\bullet \cup X_2^\bullet$, where

$$X_1^\bullet = \{\chi_\rho \mid \rho_1 \text{ is irreducible and } \rho_2 \text{ is reducible}\},$$

$X_2^\bullet$ is defined similarly, and $X_1^\bullet$ and $X_2^\bullet$ are disjoint.

For $\chi_\rho \in X_1^\bullet$, reducibility of ρ_2 and the splicing relations imply that $\rho_1(\mathscr{M}_1) = \rho_2(\mathscr{L}_2) = I$. Hence ρ_1 extends to an irreducible representation $\rho_1'\colon \pi_1(\Sigma_1) \to SL_2(\mathbb{C})$. Thus, we have a natural map $\Phi_1 : X_1^\bullet \to X^*(\Sigma_1)$ given by $\chi_\rho \mapsto \chi_{\rho_1'}$. We define a map $\Phi_2 : X_2^\bullet \to X^*(\Sigma_2)$ analogously.

Conversely, given an irreducible representation $\rho_1'\colon \pi_1(\Sigma_1) \to SL_2(\mathbb{C})$ with $\chi_{\rho_1'}$ isolated, we define a reducible representation $\rho_2\colon \pi_1(M_2) \to SL_2(\mathbb{C})$ by setting $\rho_2(\mathscr{M}_2) = \rho_1(\mathscr{L}_1)$. (Here, $\rho_1 = \rho_1'|_{\pi_1(M_1)}$.) Note that hypothesis (i) implies that ρ_2 is abelian by Proposition 6.1 of [6], so this assignment of $\rho_2(\mathscr{M}_2)$ completely determines ρ_2. A direct inspection shows that ρ_1 and ρ_2 satisfy the splicing relations; thus they give rise to an irreducible representation $\rho\colon \Sigma \to SL_2(\mathbb{C})$. Furthermore, since $\chi_{\rho_1'} \in X^\bullet(\Sigma_1)$ is isolated and ρ_2 is completely determined by $\rho_1(\mathscr{L}_1)$, it is not difficult to see that $\chi_\rho \in X_1^\bullet \subset X^\bullet(\Sigma)$ is also isolated.

This defines a map $\Psi_1\colon X^\bullet(\Sigma_1) \to X_1^\bullet$, which is an inverse to Φ_1 and gives a one-to-one correspondence between $X^\bullet(\Sigma_1)$ and $X_1^\bullet$.

The same construction with the roles of ρ_1 and ρ_2 reversed defines a map $\Psi_2\colon X^\bullet(\Sigma_2) \to X_2^\bullet$ that is an inverse to Φ_2 and gives a one-to-one correspondence between $X^\bullet(\Sigma_2)$ and $X_2^\bullet$.

It remains to show that $\chi_1 \in X_1^\bullet \subset X^\bullet(\Sigma)$ and $\Phi_1(\chi_1) \in X^\bullet(\Sigma_1)$ both contribute equally to their respective $SL_2(\mathbb{C})$ Casson invariants, and similarly for $\chi_2 \in X_2^\bullet \subset X^\bullet(\Sigma)$ and $\Phi_2(\chi_2) \in X^\bullet(\Sigma_2)$.

Choose a triangulation of Σ_1 such that the 1-skeleton contains K_1. Build a Heegaard decomposition (U_1, U_2) of Σ_1 by letting U_1 be a tubular neighborhood of this 1-skeleton. (See Theorem 2.5 of [11].) Call the Heegaard surface for this Heegaard decomposition F_1. Similarly choose a triangulation for Σ_2 whose 1-skeleton contains K_2 and build a Heegaard splitting (V_1, V_2) of Σ_2 by letting V_2 be a neighborhood of the 1-skeleton. Call the Heegaard surface F_2.

Choose a symplectic basis for F_1 consisting of curves $a_1, b_1, \dots, a_j, b_j, \mathscr{M}_1, \mathscr{L}_1$, where the curves $a_1, \dots, a_j$, and $\mathscr{M}_1$ are homotopically trivial in U_1. Choose a symplectic basis for F_2 consisting of curves $\mathscr{L}_2, \mathscr{M}_2, c_1, d_1, \dots, c_k, d_k$, where the curves $\mathscr{M}_2$ and $d_1, \dots, d_k$ are homotopically trivial in V_2.

Note that U_1 is the union of a tubular neighborhood $N(K_1)$ of the knot K_1 and a handlebody H_1 of genus j spanned by the curves $a_1, b_1, \dots, a_j, b_j$ in the obvious way. Similarly V_2 is the union of a tubular neighborhood $N(K_2)$ of the knot K_2 and a handlebody H_2 of genus k spanned by the curves $d_1, c_1, \dots, d_k, c_k$ in the obvious way. On the other hand, U_2 is a subset of M_1 and V_1 is a subset of M_2. From this, one sees that the restrictions of these Heegaard splittings to M_1 and M_2 glue together to form a Heegaard splitting (W_1, W_2) of Σ, where W_1 is the connected sum of H_1 and V_1, and W_2 is the connected sum of U_2 and H_2. Denote by F the Heegaard surface of this Heegaard decomposition of Σ. Note that F can be viewed as the connected sum of F_1 and the boundary of H_2 or as the connected sum of the boundary of H_1 and F_2.

Now given $\chi_\rho \in X_1^\bullet$, we see that $X(V_2 - N(K_2))$ and $X(V_1)$ are transverse in $X(F_2)$ at χ_{ρ_2} since the dimension of $H^1(M_2; \mathfrak{sl}_2(\mathbb{C})_{\operatorname{Ad}\rho})$ is 1. This follows from the Mayer-Vietoris sequence for $M_2 = (V_2 - N(K_2)) \cup V_1$, using the fact that condition (i) of the theorem is satisfied at ρ.

It follows that there is an isotopy h_t of $X(F)$, $t \in [0,1]$, such that h_0 is the identity; $h_t(\phi(c_i)) = \phi(c_i)$, $h_t(\phi(d_i)) = \phi(d_i)$, $h_t(\chi_\phi(\mathscr{M}_2)) = \chi_\phi(\mathscr{M}_2)$, and $h_t(\chi_\phi(\mathscr{L}_2)) = \chi_\phi(\mathscr{L}_2)$ for every χ_ϕ and every i; and $h_1(X(W_2))$ meets $X(W_1)$

transversely in a neighborhood of χ_ρ. In fact, h_t can be chosen to have support in a neighborhood N of χ_ρ such that N meets $X(\Sigma)$ only in χ_ρ and for any χ_ϕ in N, if μ is an eigenvalue of $\phi(\mathscr{L}_1)$, then $\Delta_{K_2}(\mu^2) \neq 0$. Then the contribution of χ_ρ to $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ is precisely the number of points in the intersection of $X(W_1)$ and $h_1(X(W_2))$ in N.

Now since $h_t(\phi(c_i)) = \phi(c_i)$ and $h_t(\phi(d_i)) = \phi(d_i)$ for every i and every ϕ, we see that h_t preserves the subvariety of $X(F)$ consisting of characters χ_ϕ for which $[\phi(c_1), \phi(d_1)][\phi(c_2), \phi(d_2)] \dots [\phi(c_k), \phi(d_k)] = I$. But these characters are precisely the characters for which $[\phi(a_1), \phi(b_1)][\phi(a_2), \phi(b_2)] \dots [\phi(a_j), \phi(b_j)][\phi(\mathscr{M}_1), \phi(\mathscr{L}_1)] = I$, i.e., the characters that are the images of characters in $X(F_1)$. It follows that h_t induces an isotopy $\tilde{h}_t$ of $X(F_1)$. Moreover $X(U_1)$ and $\tilde{h}_1(X(U_2))$ intersect transversely in a neighborhood of the image of $\Phi_1(\chi_\rho)$ in $X(F_1)$ since $X(W_1)$ and $h_t(X(W_2))$ are transverse. Thus the contribution of $\Phi_1(\chi_\rho)$ to $\lambda_{SL_2(\mathbb{C})}(\Sigma_1)$ is precisely the number of points of intersection of $X(U_1)$ and $\tilde{h}_1(X(U_2))$ in the neighborhood of $\Phi_1(\chi_\rho)$ that is the support of $\tilde{h}$.

It remains to be shown that the points of intersection of $X(U_1)$ and $\tilde{h}_1(X(U_2))$ in the neighborhood of $\Phi_1(\chi_\rho)$ that is the support of $\tilde{h}$ are in one-to-one correspondence with the points of intersection of $X(W_1)$ and $h_1(X(W_2))$ in N. This follows since every point χ_ϕ in the intersection of $X(W_1)$ and $h_1(X(W_2))$ in N satisfies $\phi(d_1) = \phi(d_2) = \dots = \phi(d_k) = I$ since h_t did not affect the values of ϕ at $d_1, d_2, \dots, d_k$ and $\chi_\phi \in h_t(X(W_2))$, and so $[\phi(c_1), \phi(d_1)][\phi(c_2), \phi(d_2)] \dots [\phi(c_k), \phi(d_k)] = I = [\phi(a_1), \phi(b_1)][\phi(a_2), \phi(b_2)] \dots [\phi(a_j), \phi(b_j)][\phi(\mathscr{M}_1), \phi(\mathscr{L}_1)]$.

Thus, χ_ρ and $\Phi_1(\chi_\rho)$ contribute equally to their respective Casson invariants. That points in $\chi_2 \in X^\bullet(\Sigma_2)$ and $\Psi_2(\chi_2) \in X_2^\bullet \subset X^\bullet(\Sigma)$ also contribute equally to their respective Casson invariants can be proved analogously. □

Remark 3.5. A useful observation is that the two hypotheses in Theorem 3.4, are equivalent to the following conditions:

(i) If t^{2k} is a root of the Alexander polynomial of K_2, then $A_{K_1}(t, t^{-k}) \neq 0$, where A_{K_1} denotes the A-polynomial of K_1 .
(ii) If t^{2k} is a root of the Alexander polynomial of K_1, then $A_{K_2}(t, t^{-k}) \neq 0$, where A_{K_2} denotes the A-polynomial of K_2 .

We now describe the operation of a k-spliced sum for two knots K_1, K_2 in S^3. Let $M_1 = S^3 \setminus K_1$ and $M_2 = S^3 \setminus K_2$ be their complements, and denote by $\mathscr{M}_1, \mathscr{L}_1$ and $\mathscr{M}_2, \mathscr{L}_2$ the meridian and longitude of K_1 and K_2. The *k-spliced sum* is the 3-manifold $\Sigma_k = M_1 \cup_\phi M_2$, with ∂M_1 glued to ∂M_2 by a diffeomorphism ϕ identifying $\mathscr{M}_1$ to $\mathscr{L}_2$ and $\mathscr{L}_1$ to $\mathscr{M}_2 \mathscr{L}_2^k$. The diffeomorphism $\phi\colon \partial M_1 \to \partial M_2$ is represented on $\pi_1 T$ by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & k \end{smallmatrix}\right)$. It is not difficult to see that Σ_k is a homology sphere with Σ_0 the spliced sum considered previously. Furthermore, if K_1 is the unknot, then Σ_k is the homology sphere obtained by $1/k$ Dehn surgery on K_2. We apply Theorem 3.4 to determine the $SL_2(\mathbb{C})$ Casson invariant of k-spliced sums.

Corollary 3.6. *Let K_1 and K_2 be knots in S^3. Let k be an integer, and let Σ_k be the k-spliced sum of K_1 and K_2. Then $\lambda_{SL_2(\mathbb{C})}(\Sigma_k) = \lambda_{SL_2(\mathbb{C})}\Big(S^3_{1/k}(K_2)\Big)$.*

Proof. Set $\Sigma_2 = S^3_{1/k}(K_2)$ and let $\widetilde{K}_2$ be the image of K_2 in Σ_2. Then the k-spliced sum of K_1 and K_2 is homeomorphic to the spliced sum of S^3 and Σ_2 along K_1 and $\widetilde{K}_2$ since the meridian of $\widetilde{K}_2$ is $\mathscr{M}_2 \mathscr{L}_2^k$.

If $\rho_2\colon \pi_1(\Sigma_2) \to SL_2(\mathbb{C})$ is an irreducible representation, then $\rho_2(\mathscr{M}_2\mathscr{L}_2^k) = I$. We can conjugate so that $\rho_2(\mathscr{L}_2)$ is either diagonal or a matrix of the form $\left(\begin{smallmatrix} \pm 1 & a \\ 0 & \pm 1 \end{smallmatrix}\right)$ for some $a \in \mathbb{C}$. If $\rho_2(\mathscr{L}_2)$ is diagonal, then since $\rho(\mathscr{M}_2\mathscr{L}_2^k) = I$, we see that $\rho_2(\mathscr{M}_2)$ is also diagonal, which contradicts the irreducibility of ρ_2.

Hence the eigenvalues of $\rho_2(\mathscr{L}_2)$ are in $\{\pm 1\}$. Since their squares, which equal 1, are not roots of the Alexander polynomial for any knot, Theorem 3.4 applies and implies $\lambda_{SL_2(\mathbb{C})}(\Sigma) = \lambda_{SL_2(\mathbb{C})}(\Sigma_2) = \lambda_{SL_2(\mathbb{C})}(S^3_{1/k}(K_2))$. □

Finally, combining this corollary with Theorem 2.2 yields the following result.

Corollary 3.7. *If K_1 is any knot in S^3 and K_2 is a 2-bridge or a torus knot, then the k-spliced sum of K_1 and K_2 satisfies $\lambda_{SL_2(\mathbb{C})}(\Sigma_k) > 0$ for $k \neq 0$.*

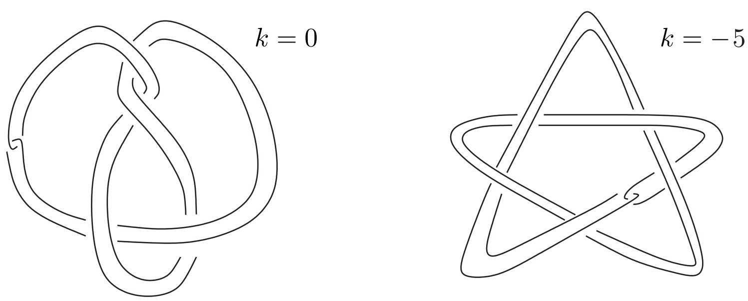

Figure 2. The k-twisted Whitehead doubles of the figure-8 knot and the $(2, 5)$ torus knot

If K_1 is the left-hand trefoil, the k-spliced sum of K_1 and K_2 is the homology sphere obtained by -1 surgery on the $-k$-twisted Whitehead double of K_2 (see Prop. 6.1, [12]). In particular, we conclude that the homology sphere Σ_k obtained by -1 surgery on a k-twisted Whitehead double of any 2-bridge or torus knot has $\lambda_{SL_2(\mathbb{C})}(\Sigma_k) > 0$ provided $k \neq 0$. For example, taking $K_2 = T(p, q)$ to be the (p, q) torus knot and denoting by L_k the $-k$-twisted Whitehead double of $T(p, q)$, shown in Figure 2 for $(p, q) = (2, 5)$, we see that for $k > 0$, we have

$$\lambda_{SL_2(\mathbb{C})}(S^3_{-1}(L_k)) = \lambda_{SL_2(\mathbb{C})}(\Sigma(p, q, pqk - 1)) = \frac{(p-1)(q-1)(pqk-2)}{4}$$

by combining the above corollary with Theorem 2.3 in [1]. A similar result follows for $k < 0$, and the same idea applies to provide explicit computations of $\lambda_{SL_2(\mathbb{C})}(S^3_{-1}(L_k))$ for L_k the $-k$-twisted Whitehead double of a twist knot as for the figure-8 knot shown in Figure 2; see Theorems 5.7 and 5.9 in [1].

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Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, L8S 4K1 Canada
E-mail address: boden@mcmaster.ca

Department of Mathematics and Statistics, The College of New Jersey, Ewing, New Jersey 08628
E-mail address: ccurtis@tcnj.edu